Mixed Integer Quadratic Programming

Part I: Structural properties

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Section 1

The problem

Mixed Integer Quadratic Programming

$$\begin{aligned} \min \quad & \mathbf{x}^\mathsf{T} Q \mathbf{x} + c^\mathsf{T} \mathbf{x} \\ \text{s. t.} \quad & A\mathbf{x} \le b \\ & \mathbf{x} \in \mathbb{Z}^p \times \mathbb{R}^{n-p} \end{aligned} \tag{MIQP}$$

- Rational data: Q, c, A, b
- Q symmetric
- We denote by $f(\mathbf{x}) := \mathbf{x}^\mathsf{T} Q \mathbf{x} + c^\mathsf{T} \mathbf{x}$
- We denote by $\mathcal{P} := \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \le b\}$

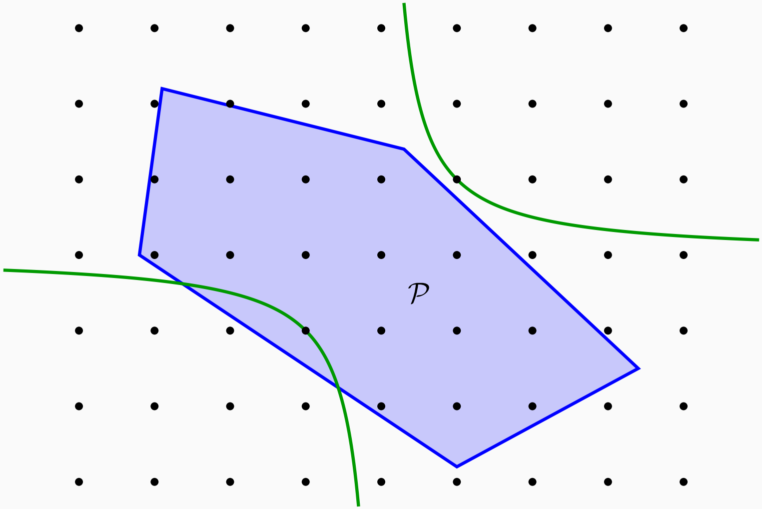
Mixed Integer Quadratic Programming

$$\begin{aligned} \min \quad & \mathbf{x}^{\mathsf{T}} Q \mathbf{x} + c^{\mathsf{T}} \mathbf{x} \\ \text{s. t.} \quad & A\mathbf{x} \leq b \\ & \mathbf{x} \in \mathbb{Z}^{p} \times \mathbb{R}^{n-p} \end{aligned} \tag{MIQP}$$

- With $Q = 0$: Mixed Integer Linear Programming (MILP)
- With $p = 0$: Quadratic Programming (QP)

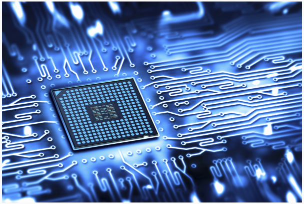
- Prototypical Mixed Integer Nonlinear Programming (MINLP)

Geometry of MIQP



Applications

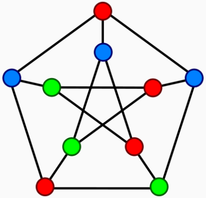
Power grid



Chip design

Pooling

Game theory



Graph theory

Social choice

Application: Hydropower Scheduling

- A hydropower plant operates n turbines fed by a common water source
- Goal: maximize total energy production minus startup costs
- Power output of turbine i is a quadratic function of its flow: $P_i(\mathbf{x}_i) = \beta_i \mathbf{x}_i - \alpha_i \mathbf{x}_i^2$ (as flow increases, hydraulic friction losses grow quadratically, reducing efficiency)
- Constraint: each turbine is either running or shut down, with startup cost γ_i

Application: Hydropower Scheduling

Variables:

- Continuous $\mathbf{x}_i \in [0, 1]$: percentage of flow through turbine i
- Binary $\mathbf{y}_i \in \{0, 1\}$: turbine i is on ($\mathbf{y}_i = 1$) or off ($\mathbf{y}_i = 0$)

Objective function:

- $c_i = -\beta_i$: linear power output coefficient of turbine i
- $Q_{ii} = \alpha_i$: quadratic power output coefficient of turbine i
- $c_{n+i} = \gamma_i$: startup cost of turbine i (associated to $\mathbf{y}_i$)

Constraints:

- Water budget: $\sum_{i=1}^{n} \mathbf{x}_i \leq 1$
- Linking: $\mathbf{x}_i \leq \mathbf{y}_i \quad \forall\, i$ (no flow if turbine is off)

Application: Portfolio Optimization

- An investor allocates a budget across n assets
- Goal: minimize risk while maximizing expected return μ
- Risk is measured by the variance of the portfolio return: $\mathbf{x}^{\mathsf{T}}\Sigma\mathbf{x}$
- Constraint: invest in at most k assets (sparse portfolio)

Application: Portfolio Optimization

Variables:

- Continuous $\mathbf{x}_i \in [0,1]$: percentage of budget in asset i
- Binary $\mathbf{y}_i \in \{0,1\}$: asset i is selected ($\mathbf{y}_i = 1$) or not ($\mathbf{y}_i = 0$)

Objective function:

- $Q = \Sigma$: covariance matrix of asset returns
- $c = -\mu$: negative expected returns

Constraints:

- Budget: $\sum_{i=1}^{n} \mathbf{x}_i \leq 1$
- Linking: $\mathbf{x}_i \leq \mathbf{y}_i \quad \forall\, i$
- Cardinality: $\sum_{i=1}^{n} \mathbf{y}_i \leq k$ (invest in at most k assets)

Section 2

Some fundamental properties

Some fundamental properties of MILP

$$\begin{aligned} \min \quad & c^{\mathsf{T}} \mathbf{x} \\ \text{s. t.} \quad & A\mathbf{x} \le b \\ & \mathbf{x} \in \mathbb{Z}^p \times \mathbb{R}^{n-p} \end{aligned} \tag{MILP}$$

Two fundamental properties:

1. If feasible and optimal cost is not $-\infty$, there exist an optimal solution
2. If optimal cost is $-\infty$, there exist an unbounded ray

Furthermore, optimal solutions and unbounded ray can always be chosen

- rational, and
- of polynomial size

Property 1: Attainability

$$\begin{array}{ll} \inf & f(\mathbf{x}) \\ \text{s.t.} & \mathbf{x} \in \mathcal{F} \end{array}$$

Attainability:

- Assume problem feasible and optimal cost is not $-\infty$
- Does there exist an optimal solution?
- If so, we can replace inf with min

Some more details:

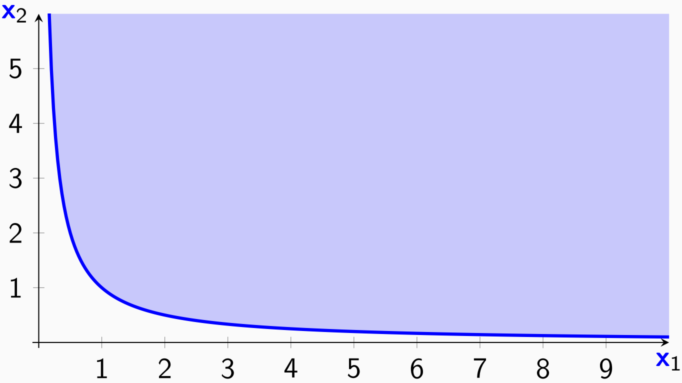
- Assume problem feasible
- Assume $\exists \gamma \in \mathbb{R}$ such that $f(\mathbf{x}) \geq \gamma, \forall \mathbf{x} \in \mathcal{F}$
- By completeness of $\mathbb{R}$, $\{f(\mathbf{x}) \mid \mathbf{x} \in \mathcal{F}\}$ has an infimum, say $\bar{\gamma}$
- We want to know if $\exists \bar{\mathbf{x}}$ with $f(\bar{\mathbf{x}}) = \bar{\gamma}$

Attainability does not always hold

- False with one quadratic inequality:

$$\begin{array}{rl} \inf & x_1 \\ \text{s. t.} & x_1 x_2 \geq 1 \\ & x_1, x_2 \geq 0 \end{array}$$

- Optimal cost is not $-\infty$, but no optimal solution exists



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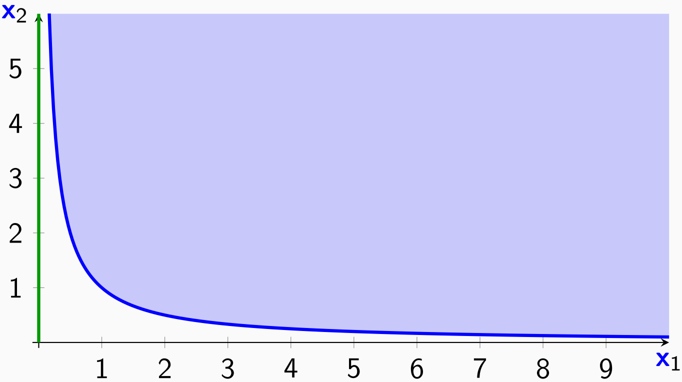

Attainability does not always hold

- False with quartic objective:

$$\begin{aligned} \inf \quad & x_1^2 + (x_1 x_2 - 1)^2 \\ \text{s.t.} \quad & x_1, x_2 \geq 0 \end{aligned}$$

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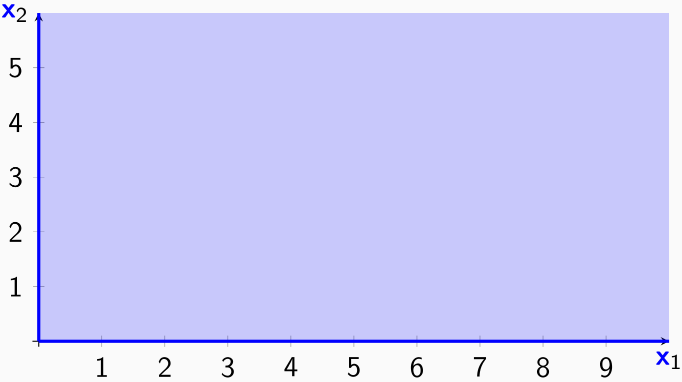

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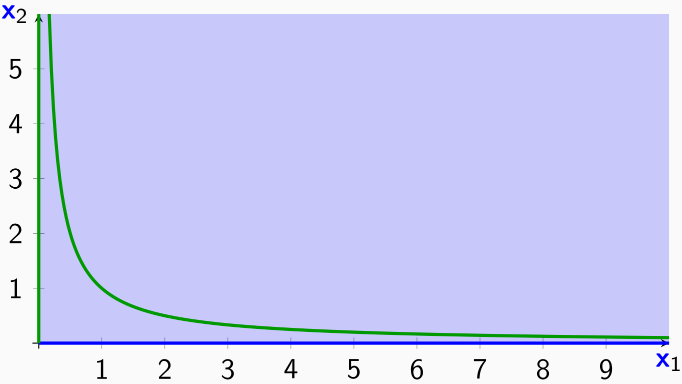


Rationality and polynomial size

$$\begin{aligned} \min \quad & f(\mathbf{x}) \\ \text{s. t.} \quad & \mathbf{x} \in \mathcal{F} \end{aligned}$$

- Assume there exits an optimal solution

Rationality of optimal solutions:

- Does there exist a rational optimal solution?
- Meaning, an optimal solution $\bar{\mathbf{x}} \in \mathbb{Q}^n$

Optimal solutions of polynomial size:

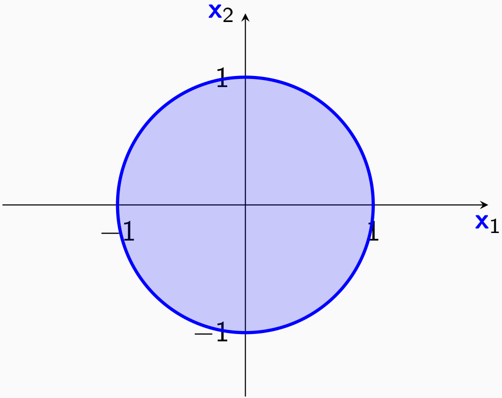
- Does there exist an optimal solution of polynomial size?
- Meaning, an optimal solution $\bar{\mathbf{x}}$ of bit length polynomial in the bit length of the instance

There may be no rational optimal solution

- False with one quadratic inequality:

$$\begin{aligned} \min \quad & -x_1 - x_2 \\ \text{s.t.} \quad & x_1^2 + x_2^2 \leq 1 \end{aligned}$$

- The unique optimal solution is $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$
- It is not rational



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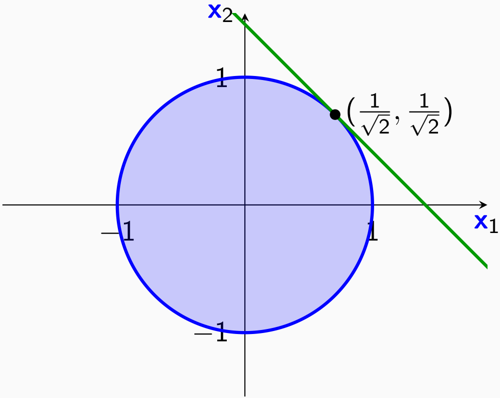

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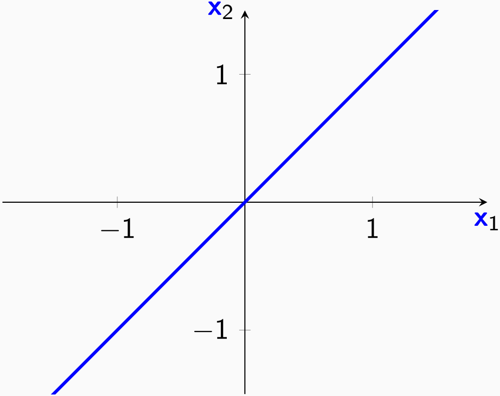
$$\begin{aligned} \min \quad & (x_1^2 + x_2^2 - 1)^2 \\ \text{s.t.} \quad & x_1 - x_2 = 0 \end{aligned}$$

- The optimal solutions are $\pm(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$
- They are not rational



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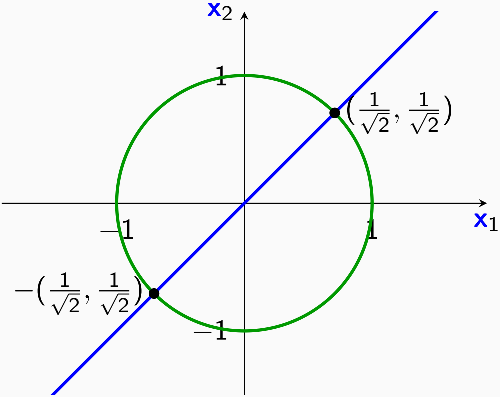
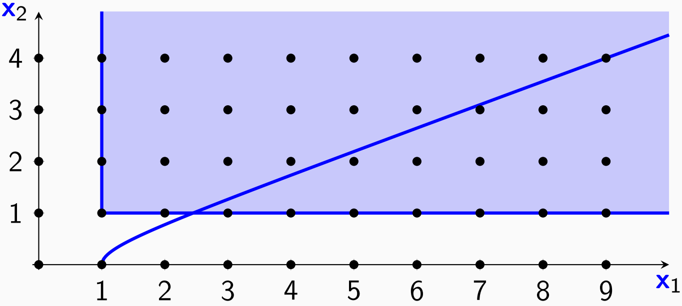


There may be no optimal solution of polynomial size

- False with one quadratic equality:

$$
\begin{aligned}
\min \quad & \mathbf{x}_1 \\
\text{s.\,t.} \quad & \mathbf{x}_1^2 - N\mathbf{x}_2^2 = 1 \\
& \mathbf{x}_1, \mathbf{x}_2 \geq 1 \\
& \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{Z}
\end{aligned}
$$

- For $N = 5^{2k+1}$, $k \in \mathbb{N}$, all feasible solutions have size $\Omega(5^k)$



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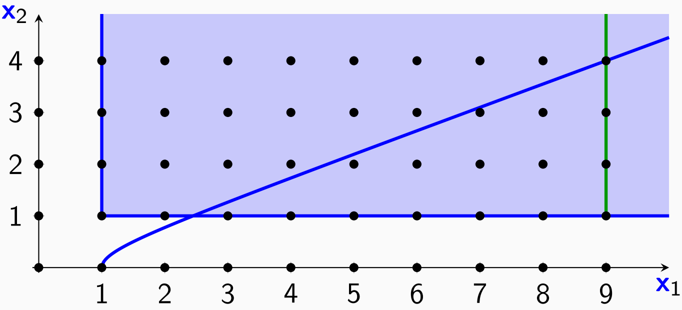

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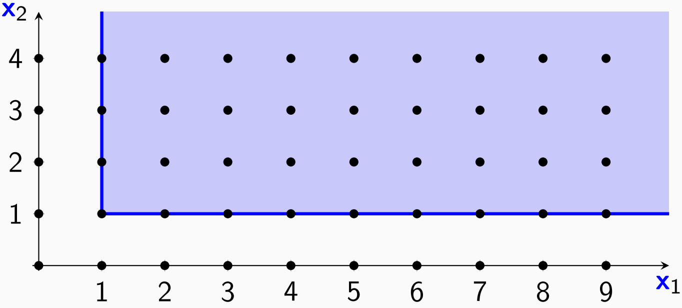

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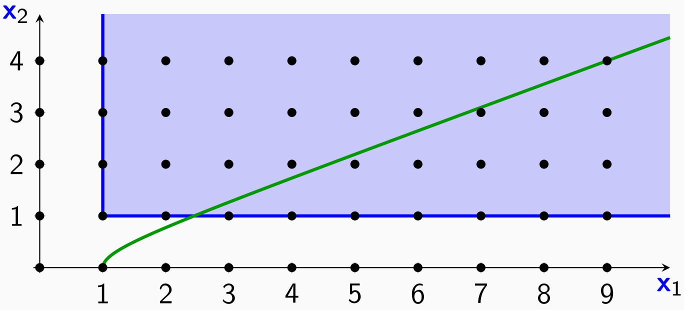


Property 2: Unbounded rays

$$\begin{array}{ll} \inf & f(\mathbf{x}) \\ \text{s. t.} & \mathbf{x} \in \mathcal{F} \end{array}$$

Unbounded rays:

- Assume optimal cost is $-\infty$
- Does there exist an unbounded ray?

Unbounded ray: A half line

$$\mathcal{R} := \left\{ \bar{\mathbf{x}} + \lambda \bar{\mathbf{d}} \mid \lambda \geq 0 \right\}, \qquad \text{for } \bar{\mathbf{x}}, \bar{\mathbf{d}} \in \mathbb{R}^n,$$

such that the optimal cost of the problem restricted to $\mathcal{R}$ is $-\infty$:

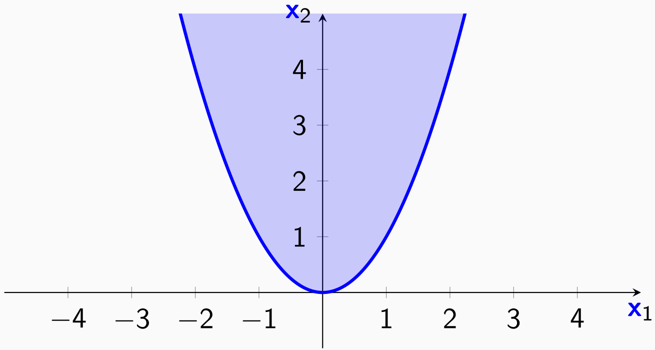
$$\begin{array}{ll} \inf & f(\mathbf{x}) \\ \text{s. t.} & \mathbf{x} \in \mathcal{F} \cap \mathcal{R} \end{array}$$

Unbounded rays may not exist

- False with one quadratic inequality:

$$\begin{aligned} \inf \quad & x_1 \\ \text{s. t.} \quad & x_1^2 - x_2 \le 0 \end{aligned}$$

- Optimal cost is $-\infty$, but there is no unbounded ray



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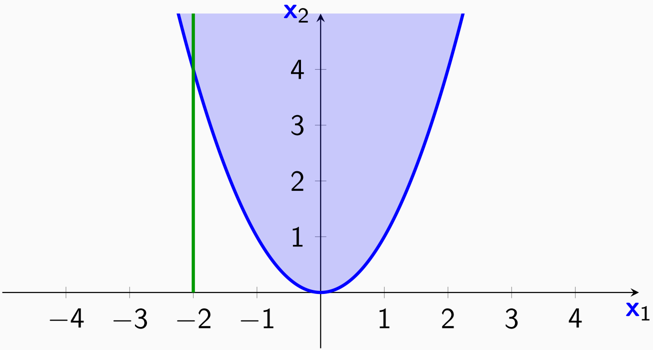

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$$\begin{aligned} \inf \quad & x_1 + (x_1^2 - x_2)^2 \\ \text{s.t.} \quad & (x_1, x_2) \in \mathbb{R}^2 \end{aligned}$$

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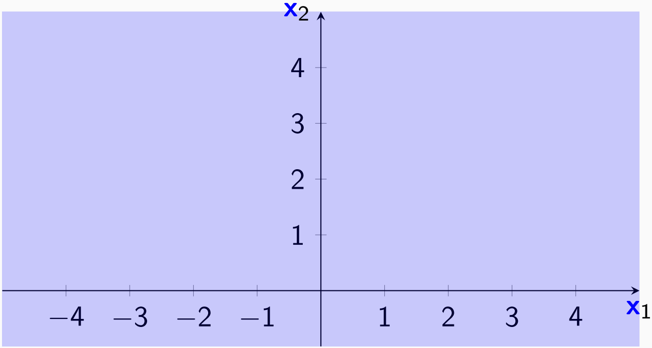

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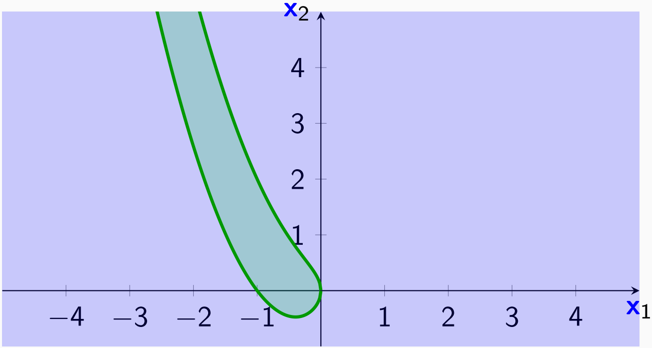

Some remarks

Remark 1: All examples in dimension two, more general than MIQP because of:

- Either one quadratic constraint
- Or a quartic objective function

Remark 2: Why quartic objective functions?

- Examples easy to construct and understand
- We also know some examples with cubic objective functions

One more example: more quadratic inequalities

Hilbert's 10^{th} problem [1900]:

- *It is the challenge to provide a general algorithm that, for any given Diophantine equation*[1]*, can decide whether the equation has an integral solution*
- The solution to Hilbert's tenth problem shows that this problem is undecidable [Davis - Matiyasevich - Putnam - Robinson 70]
- That is, such a general algorithm cannot exist

- [Jeroslow 73] used this result to show that integer feasibility of a set defined by a fixed number of quadratic inequalities is undecidable
  $\Rightarrow$ It is not possible to bound the size of any feasible solution

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[1] A polynomial equation with integer coefficients

WE’LL BE
BACK IN

5

MINUTES

Section 3

A proof for MILP

Proof for MILP

We show that all properties hold for MILP:

Theorem

For any feasible MILP, exactly one of the following holds:

1. *There is an optimal solution of polynomial size*
2. *There is an unbounded ray of polynomial size*

Remark: This is a proof sketch, and we focus on existence

Proof for MILP

- We can assume $\mathcal{P} \subseteq \mathbb{R}^n_{\geq 0}$
- To see that, intersect $\mathcal{P}$ with each orthant of $\mathbb{R}^n$

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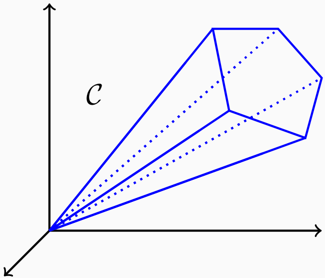
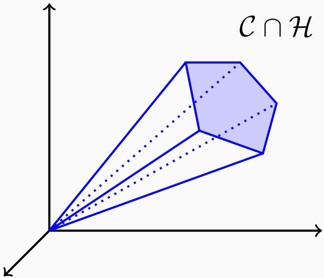
- We consider the recession cone $\mathcal{C} := \operatorname{rec.cone}(\mathcal{P}) \subseteq \mathbb{R}^n_{\geq 0}$



Proof for MILP

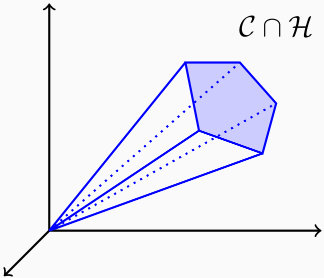
- We can assume $\mathcal{P} \subseteq \mathbb{R}^n_{\geq 0}$
- To see that, intersect $\mathcal{P}$ with each orthant of $\mathbb{R}^n$
- We consider the recession cone $\mathcal{C} := \operatorname{rec.cone}(\mathcal{P}) \subseteq \mathbb{R}^n_{\geq 0}$
- We slice it with hyperplane $\mathcal{H} := \{x \in \mathbb{R}^n \mid \sum_{i=1}^n \mathbf{x}_i = 1\}$



Proof for MILP

- Let $\bar{\mathbf{d}}$ be an optimal solution to the problem

$$\begin{aligned} \min \quad & c^\mathsf{T} \mathbf{d} \\ \text{s. t.} \quad & \mathbf{d} \in \mathcal{C} \cap \mathcal{H} \end{aligned}$$



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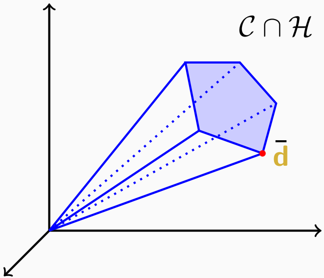
- $\bar{\mathbf{d}}$ can be chosen as a vertex of the polytope $\mathcal{C} \cap \mathcal{H}$



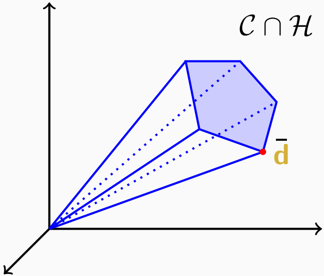
Proof for MILP

Case $c^\mathsf{T}\bar{\mathbf{d}} < 0$:

- Let $\bar{\mathbf{x}} \in \mathcal{P} \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$
- Then $\{\bar{\mathbf{x}} + \lambda\bar{\mathbf{d}} \mid \lambda \geq 0\}$ is an unbounded ray:

  It contains infinitely many feasible points and

$$c^\mathsf{T}(\bar{\mathbf{x}} + \lambda\bar{\mathbf{d}}) = c^\mathsf{T}\bar{\mathbf{x}} + \lambda\underbrace{(c^\mathsf{T}\bar{\mathbf{d}})}_{<0} \to -\infty$$

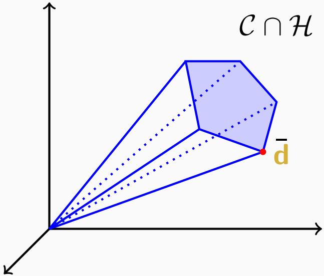


Proof for MILP

Case $c^\mathsf{T}\bar{\mathbf{d}} \geq 0$:

- Then $c^\mathsf{T}\mathbf{d} \geq 0$ for every $\mathbf{d} \in \mathcal{C}$, thus

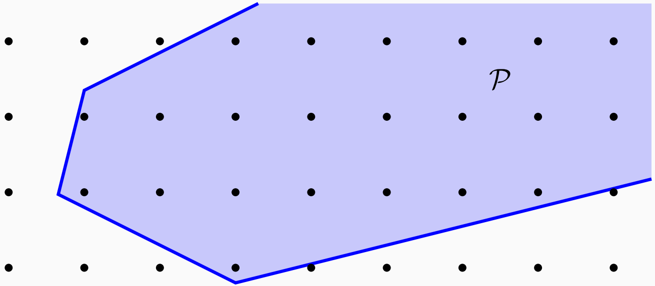
$$\begin{aligned} c^\mathsf{T}(\mathbf{x} + \lambda\mathbf{d}) &= c^\mathsf{T}\mathbf{x} + \lambda\underbrace{(c^\mathsf{T}\mathbf{d})}_{\geq 0} \\ &\geq c^\mathsf{T}\mathbf{x} \qquad \forall\mathbf{x},\ \forall\mathbf{d} \in \mathcal{C} \end{aligned}$$



Proof for MILP

Case $c^\mathsf{T}\bar{\mathbf{d}} \geq 0$:

- Fundamental Theorem of MILP: There exists a polytope $\mathcal{Q} \subset \mathcal{P}$ such that each feasible point is of the form $\mathbf{x} + \lambda\mathbf{d}$ with $\mathbf{x} \in \mathcal{Q} \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$, $\mathbf{d} \in \mathcal{C}$, $\lambda \geq 0$
- Infima over $\mathcal{P} \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$ and $\mathcal{Q} \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$ coincide
- $\mathcal{Q} \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$ is compact, so there is an optimal solution ☐



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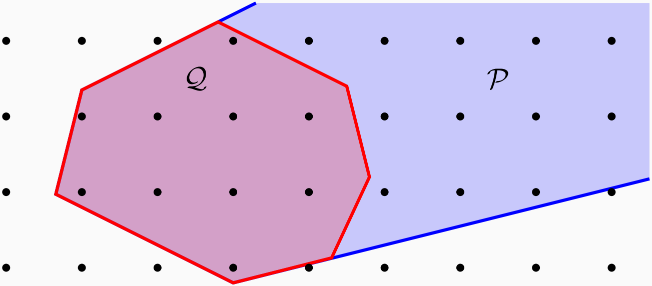

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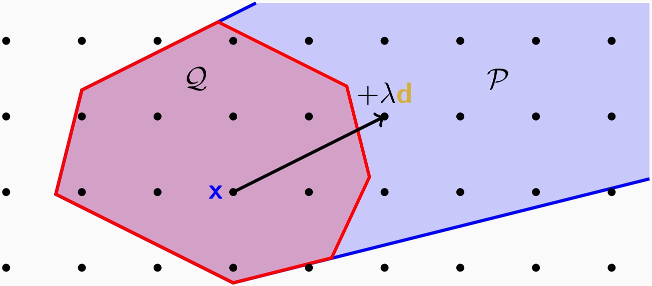
- $\mathcal{Q} \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$ is compact, so there is an optimal solution □



Section 4

Two ingredients

First ingredient: QP on polytope

We need to generalize two ingredients from the MILP proof

First ingredient

- There is an optimal solution of LP on a polytope on a vertex of the polytope
- This is not true with a quadratic objective
- No problem:

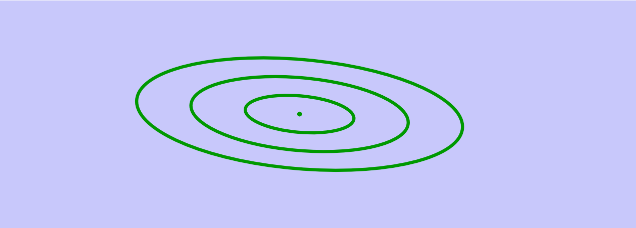
Proposition

If $\mathcal{P}$ is bounded, QP has a rational optimal solution

First ingredient: QP on polytope

Simplest case: Q positive definite, $\mathcal{P} = \mathbb{R}^n$

- The function
$$f(x) = \mathbf{x}^\mathsf{T} Q \mathbf{x} + c^\mathsf{T} \mathbf{x}$$
is strictly convex, so there is a unique global minimum
- Setting the gradient to zero:
$$\nabla f(x) = 2Q\mathbf{x} + c = 0 \implies \bar{\mathbf{x}} = -\tfrac{1}{2} Q^{-1} c$$
- This is the unique global minimizer, and it is rational



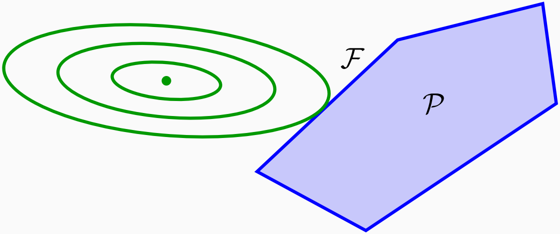
First ingredient: QP on polytope

Easy case: Q positive definite

- Since $\mathcal{P}$ is bounded, there is an optimal solution
- Let $\mathcal{F}$ be a face of $\mathcal{P}$ of smallest dimension containing an optimal solution
- The optimal solution is in the interior of $\mathcal{F}$

$$\begin{aligned}\min\{f(\mathbf{x}) \mid \mathbf{x} \in \mathcal{P}\} &= \min\{f(\mathbf{x}) \mid \mathbf{x} \in \mathcal{F}\} \\ &= \min\{f(\mathbf{x}) \mid \mathbf{x} \in \text{aff. hull}(\mathcal{F})\}\end{aligned}$$

- We reduced the case to the previous unconstrained case



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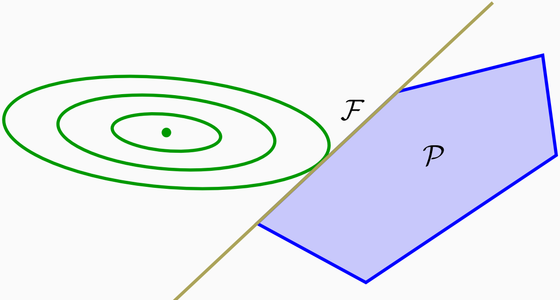

First ingredient: QP on polytope

Remaining case: Q not positive definite

- Let $\bar{\mathbf{x}} \in \text{int}(\mathcal{P})$
- There exists $\bar{\mathbf{d}} \in \mathbb{R}^n$ such that

$$f(\bar{\mathbf{x}} + \lambda \bar{\mathbf{d}}) \leq f(\bar{\mathbf{x}}) \qquad \forall \lambda \geq 0$$

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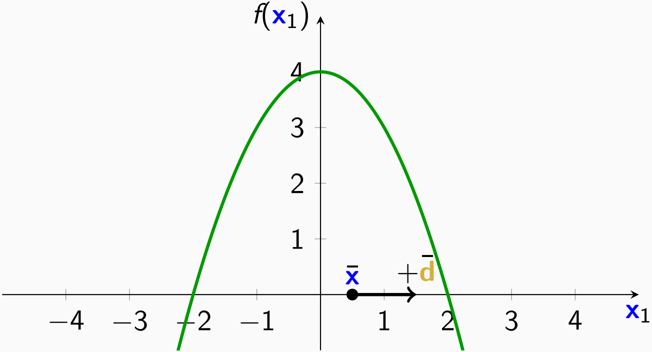

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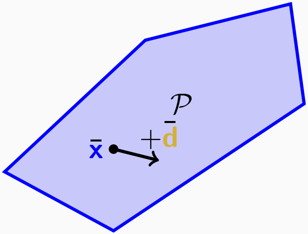
- Travel from $\bar{\mathbf{x}}$ in direction $\bar{\mathbf{d}}$, hit the boundary
- It suffices to consider the proper faces of $\mathcal{P}$



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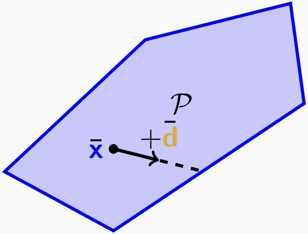


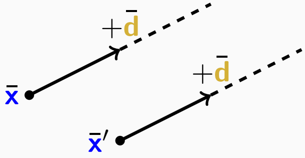
Second ingredient: ray translations

For a linear function, translating a ray maintains limit behavior

- Fix $\bar{\mathbf{d}}$ and restrict $f(\mathbf{x}) = c^\mathsf{T}\mathbf{x}$ to the ray $\{\bar{\mathbf{x}} + \lambda\bar{\mathbf{d}} \mid \lambda \geq 0\}$:

$$\begin{aligned} f(\bar{\mathbf{x}} + \lambda\bar{\mathbf{d}}) &= c^\mathsf{T}(\bar{\mathbf{x}} + \lambda\bar{\mathbf{d}}) \\ &= c^\mathsf{T}\bar{\mathbf{x}} + \lambda(c^\mathsf{T}\bar{\mathbf{d}}) \end{aligned}$$

- Linear function in λ
- Linear term: $c^\mathsf{T}\bar{\mathbf{d}}$, independent on $\bar{\mathbf{x}}$
- Constant term: $c^\mathsf{T}\bar{\mathbf{x}}$



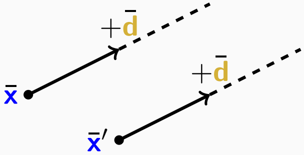
Second ingredient: ray translations

For a quadratic function, translating a ray may change limit behavior

- Fix $\bar{\mathbf{d}}$ and restrict $f(\mathbf{x}) = \mathbf{x}^\mathsf{T} Q\mathbf{x} + c^\mathsf{T}\mathbf{x}$ to the ray $\{\bar{\mathbf{x}} + \lambda\bar{\mathbf{d}} \mid \lambda \geq 0\}$:

$$\begin{aligned} f(\bar{\mathbf{x}} + \lambda\bar{\mathbf{d}}) &= (\bar{\mathbf{x}} + \lambda\bar{\mathbf{d}})^\mathsf{T} Q(\bar{\mathbf{x}} + \lambda\bar{\mathbf{d}}) + c^\mathsf{T}(\bar{\mathbf{x}} + \lambda\bar{\mathbf{d}}) \\ &= \lambda^2(\bar{\mathbf{d}}^\mathsf{T} Q\bar{\mathbf{d}}) + \lambda(2\bar{\mathbf{d}}^\mathsf{T} Q\bar{\mathbf{x}} + c^\mathsf{T}\bar{\mathbf{d}}) + \underbrace{(\bar{\mathbf{x}}^\mathsf{T} Q\bar{\mathbf{x}} + c^\mathsf{T}\bar{\mathbf{x}})}_{=f(\bar{\mathbf{x}})} \end{aligned}$$

- Quadratic function in λ
- Quadratic term: $\bar{\mathbf{d}}^\mathsf{T} Q\bar{\mathbf{d}}$, independent on $\bar{\mathbf{x}}$
- Linear term: $2\bar{\mathbf{d}}^\mathsf{T} Q\bar{\mathbf{x}} + c^\mathsf{T}\bar{\mathbf{d}}$, depends on $\bar{\mathbf{x}}$
- Constant term: $f(\bar{\mathbf{x}})$



Section 5

A proof for MIQP

Proof for MIQP

We show that all properties hold for MIQP:

Theorem

For any feasible MIQP, exactly one of the following holds:

1. *There is an optimal solution of polynomial size*
2. *There is an unbounded ray of polynomial size*

Remark: This is a proof sketch, and we focus on existence

Proof for MIQP

- We can assume $\mathcal{P} \subseteq \mathbb{R}^n_{\geq 0}$
- To see that, intersect $\mathcal{P}$ with each orthant of $\mathbb{R}^n$

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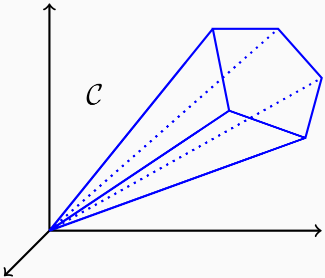

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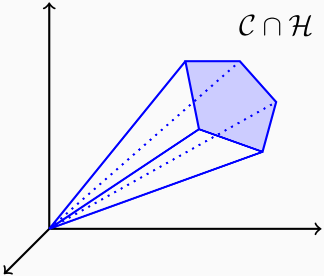
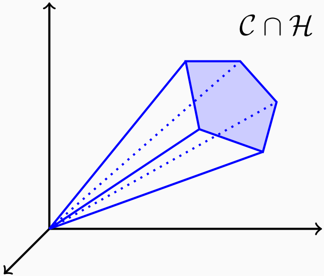
- To see that, intersect $\mathcal{P}$ with each orthant of $\mathbb{R}^n$
- We consider the recession cone $\mathcal{C} := \operatorname{rec.cone}(\mathcal{P}) \subseteq \mathbb{R}^n_{\geq 0}$
- We slice it with hyperplane $\mathcal{H} := \{x \in \mathbb{R}^n \mid \sum_{i=1}^n \mathbf{x}_i = 1\}$



Proof for MIQP

- Let $\bar{\mathbf{d}}$ be an optimal solution to the problem

$$\begin{aligned} \min \quad & \mathbf{d}^\mathsf{T} Q \mathbf{d} \\ \text{s.t.} \quad & \mathbf{d} \in \mathcal{C} \cap \mathcal{H} \end{aligned}$$



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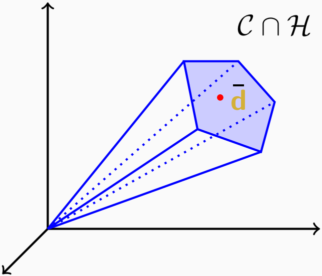
- **First ingredient:** $\bar{\mathbf{d}}$ can be chosen rational



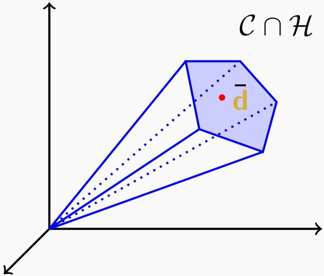
Proof for MIQP

Case $\bar{\mathbf{d}}^\mathsf{T} Q \bar{\mathbf{d}} < 0$:

- Let $\bar{\mathbf{x}} \in \mathcal{P} \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$
- Then $\left\{\bar{\mathbf{x}} + \lambda \bar{\mathbf{d}} \mid \lambda \geq 0\right\}$ is an unbounded ray:

  It contains infinitely many feasible points and

$$f(\bar{\mathbf{x}} + \lambda \bar{\mathbf{d}}) = \lambda^2 \underbrace{(\bar{\mathbf{d}}^\mathsf{T} Q \bar{\mathbf{d}})}_{<0} + O(\lambda) \to -\infty$$



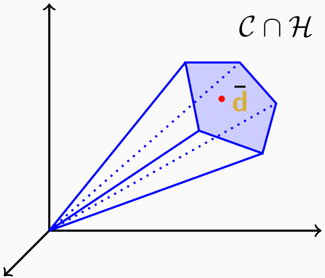
Proof for MIQP

Case $\bar{\mathbf{d}}^\mathsf{T} Q \bar{\mathbf{d}} > 0$:

- Then $\mathbf{d}^\mathsf{T} Q \mathbf{d} \geq \epsilon > 0$ for every $\mathbf{d} \in \mathcal{C}$, thus

$$f(\mathbf{x} + \lambda \mathbf{d}) = \underbrace{\lambda^2(\mathbf{d}^\mathsf{T} Q \mathbf{d}) + \lambda(2\mathbf{d}^\mathsf{T} Q \mathbf{x} + c^\mathsf{T} \mathbf{d})}_{\geq 0 \text{ for } \lambda \geq \bar{\lambda}} + f(\mathbf{x})$$

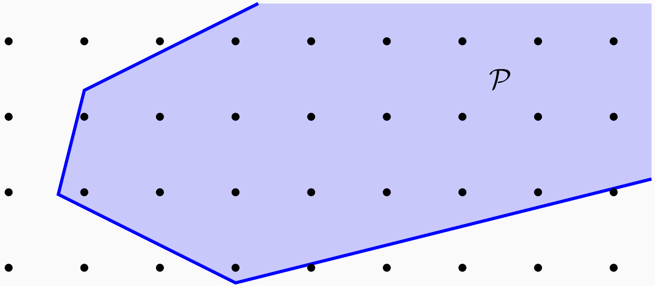
$$\geq f(\mathbf{x}) \qquad \forall \lambda \geq \bar{\lambda},\ \forall \mathbf{x},\ \forall \mathbf{d} \in \mathcal{C}$$



Proof for MIQP

Case $\bar{\mathbf{d}}^\top Q\bar{\mathbf{d}} > 0$:

- Fundamental Theorem of MILP: There exists a polytope $\mathcal{Q} \subset \mathcal{P}$ such that each feasible point is of the form $\mathbf{x} + \lambda\mathbf{d}$ with $\mathbf{x} \in \mathcal{Q} \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$, $\mathbf{d} \in \mathcal{C}$, $\lambda \geq 0$
- Infima over $\mathcal{P} \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$ and $\mathcal{Q}' \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$ coincide, where $\mathcal{Q}'$ is a polytope "close" to $\mathcal{Q}$
- $\mathcal{Q}' \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$ is compact, so there is an optimal solution



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Case $\bar{\mathbf{d}}^\mathsf{T} Q \bar{\mathbf{d}} > 0$:

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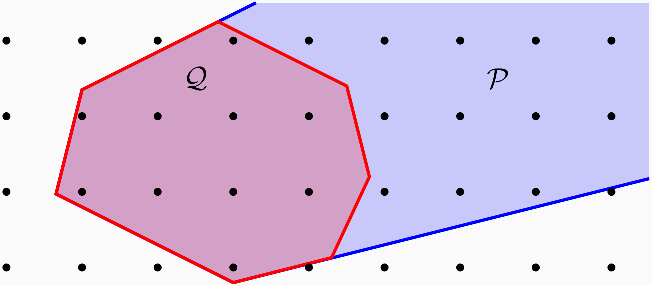

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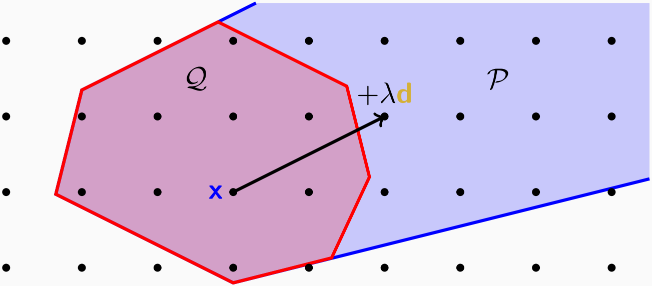

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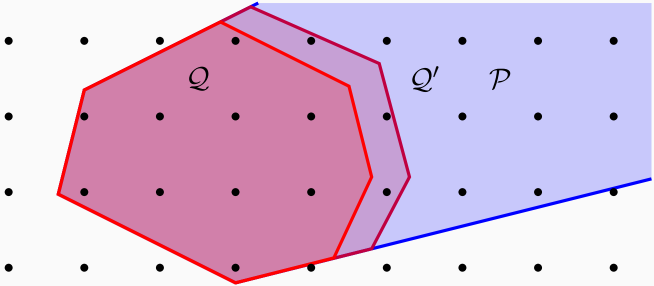
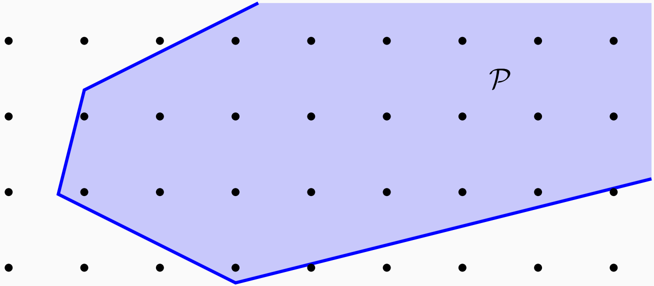


Proof for MIQP

Case $\bar{\mathbf{d}}^{\mathsf{T}} Q \bar{\mathbf{d}} = 0$:

- In this case the linear term $\lambda(2\bar{\mathbf{d}}^{\mathsf{T}} Q\mathbf{x} + c^{\mathsf{T}}\bar{\mathbf{d}})$ becomes dominant, and it depends on $\mathbf{x}$
- Let $\bar{\mathbf{x}}$ be an optimal solution to the MILP

$$\begin{array}{ll} \min & 2\bar{\mathbf{d}}^{\mathsf{T}} Q\mathbf{x} + c^{\mathsf{T}}\bar{\mathbf{d}} \\ \text{s.t.} & \mathbf{x} \in \mathcal{P} \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p}) \end{array}$$



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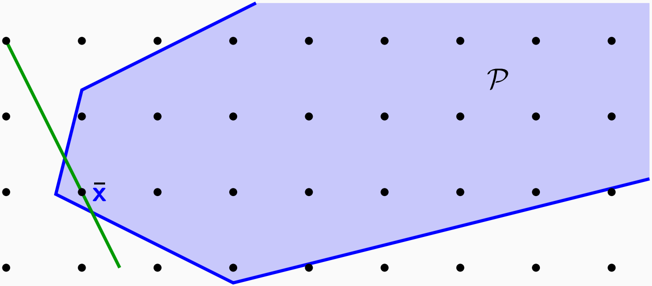
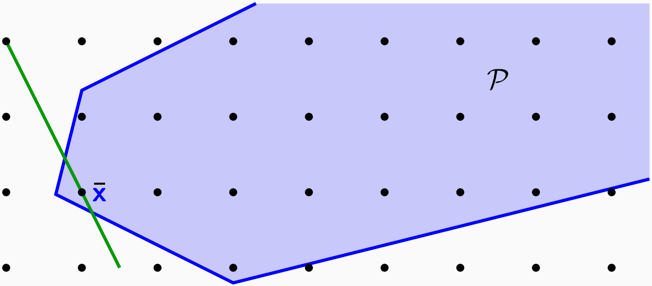


Proof for MIQP

Sub-case $2\bar{\mathbf{d}}^\mathsf{T} Q\bar{\mathbf{x}} + c^\mathsf{T}\bar{\mathbf{d}} < 0$**:**

- Then $\left\{\bar{\mathbf{x}} + \lambda\bar{\mathbf{d}} \mid \lambda \geq 0\right\}$ is an unbounded ray



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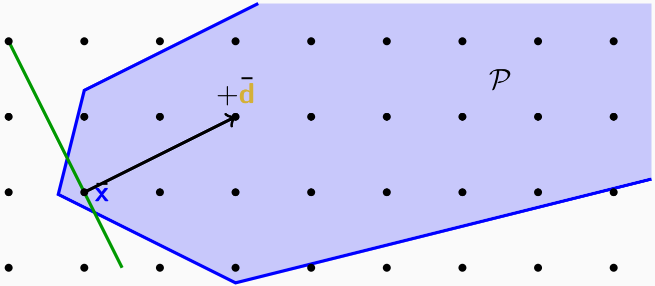

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Sub-case $2\bar{\mathbf{d}}^\mathsf{T} Q\bar{\mathbf{x}} + c^\mathsf{T}\bar{\mathbf{d}} \geq 0$:

- Then $2\bar{\mathbf{d}}^\mathsf{T} Q\mathbf{x} + c^\mathsf{T}\bar{\mathbf{d}} \geq 0$ for every $\mathbf{x} \in \mathcal{P} \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$, thus

$$f(\mathbf{x} + \lambda\bar{\mathbf{d}}) = \lambda^2 \underbrace{(\bar{\mathbf{d}}^\mathsf{T} Q\bar{\mathbf{d}})}_{=0} + \lambda \underbrace{(2\bar{\mathbf{d}}^\mathsf{T} Q\mathbf{x} + c^\mathsf{T}\bar{\mathbf{d}})}_{\geq 0} + f(\mathbf{x})$$
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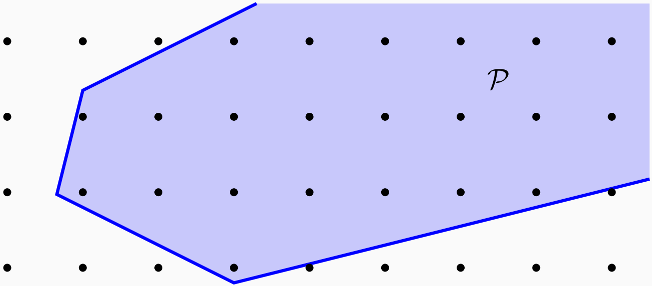

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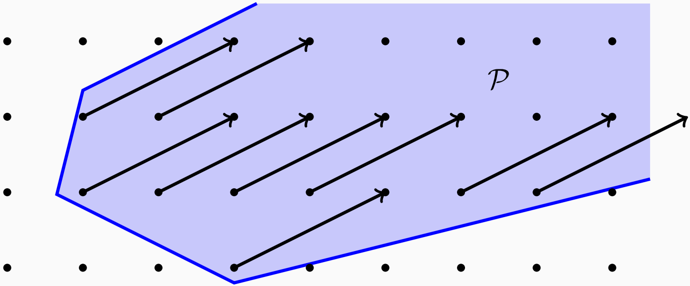

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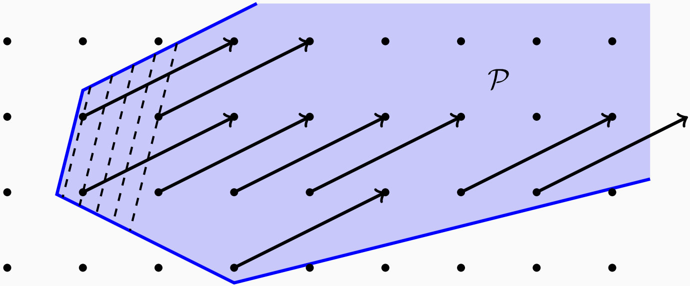
Sub-case $2\bar{\mathbf{d}}^\mathsf{T} Q\bar{\mathbf{x}} + c^\mathsf{T}\bar{\mathbf{d}} \geq 0$**:**

- It suffices to consider vectors "close" to the facets of $\mathcal{P}$
- Namely, on few translates of the facets of $\mathcal{P}$
- The result follows by induction on the dimension □



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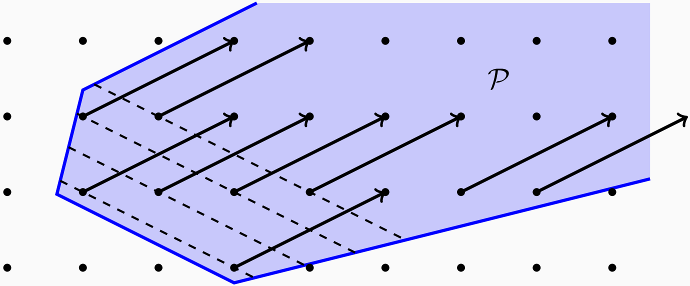

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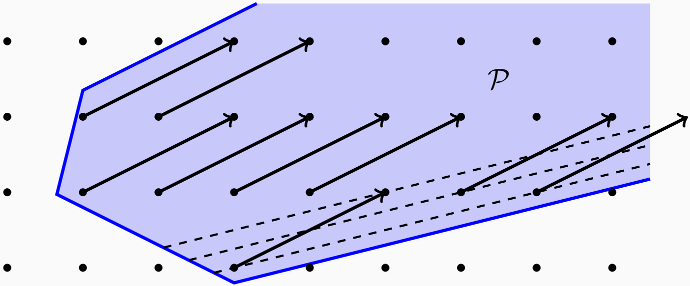
- Namely, on few translates of the facets of $\mathcal{P}$
- The result follows by induction on the dimension $\square$



Enough for today!

Questions or counterexamples?