

Chapter 1

The Characteristic Matrix of Nonuniqueness for First-Kind Equations

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1.1 Introduction

Let S be a finite domain in \mathbb{R}^2 , bounded by a simple, closed, C^2 curve ∂S . We denote by x and y generic points in $S \cup \partial S$ and by $|x - y|$ the distance between x and y in the Cartesian metric. Also, let $C^{0,\alpha}(\partial S)$ and $C^{1,\alpha}(\partial S)$, $\alpha \in (0, 1)$, be, respectively, the spaces of Hölder continuous and Hölder continuously differentiable functions on ∂S . In what follows, Greek and Latin indices take the values 1, 2 and 1, 2, 3, respectively, and a superscript T denotes matrix transposition.

For any function f continuous on ∂S , we define the ‘calibration’ functional p by

$$pf = \int_{\partial S} f ds.$$

Using the fundamental solution for the two-dimensional Laplacian

$$g(x, y) = -\frac{1}{2\pi} \ln |x - y|,$$

we define the single-layer harmonic potential of density φ by

$$(V\varphi)(x) = \int_{\partial S} g(x, y)\varphi(y) ds(y).$$

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The proof of the following assertion can be found, for example, in [Co94] or [Co00].

Theorem 1. *For any $\alpha \in (0, 1)$, there are a unique nonzero function $\Phi \in C^{0,\alpha}(\partial S)$ and a unique number ω such that*

$$V\Phi = \omega \quad \text{on } \partial S, \quad p\Phi = 1.$$

It is easy to see that Φ and ω depend on g and ∂S .

The numbers $2\pi\omega$ and $e^{-2\pi\omega}$ are called *Robin's constant* and the *logarithmic capacity* of ∂S .

For a circle with the center at the origin and radius R , both Φ and ω can be determined explicitly:

$$\Phi = \frac{1}{2\pi R}, \quad \omega = -\frac{1}{2\pi} \ln R.$$

For other boundary curves, Φ and ω are practically impossible to determine analytically and must be computed by numerical methods.

If the solution of the Dirichlet problem in S with data function f on ∂S is sought as $u = V\varphi$, then φ is a solution of the (weakly singular) first-kind boundary integral equation

$$V\varphi = f \quad \text{on } \partial S.$$

This is a well-posed problem if and only if $\omega \neq 0$. If $\omega = 0$, the above equation has infinitely many solutions, which are expressed in terms of Φ .

1.2 Plane Elastic Strain

Consider a plate made of a homogeneous and isotropic material with Lamé constants λ and μ which undergoes deformations in the (x_1, x_2) -plane. If the body forces are negligible, then its (static) displacement vector $u = (u_1, u_2)^T$ satisfies the equilibrium system of equations [Co00]

$$Au = 0 \quad \text{in } S,$$

where

$$A(\partial_1, \partial_2) = \begin{pmatrix} \mu\Delta + (\lambda + \mu)\partial_1^2 & (\lambda + \mu)\partial_1\partial_2 \\ (\lambda + \mu)\partial_1\partial_2 & \mu\Delta + (\lambda + \mu)\partial_2^2 \end{pmatrix}.$$

It is not difficult to show [Co00] that the columns $F^{(i)}$ of the matrix

$$F = \begin{pmatrix} 1 & 0 & x_2 \\ 0 & 1 & -x_1 \end{pmatrix}$$

form a basis for the space of rigid displacements.

The ‘calibrating’ vector-valued functional p is defined for continuous 2×1 vector functions f by

$$pf = \int_{\partial S} F^T f ds.$$

A matrix of fundamental solutions for A is [Co00]

$$D(x, y) = -\frac{1}{4\pi\mu(\gamma+1)} \times \begin{pmatrix} 2\gamma \ln|x-y| + 2\gamma + 1 - \frac{2(x_1-y_1)^2}{|x-y|^2} & -\frac{2(x_1-y_1)(x_2-y_2)}{|x-y|^2} \\ -\frac{2(x_1-y_1)(x_2-y_2)}{|x-y|^2} & 2\gamma \ln|x-y| + 2\gamma + 1 - \frac{2(x_2-y_2)^2}{|x-y|^2} \end{pmatrix},$$

$$\gamma = \frac{\lambda + 3\mu}{\lambda + \mu}.$$

The single-layer potential of density φ is defined by

$$(V\varphi)(x) = \int_{\partial S} D(x, y)\varphi(y) ds(y).$$

The proof of the following assertion can be found in [Co00].

Theorem 2. *There is a unique 2×3 matrix function $\Phi \in C^{0,\alpha}(\partial S)$ and a unique 3×3 constant symmetric matrix \mathcal{C} such that the columns $\Phi^{(i)}$ of Φ are linearly independent and*

$$V\Phi = F\mathcal{C} \quad \text{on } \partial S, \quad p\Phi = I,$$

where I is the identity matrix.

Clearly, Φ and \mathcal{C} depend on A , D , and ∂S .

In the so-called alternative indirect method [Co00], the solution of the Dirichlet problem in S with data function f on ∂S is sought in the form

$$u = V\varphi. \tag{1.1}$$

Then the problem reduces to the (weakly singular) boundary integral equation

$$V\varphi = f \quad \text{on } \partial S. \tag{1.2}$$

Theorem 3. *Equation (1.1) has a unique solution $\varphi \in C^{0,\alpha}(\partial S)$, $\alpha \in (0, 1)$, for any $f \in C^{1,\alpha}(\partial S)$ if and only if $\det \mathcal{C} \neq 0$. In this case, (1.2) is the unique solution of the Dirichlet problem.*

If $\det \mathcal{C} = 0$, then the unique solution of the Dirichlet problem is obtained by solving an ill-posed modified boundary integral equation whose infinitely many solutions are constructed with Φ and \mathcal{C} .

In the so-called refined indirect method [Co00], the solution of the Dirichlet problem is sought as a pair $\{\varphi, c\}$ such that

$$u = V\varphi - Fc \quad \text{in } S, \quad p\varphi = s,$$

where s a constant 3×1 vector chosen (arbitrarily) a priori and c is a constant 3×1 vector. This leads to the system of boundary integral equations

$$V\varphi - Fc = f \quad \text{on } \partial S, \quad p\varphi = s. \quad (1.3)$$

Theorem 4. *System (1.3) has a unique solution $\{\varphi, c\}$ with $\varphi \in C^{0,\alpha}(\partial S)$ for any $f \in C^{1,\alpha}(\partial S)$, $\alpha \in (0, 1)$, and any s .*

It is important to evaluate the arbitrariness in the representation of the solution with respect to the prescribed ‘calibration’ s .

Theorem 5. *If $\{\varphi^{(1)}, c^{(1)}\}, \{\varphi^{(2)}, c^{(2)}\}$ are two solutions of (1.3) constructed with $s^{(1)}$ and $s^{(2)}$, respectively, then*

$$\begin{aligned} \varphi^{(2)} &= \varphi^{(1)} + \Phi(s^{(2)} - s^{(1)}), \\ c^{(2)} &= c^{(1)} + \mathcal{C}(s^{(2)} - s^{(1)}). \end{aligned}$$

It is not easy to compute Φ and \mathcal{C} analytically, or even numerically, in arbitrary domains S , but this can be accomplished if S is a circular disk. Let ∂S be the circle with center at the origin and radius R . In this case, Φ and \mathcal{C} can be determined analytically as

$$\Phi = \frac{1}{2\pi R} \begin{pmatrix} 1 & 0 & R^{-2}x_2 \\ 0 & 1 & -R^{-2}x_1 \end{pmatrix},$$

$$\begin{aligned} \mathcal{C} &= -\frac{1}{4\pi\mu(\lambda + 2\mu)R^2} \\ &\times \begin{pmatrix} (\lambda + 3\mu)R^2(\ln R + 1) & 0 & 0 \\ 0 & (\lambda + 3\mu)R^2(\ln R + 1) & 0 \\ 0 & 0 & -(\lambda + \mu) \end{pmatrix}. \end{aligned}$$

Clearly, $\det \mathcal{C} = 0$ if and only if $R = e^{-1}$.

Analytic computation of Φ and \mathcal{C} is practically impossible for non-circular domains, and must be performed numerically.

We choose four 3×1 constant vectors $s^{(0)}, s^{(i)}$ such that the set $\{s^{(i)} - s^{(0)}\}$ is linearly independent, and form the 3×3 matrix Σ with columns $s^{(i)} - s^{(0)}$. Also, we choose an arbitrary function f . Next, we compute the solutions $\{\varphi^{(0)}, c^{(0)}\}, \{\varphi^{(i)}, c^{(i)}\}$ of (1.3) corresponding to $s^{(0)}, s^{(i)}$, respectively, and f by the refined indirect method, then form the 2×3 matrix function Ψ with columns $\varphi^{(i)} - \varphi^{(0)}$ and the constant 3×3 matrix Γ with columns $c^{(i)} - c^{(0)}$.

From Theorem 4 it follows that

$$\begin{aligned}\varphi^{(i)} - \varphi^{(0)} &= \Phi(s^{(i)} - s^{(0)}), \\ c^{(i)} - c^{(0)} &= \mathcal{C}(s^{(i)} - s^{(0)}),\end{aligned}$$

or, what is the same,

$$\Phi\Sigma = \Psi, \quad \mathcal{C}\Sigma = \Gamma;$$

hence,

$$\Phi = \Psi\Sigma^{-1}, \quad \mathcal{C} = \Gamma\Sigma^{-1}.$$

A similar analysis can be performed for other two-dimensional linear elliptic systems with constant coefficients—for example, the system modeling bending of elastic plates with transverse shear deformation [Co00]. No apparent connection exists between the matrix \mathcal{C} and the characteristic constant ω of ∂S .

1.3 Numerical Examples

Consider a steel plate with scaled Lamé coefficients

$$\lambda = 11.5, \quad \mu = 7.69,$$

and let ∂S (see Fig. 1.1) be the curve of parametric equations

$$\begin{aligned}x_1(t) &= 2\cos(\pi t) - \frac{4}{3}\cos(2\pi t) + \frac{10}{3}, \\ x_2(t) &= 2\sin(\pi t) + 2, \quad 0 \leq t \leq 2.\end{aligned}$$

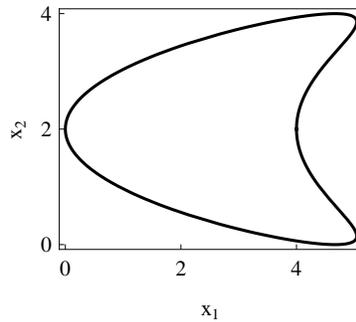


Fig. 1.1 The boundary curve ∂S .

We choose the vectors

$$s^{(0)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad s^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad s^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad s^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$f(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The approximating functions for computing $\varphi^{(0)}(t)$ and $\varphi^{(i)}(t)$ are piecewise cubic Hermite splines on 12 knots; that is, the interval $0 \leq t \leq 2$ is divided into 12 equal subintervals. Then the characteristic matrix (with entries rounded off to 5 decimal places) is

$$\mathcal{C} = \begin{pmatrix} -0.01627 & -0.01083 & -0.00370 \\ -0.01083 & -0.00892 & 0.00542 \\ -0.00370 & 0.00542 & 0.00185 \end{pmatrix}.$$

Here,

$$\det \mathcal{C} = 1.08273 \times 10^{-6}.$$

The graphs of the components $\Phi_{\alpha i}$ of Φ are shown in Fig. 1.2.

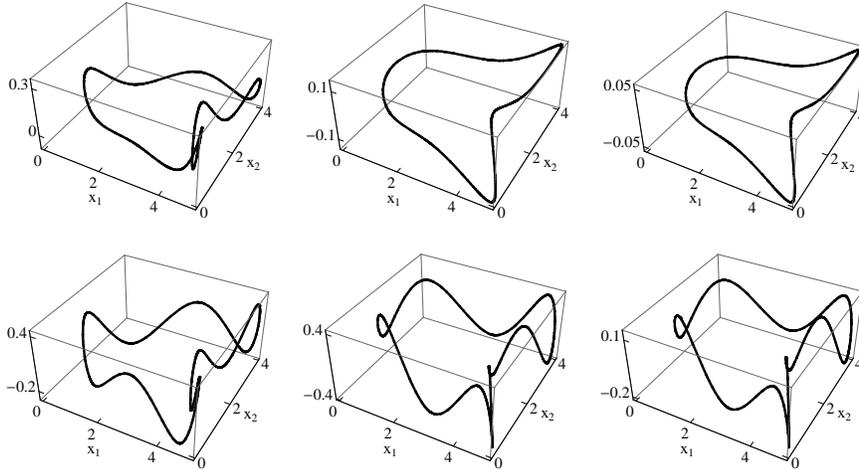


Fig. 1.2 Graphs of the $\Phi_{\alpha i}$.

As a second example, consider the ‘expanding’ ellipse ∂S of parametric equations

$$x_1(t) = 2k \cos(\pi t),$$

$$x_2(t) = k \sin(\pi t), \quad 0 \leq t \leq 2.$$

The graph of $\det \mathcal{C}$ as a function of k is shown in Fig. 1.3.

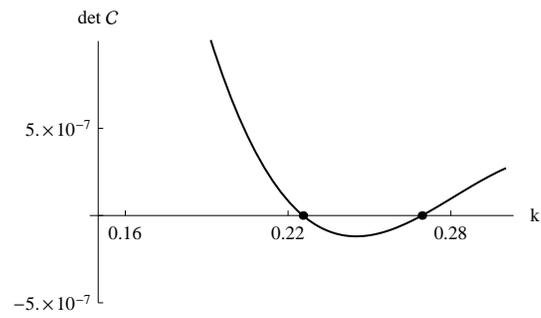


Fig. 1.3 Graph of $\det \mathcal{C}$.

Here, $\det \mathcal{C} = 0$ for $k = 0.22546$ and $k = 0.26934$.

References

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