Duality-based reformulations in bilevel optimization

Part I

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Set subsidy

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Set subsidy

Decides production

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Foreign producers may join domestic markets.

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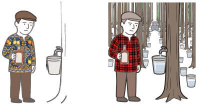

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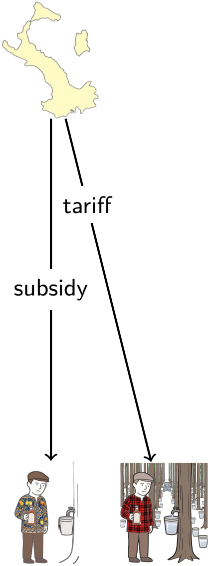

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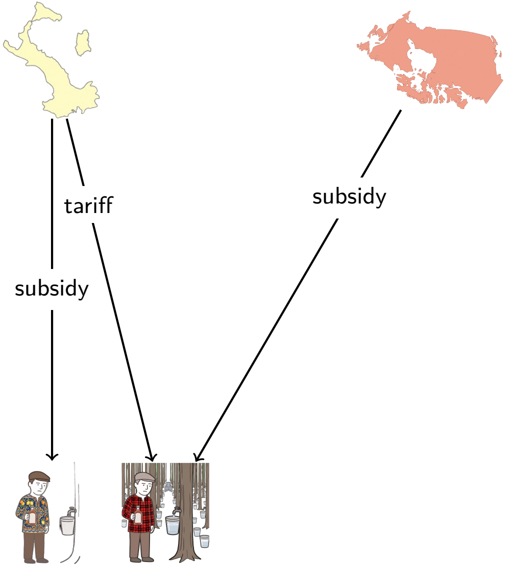

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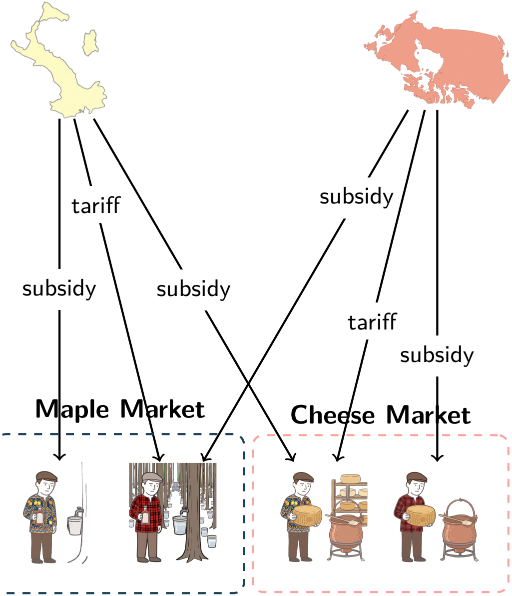
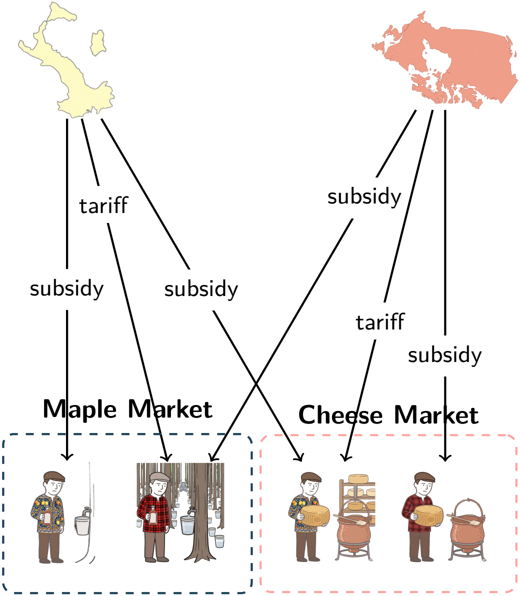


Imagine we govern the *Repubblica di Mozzarella*...

Foreign producers may join domestic markets.

We get a **game with multiple leaders and followers**.



More formally: the problem of is

$$\max_{\;,\;,\;} \quad L(\;,\;,\;,\;)$$

subject to $(\;,\;) \in X$

where solves an optimization problem

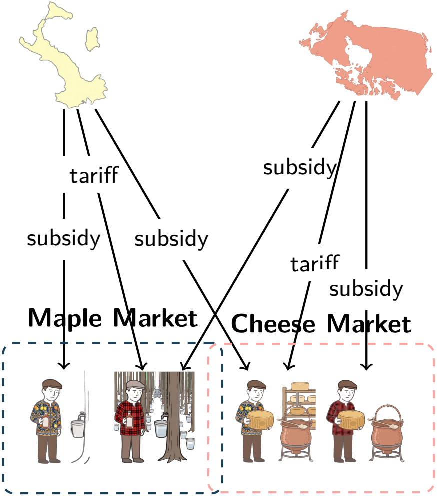
$$\min F(\;,\;,\;)$$

s.t. $\in Y$

where solves an optimization problem

$$\min F(\;,\;,\;,\;)$$

s.t. $\in Z$



Our goal

1. To understand how to use dualize-and-combine to tackle bilevel optimization problems;
2. To extend the technique for non-linear and mixed-integer cases.

Contents

Formalization

$$\begin{aligned}
\min_{x,y} \quad & L(x,y) \\
\text{s.t.} \quad & (x,y) \in \mathcal{X} \\
& y \in \arg\min_{\hat{y}} \quad F(x,\hat{y}) \\
& \qquad\qquad \text{s.t} \quad (x,\hat{y}) \in \mathcal{Y}
\end{aligned}$$

- Upper-level/leader variables x

Formalization

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- Set $\mathcal{X}$ can contain coupling (linking) constraints

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- Upper-level/leader variables x
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Optimal/best-response

$$BR(x) = \{y : y \in \arg\min_{\hat{y}}\{F(x,\hat{y}) : (x,\hat{y}) \in \mathcal{Y}\}\}$$

Formalization

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$$

- Upper-level/leader variables x
- Lower-level/follower variables y
- Set $\mathcal{X}$ can contain coupling (linking) constraints

Inducible region (bilevel feasible solutions)

$$IR = \{(x,y) : (x,y) \in \mathcal{X}, y \in BR(x)\}$$

Formalization

Optimistic model

$$\begin{aligned}
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Formalization

Optimistic model

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Formalization

Optimistic model

$$\begin{aligned} \min_{x,y} \quad & L(x,y) \\ \text{s.t.} \quad & (x,y) \in \mathcal{X} \\ & y \in BR(x) \end{aligned}$$

Pessimistic model

$$\begin{aligned} \min_{x,y} \quad & L(x,y) \\ \text{s.t.} \quad & (x,y) \in \mathcal{X} \\ & y \in BR(x) \\ & L(x,y) \geq L(x,\hat{y}) \quad \forall \hat{y} \in BR(x) \end{aligned}$$

Formalization

Optimistic model

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In both models above, for (x,y) to be bilevel feasible, $BR(x) \neq \emptyset$.
There are alternative pessimistic models, e.g., [Wiesemann et al., 2013].

Formalization

Optimistic model

$$\begin{aligned} \min_{x,y} \quad & L(x,y) \\ \text{s.t.} \quad & (x,y) \in \mathcal{X} \\ & y \in BR(x) \end{aligned}$$

(Another) Pessimistic model

$$\begin{aligned} \min_{x,y} \quad & L(x,y) \\ \text{s.t.} \quad & (x,y) \in \mathcal{X} \\ & \cancel{y \in BR(x)} \\ & L(x,y) \geq L(x,\hat{y}) \quad \forall \hat{y} \in BR(x) \end{aligned}$$

In both models above, for (x,y) to be bilevel feasible, $BR(x) \neq \emptyset$.
There are alternative pessimistic models, e.g., [Wiesemann et al., 2013].

Formalization

Besides the extreme tie-breakers (optimistic and pessimistic), we have the general intermediate case:

$$\begin{aligned} \min_{x} \quad & \mathbb{E}_{\hat{y}\sim\mu_x}\left[L(x,\hat{y})\right] \\ \text{s.t.} \quad & x \in \mathcal{X} \end{aligned}$$

where μ_x is a probability measure with $\text{supp}(\mu_x) \subseteq BR(x)$.

First studied
by [Aboussoror and Loridan, 1995, Mallozzi and Morgan, 1996].
Most works study the existence of optimal solution.

Formalization

Bilevel programming problem

$$\begin{aligned} \min_{x,y} \quad & L(x,y) \\ \text{s.t.} \quad & (x,y) \in \mathcal{X} \\ & y \in \arg\min_{\hat{y}} \quad F(x,\hat{y}) \\ & \qquad \text{s.t} \quad (x,\hat{y}) \in \mathcal{Y} \end{aligned}$$

Formalization

Single-level relaxation, also known as high-point relaxation

$$\begin{aligned} \min_{x,y} \quad & L(x,y) \\ \text{s.t.} \quad & (x,y) \in \mathcal{X} \\ & (x,y) \in \mathcal{Y} \end{aligned}$$

Formalization

Bilevel programming problem

$$\begin{aligned} \min_{x,y} \quad & L(x,y) \\ \text{s.t.} \quad & (x,y) \in \mathcal{X} \\ & y \in \arg\min_{\hat{y}} \quad F(x,\hat{y}) \\ & \qquad\qquad \text{s.t} \quad (x,\hat{y}) \in \mathcal{Y} \end{aligned}$$

Optimal value function reformulation

$$\begin{aligned} \min_{x,y} \quad & L(x,y) \\ \text{s.t.} \quad & (x,y) \in \mathcal{X} \\ & (x,y) \in \mathcal{Y} \\ & F(x,y) \leq \phi(x) \end{aligned}$$

where $\phi(x) = \min_{\hat{y}}\{F(x,\hat{y}) : (x,\hat{y}) \in \mathcal{Y}\}$

Example from [Moore and Bard, 1990]

$$\min_{x,y} \quad -x - 10y$$

s.t. $x \in P_U$

where y solves

$$\min_{y} \quad y$$

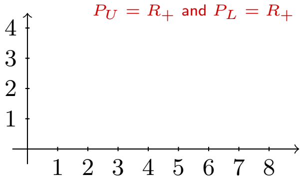
s.t $5x - 4y \geq -6$

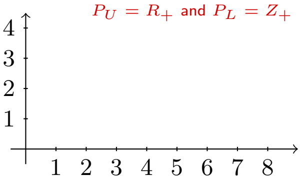
$-x - 2y \geq -10$

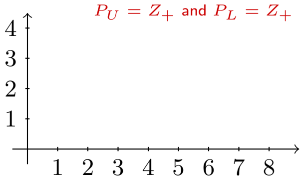
$-2x + y \geq -15$

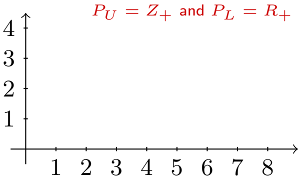
$2x + 10y \geq 15$

$y \in P_L$









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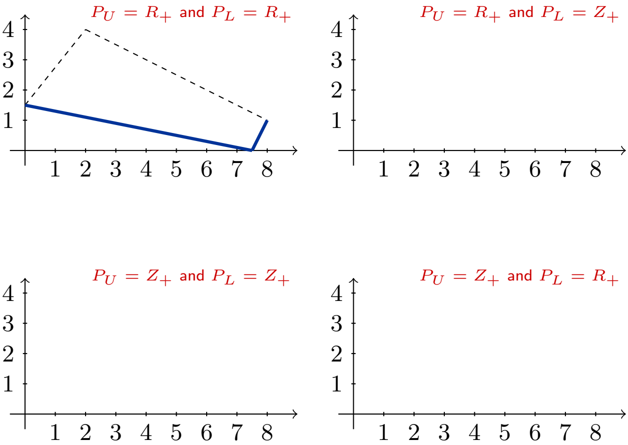

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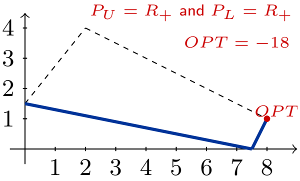

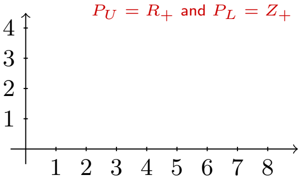

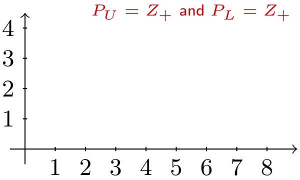

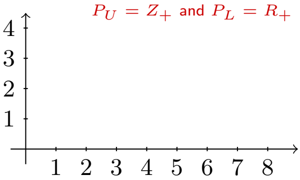

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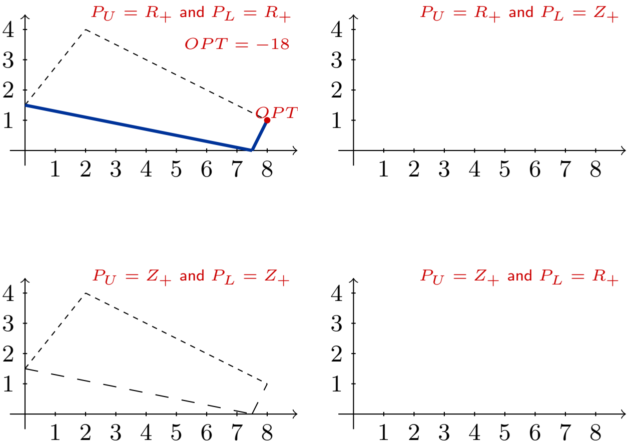

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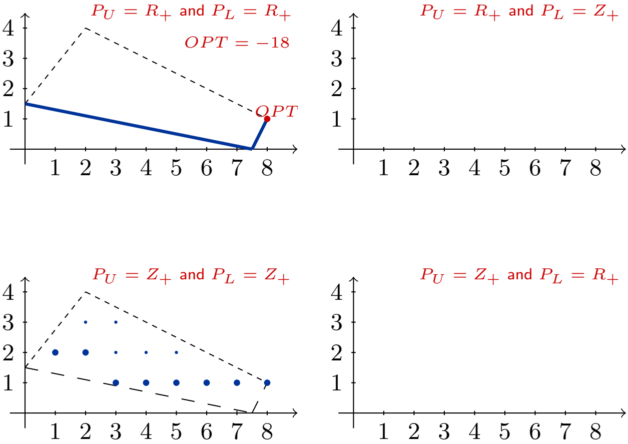

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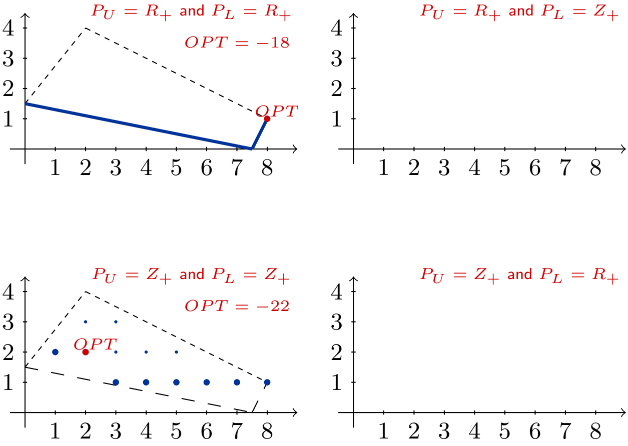

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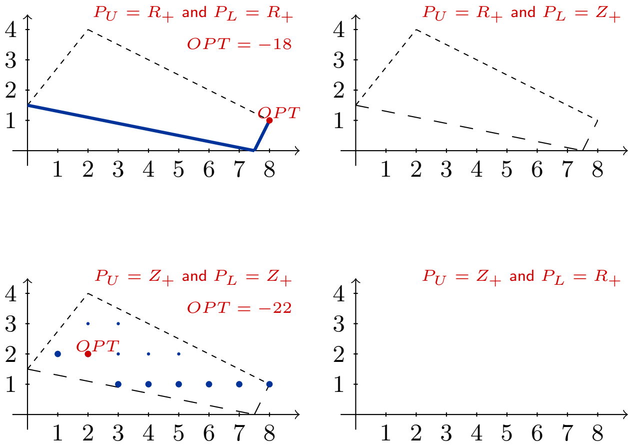

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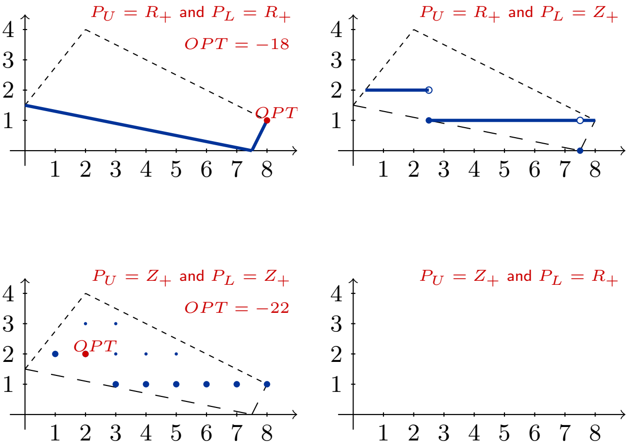

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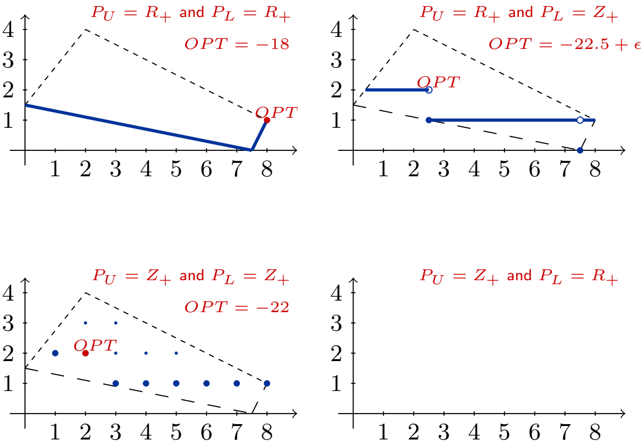

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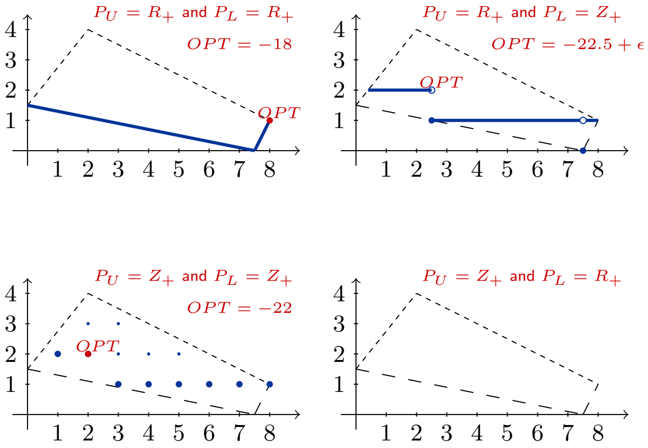

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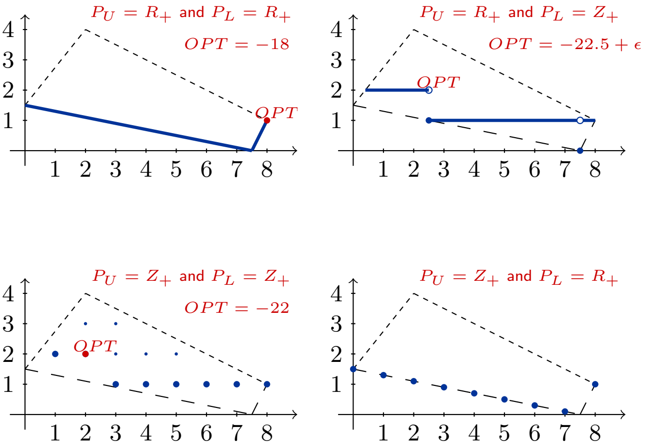

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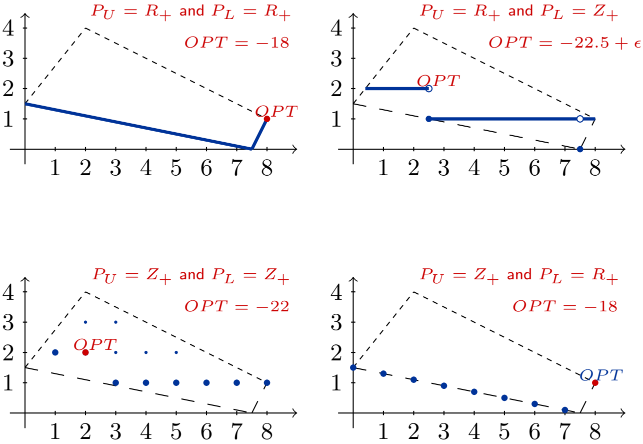

Contents

Linear case

The bilevel linear program

$$\begin{aligned}\max_{x,y} \quad & c^\top x + d^\top y \\ \text{s. t.} \quad & A_1 x + A_2 y \leq b_1 \\ & \text{where y solves} \\ & \qquad \max_{y} \quad d_2^\top y \\ & \qquad \text{s. t.} \quad A_3 x + A_4 y \leq b_2\end{aligned}$$

Bilevel LP

The bilevel linear program

$$\begin{aligned} \max_{x,y} \quad & c^\top x + d^\top y \\ \text{s. t.} \quad & A_1 x + A_2 y \leq b_1 \\ & \text{where } y \text{ solves:} \\ & \max_{y} \quad d_2^\top y \\ & \text{s. t.} \quad A_3 x + A_4 y \leq b_2 \end{aligned}$$

The tractable cases

- Fixed number of **lower-level** variables [Deng, 1998]
- Fixed number of **lower-level** constraints [Basu et al., 2021, Buchheim, 2023] *(including the pessimistic case [Ketkov and Prokopyev, 2025])*

Bilevel LP

The bilevel linear program

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The NP-hard cases

- Leader's objective $\geq$ target [Jeroslow, 1985, Ben-Ayed and Blair, 1990, Hansen et al., 1992] (also in NP [Buchheim, 2023])

Bilevel LP

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The NP-hard cases

- Leader's objective $\geq$ target [Jeroslow, 1985, Ben-Ayed and Blair, 1990, Hansen et al., 1992] (also in NP [Buchheim, 2023])
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Bilevel LP

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- Pessimistic & fixed number of lower-level variables [Ketkov and Prokopyev, 2025]
- Single upper-level variable [Sugishita and Carvalho, 2026a]

Bilevel LP

The bilevel linear program

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The NP-hard cases

- Leader's objective $\geq$ target [Jeroslow, 1985, Ben-Ayed and Blair, 1990, Hansen et al., 1992] (also in NP [Buchheim, 2023])
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Bilevel LP

The bilevel linear program

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- Verifying local optimality [Vicente et al., 1994]
- Finding solution within $c^{\text{numb. var.}}$ of local optimal solution [Prokopyev and Ralphs, 2026]

Bilevel LP

BILEVEL LP-DECISION

INPUT: Parameters of bilevel program in $\mathbb{Q}$

$$\begin{aligned} \max_{x,y} \quad & c^\top x + d^\top y \\ \text{s. t.} \quad & A_1 x + A_2 y \leq b_1 \\ & y \in \arg\max_{\hat{y}} \quad d_2^\top \hat{y} \\ & \qquad \text{s. t.} \quad A_4 \hat{y} \leq b_2 - A_3 x \end{aligned}$$

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BILEVEL LP-DECISION *is NP-hard.*

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Consider your favorite 0-1 problem (TSP, Knapsack, SAT, etc).

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Reformulation of S as BILEVEL LP-DECISION

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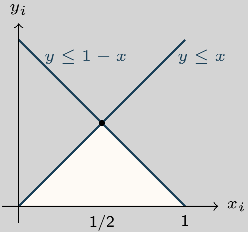
\end{aligned}$$



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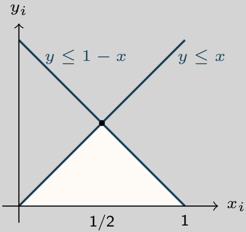

Make the target $k = 0$.

Bilevel bilinear problem

BILEVEL BILINEAR-DECISION (Quadric program... not really bilevel)

INPUT: Parameters of bilevel program in $\mathbb{Q}$

$$\begin{aligned} \min_{x,\lambda} \quad & (d_2 - D_1 x)^\top \lambda \\ \text{s. t.} \quad & A_1 x \leq b_1 \\ & \lambda \in \arg\min_{\hat{\lambda} \geq 0} \quad (d_2 - D_1 x)^\top \hat{\lambda} \\ & \qquad \text{s. t.} \quad A_4 \hat{\lambda} = b_2 \end{aligned}$$

and a target value $k \in \mathbb{Q}$.

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Consider the lower-level minimization problem for a fixed leader decision x:

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LP dual of the lower level

$$\max_{y} \left\{ b_2^\top y \ : \ A_4^\top y \leq d_2 - D_1 x \right\}$$

□

Proof.

Primal and dual form last slide:

$$\min_{\lambda \geq 0} \left\{ (d_2 - D_1 x)^\top \lambda \ : \ A_4 \lambda = b_2 \right\} \quad \max_{y} \left\{ b_2^\top y \ : \ A_4^\top y \leq d_2 - D_1 x \right\}$$

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Substituting this dual problem into the leader's formulation:

$$\begin{aligned} \min_{x,y} \quad & b_2^\top y \\ \text{s. t.} \quad & A_1 x \leq b_1 \\ & y \in \arg\max_{\hat{y}} \quad b_2^\top \hat{y} \\ & \qquad \text{s. t.} \quad D_1 x + A_4^\top \hat{y} \leq d_2 \end{aligned}$$

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This fits the Bilevel LP of the previous proof:

$$\begin{aligned} \min_{x,y} \quad & \sum_{i=1}^{n} y_i \\ \text{s.t.} \quad & Ax \leq b \\ & 0 \leq x \leq 1 \\ & y \in \arg\max_{\hat{y}} \left\{ \sum_{i=1}^{n} \hat{y}_i : \begin{array}{ll} \hat{y}_i \leq x_i & \\ \hat{y}_i \leq 1 - x_i & \end{array} \forall i \right\} \end{aligned}$$

Lower-level has finite optimal solution.

Binary bilevel LP

Theorem ([Lodi et al., 2014])

The decision version of a binary bilevel linear program is Σ_2^p-complete.

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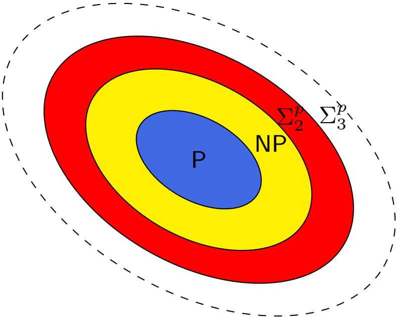

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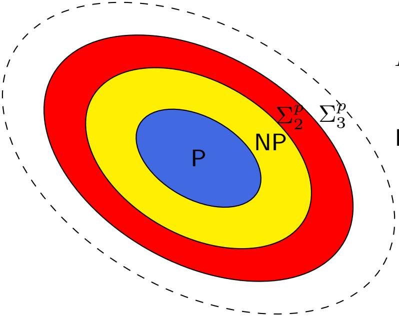

$$\max_{x,y}\{L(x,y) : (x,y) \in \mathcal{X}, y \in BR(x)\}$$

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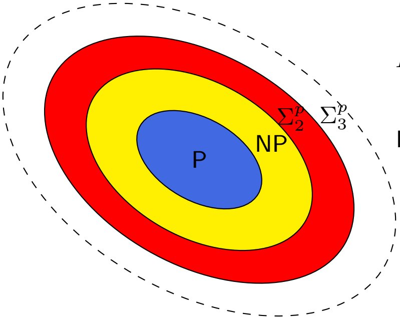

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Membership in Σ_2^p

$$\exists (x,y) \in \mathcal{X} \cap \mathcal{Y}, \forall \hat{y} \in \mathcal{Y}(x) :$$

$$L(x,y) \geq k \wedge F(x,y) \geq F(x,\hat{y})$$

Complexity landscape: Multilevel linear programming

			With Linking Constraints		Without Linking Constraints	
	k		Unbounded x	Bounded x	Unbounded x	Bounded x
k-VAL	≥ 2	■, ■	Σ^p_{k-1}-complete	Σ^p_{k-1}-complete	Σ^p_{k-1}-complete	Σ^p_{k-1}-complete
k-UNB	≥ 2	■, ■	Σ^p_{k-1}-complete	-	Σ^p_{k-1}-complete	-
k-FEAS	2	■	NP-complete	NP-complete	P	P
	3	■	Σ^p_2-complete	Σ^p_2-complete	coNP-complete	P
	4	■	Σ^p_3-complete	Σ^p_3-complete	Π^p_2-hard	P
	≥ 5	■	Σ^p_{k-1}-complete	Σ^p_{k-1}-complete	Σ^p_{k-1}-complete	Σ^p_{k-1}-complete
k-ATTAIN	2	■	DP-complete	NP-complete	coNP-complete	P
	3	■	Δ^p_3-complete	Δ^p_3-complete	Π^p_2-complete	P
	≥ 4	■	Δ^p_k-complete	Δ^p_k-complete	Δ^p_k-complete	Δ^p_k-complete
k-SEARCH	≥ 2	■	FΔ^p_k-complete	FΔ^p_k-complete	FΔ^p_k-complete	FΔ^p_k-complete

- ■ [Jeroslow, 1985]
- ■ [Rodrigues et al., 2026]
- ■ [Sugishita and Carvalho, 2026b]
- ■ [Sugishita and Carvalho, 2026c]

Complexity landscape: Mixed-binary multilevel linear programming

	k		With Linking Constraints		Without Linking Constraints	
			Unbounded x	Bounded x	Unbounded x	Bounded x
k-VAL	≥ 2	■ (green)	Σ_k^p-complete	Σ_k^p-complete	Σ_k^p-complete	Σ_k^p-complete
k-UNB	2		Σ_2^p-complete	-	NP-hard	-
	≥ 3	■ (red)	Σ_k^p-complete	-	Σ_k^p-complete	-
k-FEAS	2	■ (red)	Σ_2^p-complete	Σ_2^p-complete	NP-complete	NP-complete
	≥ 3	■ (red)	Σ_k^p-complete	Σ_k^p-complete	Σ_k^p-complete	Σ_k^p-complete
k-ATTAIN	≥ 2	■ (red)	Δ_{k+1}^p-complete	Δ_{k+1}^p-complete	Δ_{k+1}^p-complete	Δ_{k+1}^p-complete
k-SEARCH	≥ 2	■ (red)	FΔ_{k+1}^p-complete	FΔ_{k+1}^p-complete	FΔ_{k+1}^p-complete	FΔ_{k+1}^p-complete

- ■ (green) [Sugishita and Carvalho, 2026b]
- ■ (red) [Sugishita and Carvalho, 2026c]

Contents

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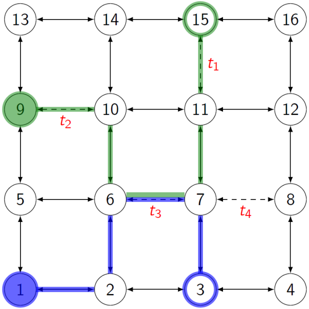
One approach to tackle bilevel programs is to represent them as optimization problems for which there are off-the-shelf solvers.

In this context there are typically 4 important components:

- primal representation
- dual representation
- optimality condition
- linearization method

Example: Network Pricing Problem [Labbé et al., 1998, Bui et al., 2022]



- Leader prices t_1, t_2, t_3, t_4.
- Follower 1 goes from 1 to 3.
  Follower 2 goes from 9 to 15.
- For each follower there is a toll-free path.

Example: Network pricing problem [Labbé et al., 1998, Bui et al., 2022]

$$
\max_{t\geq 0, x, y} \sum_{k\in\mathcal{K}} \eta^k t_a x_a^k
$$

$$
\forall k \in \mathcal{K} \begin{cases}
(x^k, y^k) \in \arg\min_{\hat{x},\hat{y}} \sum_{a\in A_1} (c_a + t_a)\hat{x}_a^k + \sum_{a\in A_2} c_a \hat{y}_a \\
\qquad \sum_{a\in A_1^+(i)} \hat{x}_a + \sum_{a\in A_2^+(i)} \hat{y}_a - \sum_{a\in A_1^-(i)} \hat{x}_a + \sum_{a\in A_2^-(i)} \hat{y}_a = b_i^k, & i \in V, \\
\qquad \hat{x}_a \in \{0,1\}, & a \in A_1, \\
\qquad \hat{y}_a \in \{0,1\}, & a \in A_2,
\end{cases}
$$

where $b_i^k = 1$ if $i = o^k$, -1 if $i = d^k$, and 0 otherwise.

Example: Network pricing problem [Labbé et al., 1998, Bui et al., 2022]

Network pricing problem:

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Dual for each follower:

$$\forall k \in \mathcal{K} \begin{cases} y^k \in \arg\max_{\hat{y}^k} (b^k)^\top \hat{y}^k \\ \qquad A^\top \hat{y}^k \leq c + t \end{cases}$$

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Standard formulation: strong duality

$$\max_{t \geq 0, x, y} \sum_{k \in \mathcal{K}} \eta^k t^\top x^k$$

$$\forall k \in \mathcal{K} \begin{cases} Ax^k = b^k \\ x^k \geq 0 \\ A^\top y^k \leq c+t \\ (c+t)^\top x^k = (b^k)^\top y^k \end{cases}$$

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Standard formulation: complementary slackness

$$\max_{t \geq 0, x, y} \sum_{k \in \mathcal{K}} \eta^k t^\top x^k$$

$$\forall k \in \mathcal{K} \begin{cases} Ax^k = b^k \\ x^k \geq 0 \\ A^\top y^k \leq c + t \\ ((c+t)^\top - A^\top y^k) x^k = 0 \end{cases}$$

Follower's problem representation

Primal Arc:

$$\begin{aligned} \min_{x^k} \quad & (c+t)^\top x^k \\ & Ax^k = b^k \\ & x^k \geq 0 \end{aligned}$$

Primal Path:

$$\begin{aligned} \min_{z^k} \quad & \sum_{p \in \mathcal{P}^k} ((c+t)^\top \hat{x}^p) z_p^k \\ & \sum_{p \in \mathcal{P}^k} z_p^k = 1 \\ & z_p^k \geq 0, p \in \mathcal{P}^k \end{aligned}$$

Dual Arc:

$$\begin{aligned} \max_{y^k} \quad & (b^k)^\top y^k \\ & A^\top y^k \leq c+t \end{aligned}$$

Dual Path:

$$\begin{aligned} \max_{L^k} \quad & L^k \\ & L^k \leq (c+t)^\top \hat{x}^p, p \in \mathcal{P}^k \end{aligned}$$

Example: Network Pricing Problem

Table 1: Strong duality

Dual \ Primal	Arc	Path
Arc	Standard (STD)	Path-Arc Standard (PASTD)
Path	Value Function (VF)	Path Value Function (PVF)

Table 2: Complementary slackness

Dual \ Primal	Arc	Path
Arc	Complementary Slackness (CS)	Path-Arc Complementary Slackness (PACS)
Path	Value Function Complementary Slackness (VFCS)	Path Complementary Slackness (PCS)

Example: Network Pricing Problem

Primal arc, Dual arc, Strong duality

$$
\begin{aligned}
\max \sum_{k \in \mathcal{K}} & \eta^k t^\top x^k \\
& A x^k = b^k, \qquad k \in \mathcal{K} \\
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& A^\top y^k \leq c + t, \qquad k \in \mathcal{K} \\
& (c+t)^\top x^k = (b^k)^\top y^k, \qquad k \in \mathcal{K} \\
& t \geq 0.
\end{aligned}
$$

Example: Network Pricing Problem

Primal path, Dual path, Complementary slackness

$$
\begin{aligned}
\max \sum_{k \in \mathcal{K}} \sum_{p \in \mathcal{P}^k} & \eta^k (t^\top \hat{x}^p) z_p^k \\
& \sum_{p \in \mathcal{P}^k} z_p^k = 1, \qquad k \in \mathcal{K} \\
& z_p^k \geq 0, \qquad k \in \mathcal{K}, p \in \mathcal{P}^k \\
& L^k \leq (c+t)^\top \hat{x}^p, \qquad k \in \mathcal{K}, p \in \mathcal{P}^k \\
& ((c+t)^\top \hat{x}^p - L^k) z_p^k = 0, \qquad k \in \mathcal{K}, p \in \mathcal{P}^k \\
& t \geq 0.
\end{aligned}
$$

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& t \geq 0.
\end{aligned}
$$

Value function constraint

$$
(c+t)^\top x^k \leq (c+t)^\top \hat{x}^p
$$

Example: Network Pricing Problem

Value function formulation: natural cutting-plane method

$$\max \sum_{k\in\mathcal{K}} \eta^k t^\top x^k$$

$$Ax^k = b^k, \qquad k \in \mathcal{K}$$

$$x^k \geq 0, \qquad k \in \mathcal{K}$$

$$(c+t)^\top x^k \leq (c+t)^\top \hat{x}^p, \qquad k \in \mathcal{K}, p \in \mathcal{P}^k$$

$$t \geq 0.$$

Mixed-integer bilevel program:

$$\begin{aligned} \min \quad & c_L^\top x + c_F^\top y \\ & A_L x + A_F y \leq b_L \\ & x \in \mathcal{X} \\ & y \in \arg\min\{d^\top y' : D_L x + D_F y' \leq b_F, y \in \mathcal{Y}\}. \end{aligned}$$

where $\mathcal{X}$ and $\mathcal{Y}$ impose integrality requirements.

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where $\mathcal{X}$ and $\mathcal{Y}$ impose integrality requirements.
Value function reformulation:

$$\begin{aligned} \min \quad & c_L^\top x + c_F^\top y \\ & A_L x + A_F y \leq b_L \\ & x \in \mathcal{X} \\ & D_L x + D_F y \leq b_F \\ & y \in \mathcal{Y} \\ & d^\top y \leq \phi(x) = \{d^\top y' : D_L x + D_F y' \leq b_F, y \in \mathcal{Y}\} \end{aligned}$$

Bilevel knapsack with interdiction

Bilevel model:

Value function model:

$$\max_{y \in B^n} \sum_{i=1}^{n} p_i y_i$$

$$\text{s.t.} \sum_{i=1}^{n} w_i y_i \leq C_l$$

Bilevel knapsack with interdiction

Bilevel model:

$$\min_{(x,y)\in B^n \times B^n} \sum_{i=1}^{n} p_i y_i$$

$$\text{s. t.} \quad \sum_{i=1}^{n} v_i x_i \leq C_u$$

where $y_1, \ldots, y_n$ solves the follower's problem

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$$\sum_{i=1}^{n} w_i y_i \leq C_l$$

$$\sum_{i=1}^{n} p_i y_i (1 - x_i) \geq \sum_{i=1}^{n} p_i \hat{y}_i (1 - x_i) \forall \text{ feasible } \hat{y}$$

Can you see duality here? We will unveil that in the next lecture...

Contents

To solve bilevels, we try to reformulate them into something we know how to solve.

Reformulations tend to have non-linearities due to optimality conditions.

Reformulations of mixed-integer linear bilevel programs tend to be non-compact.

Stackelberg game

Latin leader

$$
\begin{aligned}
\min_{x,y} \quad &: \quad c^\top x + d^\top y \\
\text{subject to} \quad & Ax + By \leq b \\
& y \in \operatorname*{arg\,min}_y \left\{ f^\top y : Qy \leq g - Px \right\}
\end{aligned}
$$

Stackelberg game

Trivial NASP

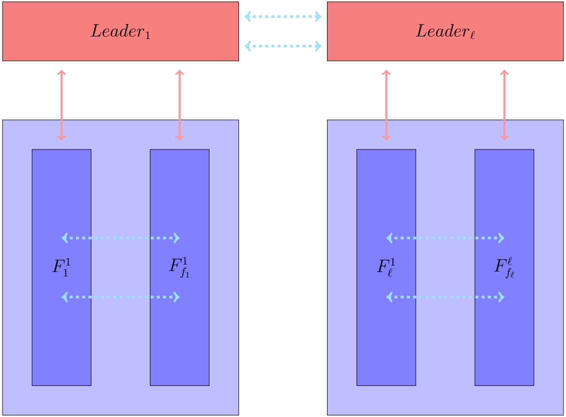
Latin leader

$$
\begin{array}{rcl}
\min\limits_{x,y} & : & c^\top x + d^\top y + \left(G \begin{pmatrix} \xi \\ \chi \end{pmatrix} \right)^\top \begin{pmatrix} x \\ y \end{pmatrix} \\
\text{subject to} & & Ax + By \quad \leq \quad b \\
 & & y \quad \in \quad \arg\min\limits_{y} \left\{ f^\top y : Qy \leq g - Px \right\}
\end{array}
$$

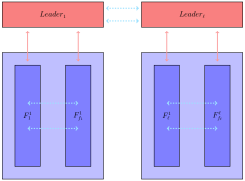
Greek leader

$$
\begin{array}{rcl}
\min\limits_{\xi,\chi} & : & \alpha^\top \xi + \beta^\top \chi + \left(\Gamma \begin{pmatrix} x \\ y \end{pmatrix} \right)^\top \begin{pmatrix} \xi \\ \chi \end{pmatrix} \\
\text{subject to} & & \Phi\xi + \Psi\chi \quad \leq \quad \rho \\
 & & \chi \quad \in \quad \arg\min\limits_{\chi} \left\{ \phi^\top \chi : \Omega\phi \leq \gamma - \Pi\xi \right\}.
\end{array}
$$

Nash Games among Stackelberg Players (Leaders) [Carvalho et al., 2023]



NASP



Definition (NASP)

A *NASP* is a linear Nash game $N = (P^1, \dots, P^k)$ where for each i, P^i is a simple Stackelberg game:

$$P^i \qquad \min_{x^i \in \mathcal{R}^{n_i}} \{f^i(x^i; x^{-i}) : x^i = (z^i, y^i) \in \mathcal{F}_i, y^i \in SOL(P(z^i))\}$$

f^i is linear
$\mathcal{F}_i$ is a polyhedron
$SOL(P(z^i))$ is the set of Nash equilibria for the game played by the followers
Followers have quadratic convex objectives and polyhedral feasible regions.

What is a Nash equilibrium?

	Rock	**Paper**	**Scissors**
Rock	$0, 0$	$-1, 1$	$1, -1$
Paper	$1, -1$	$0, 0$	$-1, 1$
Scissors	$-1, 1$	$1, -1$	$0, 0$

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Nash Equilibrium: Both players play $\mathbf{x} = \mathbf{y} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

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Nash Equilibrium: Both players play $\mathbf{x} = \mathbf{y} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

Row Player's Expected Payoff (if Column plays y):

$$\mathbb{E}[U^{Row}] = \sum_{j \in \{R,P,S\}} U_j^{Row} \cdot y_j$$

$$\mathbb{E}[U^{Row}] = \underbrace{0 \cdot \frac{1}{3}}_{\text{Tie}} + \underbrace{(-1) \cdot \frac{1}{3}}_{\text{Loss}} + \underbrace{1 \cdot \frac{1}{3}}_{\text{Win}} = 0$$

Linear complementarity problem

Definition (LCP)

Given $q \in \mathbb{R}^n$ and $M \in \mathbb{R}^{n \times n}$, the linear complementarity problem, searches for $z \in \mathbb{R}^n$ such that

$$\begin{aligned} z &\geq 0 \\ q + Mz &\geq 0 \\ z^\top (q + Mz) &= 0 \qquad \Leftrightarrow w = q + Mz, z^\top w = 0, w \geq 0 \end{aligned}$$

$$z \geq 0, \quad q + Mz \geq 0, \quad z^\top (q + Mz) = 0$$

Player X

$$\begin{aligned} \min_x \quad & c^{\top X} x + x \cdot C^X \cdot y + \frac{1}{2} x^\top Q^X x \\ s.t. \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

KKT conditions

$$\begin{aligned} \alpha &= c^X + C^X y + Q^X x - A^\top \mu \\ \nu &= -b + Ax \\ x^\top \alpha &= 0 \\ \mu^\top \nu &= 0 \\ x \geq 0, \mu \geq 0&, \alpha \geq 0, \nu \geq 0 \end{aligned}$$

Player Y

$$\begin{aligned} \min_y \quad & c^{\top Y} y + y \cdot C^Y \cdot x + \frac{1}{2} y^\top Q^Y y \\ s.t. \quad & Dy \geq f \\ & y \geq 0 \end{aligned}$$

KKT conditions

$$\begin{aligned} \beta &= c^Y + C^Y x + Q^Y y - D^\top \lambda \\ \eta &= -f + Dy \\ y^\top \beta &= 0 \\ \lambda^\top \eta &= 0 \\ y \geq 0, \lambda \geq 0&, \beta \geq 0, \eta \geq 0 \end{aligned}$$

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$$q = \begin{bmatrix} c^X \\ -b \\ c^Y \\ -f \end{bmatrix} \qquad M = \begin{bmatrix} Q^X & -A^\top & C^X & 0 \\ A & 0 & 0 & 0 \\ C^Y & 0 & Q^Y & -D^\top \\ 0 & D & 0 & 0 \end{bmatrix} \qquad z = \begin{bmatrix} x \\ \mu \\ y \\ \lambda \end{bmatrix}$$

Equilibria & LCPs

Theorem ([Cottle et al., 2009])

Let P be a facile Nash game. Then, there exist M, q such that every solution to the LCP defined by M, q is a pure Nash equilibrium for P and every pure Nash equilibrium of P solves the LCP.

Equilibria & LCPs

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Idea 1: The followers play a facile Nash game. We can find a pure equilibrium for it by solving an LCP.

$$y^i \in SOL(P(z^i)) \underbrace{\Leftrightarrow}_{KKT} 0 \leq (x^i, \lambda^i) \perp Mx^i + N\lambda^i + q \geq 0$$

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We will also show that the leader's problem can be transformed in a facile Nash game.

Stackelberb & Equilibria & LCPs

Theorem ([Basu et al., 2021])

Let S be the feasible set of a simple Stackelberg game. Then, S is a finite union of polyhedra. Conversely, let S be a finite union of polyhedra. Then, there exists a simple Stackelberg game with $P(x)$ containing exactly 1 player such that the feasible region of the simple Stackelberg game provides an extended formulation of S.

Stackelberb & Equilibria & LCPs

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Idea 2: The followers' game can be replaced by a union of polyhedra.

$$(P^i) \quad \min_{x^i \in \mathcal{R}^{n_i}} \{f^i(x^i; x^{-i}) : x^i = (z^i, y^i) \in \mathfrak{F}_i, y^i \in SOL(P(z^i))\}$$

$$\Leftrightarrow \min_{x^i \in \mathcal{R}^{n_i}} \{f^i(x^i; x^{-i}) : x^i = (z^i, y^i) \in \mathfrak{F}_i, 0 \leq w^i = (x^i, \lambda^i) \perp M' w^i + q \geq 0\}$$

L sufficiently large.

Stackelberb & Equilibria & LCPs

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$$\Leftrightarrow \min_{x^i \in \mathcal{R}^{n_i}} \{f^i(x^i; x^{-i}) : x^i = (z^i, y^i) \in \mathcal{F}_i, 0 \leq w^i_j \leq L v_j \quad \forall j = 1, \dots, k,$$

$$0 \leq \{M' w^i + q\}_j \leq (1 - v_j) L \quad \forall j = 1, \dots, k, v \in \{0,1\}^k\}$$

L sufficiently large.

Theorem ([Balas, 1985])

Given k polyhedra $S_i = \{x \in \mathcal{R}^n : A^i x \leq b^i\}$ for $i = 1, \dots k$, then $\mathrm{cl\,conv}(\bigcup_{i=1}^k S_i)$ *is given by the set* $\{x \in \mathcal{R}^n : \exists (x^1, \dots, x^k, \delta) \in (\mathcal{R}^n)^k \times \mathcal{R}^k : x \in \{A^i x^i \leq \delta_i b^i, \sum_{w=1}^k x^w = x, \sum_{w=1}^k \delta_w = 1, \delta_i \geq 0, \forall i \in [k]\}\}$

Idea 3: Leader i mixed strategy belongs to the convex hull closure of their feasible set.

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Mixed strategy: $w^i = \sum_j \eta_j \hat{w}^i_j$ with $\hat{w}^i_j \in S^i_j \cap \mathcal{F}_i$ and $\sum_j \eta_j = 1$.

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$$0 \leq \{M' w^i + q\}_j \leq (1 - v_j) L \quad \forall j = 1, \dots, k, v \in \{0,1\}^k\}$$

$$\Leftrightarrow \min_{w^i} \{f^i(x^i; x^{-i}) : x^i \in \mathcal{F}_i, w^i = (x^i, \lambda^i) \in \bigcup_{j=1}^{2^k} S^i_j\}$$

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$$
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& \quad 0 \leq \{M' w^i + q\}_j \leq (1 - v_j) L \quad \forall j = 1, \ldots, k, v \in \{0,1\}^k\} \\
\Leftrightarrow\ & \min_{w^i} \{f^i(x^i; x^{-i}) : x^i \in \mathcal{F}_i, w^i = (x^i, \lambda^i) \in \bigcup_{j=1}^{2^k} S^i_j\} \\
\Leftrightarrow\ & \min_{w^i, \eta} \{\sum_j \eta_j f^i(x^i_j; x^{-i}) : x^i_j \in \mathcal{F}_i, w^i_j \in S^i_j, \sum_j \eta_j = 1\} \quad \textit{since the objective is linear} \\
\Leftrightarrow\ & \min_{w^i} \{f^i(x^i; x^{-i}) : w^i = (\overbrace{(z^i, y^i)}^{x^i}, \lambda^i) \in \mathrm{cl\,conv}(\bigcup_{j=1}^{k'} \left(S^i_j \cap \mathcal{F}_i\right))\}
\end{aligned}
$$

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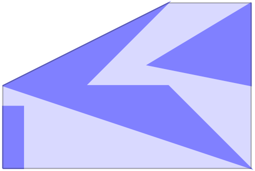
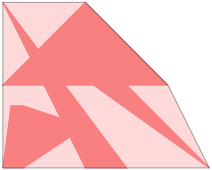
Enumeration algorithm

Step 1: enumerate all polyhedra for each leader

Enumeration algorithm

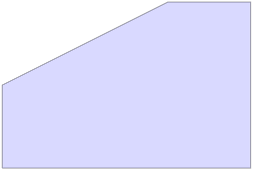
Step 2: compute the convex-hull of each leader

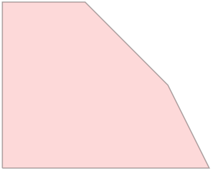




Enumeration algorithm

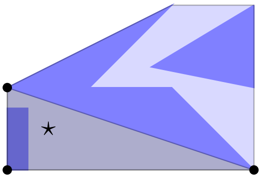
Step 3: the leaders' game is equivalent to an LCP (which can be converted in a MIP)

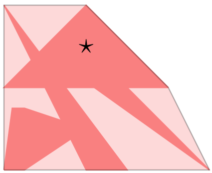




Enumeration algorithm

Step 4: the solution can be interpreted as a mixed strategy





Enumeration algorithm

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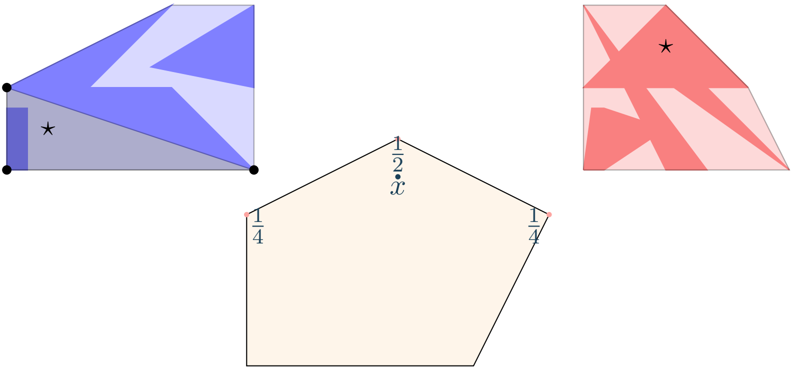

$$f^i(\sigma) = f^i(x)$$

Inner approximation algorithm

Challenge: There can be exponentially many polyhedra!

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$$S = \left\{ x : \begin{array}{c} Ax \leq b \\ z = Mx + q \\ 0 \leq x_i \perp z_i \geq 0, \quad \forall\, i \in \mathcal{C} \end{array} \right\} = \bigcup_{j=1}^{2^{|\mathcal{C}|}} S_j$$

$$\operatorname{cl}\operatorname{conv}(S) \subseteq \mathcal{O}_0 = \{x : Ax \leq b,\ z = Mx + q,\ x_i \geq 0,\ z_i \geq 0\ \ \forall\, i \in \mathcal{C}\}$$

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$$\operatorname{cl}\operatorname{conv}\left(\bigcup_{b \in J} \mathcal{P}(b) \cap \mathcal{O}_0 \right) \subseteq \operatorname{cl}\operatorname{conv}(S)$$

where
$\mathcal{P}(b) = \{x_{c_i} \leq 0,\ \forall\, i \in \{i : b_i = 0\}\} \bigcap \{[Mx + q]_{c_i} \leq 0,\ \forall\, i \in \{i : b_i = 1\}\}$

Inner approximation algorithm

1. Construct an initial inner approximation $\hat{\mathcal{F}}^i$ of each leader i feasible region
2. Solve the Nash game for the feasible strategies $\hat{\mathcal{F}}^i$
3. If step 2 found an equilibrium, verify if a player has incentive to deviate: if not, return equilibrium; otherwise go to step 4.

Inner approximation algorithm

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3. If step 2 found an equilibrium, verify if a player has incentive to deviate: if not, return equilibrium; otherwise go to step 4.
   Otherwise, for each player i, add a new set of polyhedra to $\hat{\mathcal{F}}^i$ and go to step 2.
4. Add the polyhedra corresponding to a player deviation. Go to step 2.

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