Duality-based reformulations in bilevel optimization

Part II

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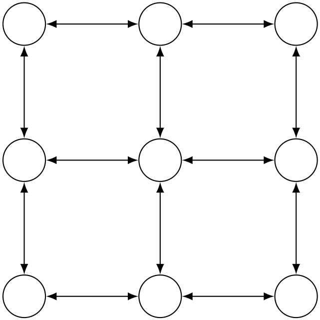
Contents

Contents

- Directed graph (V, A)



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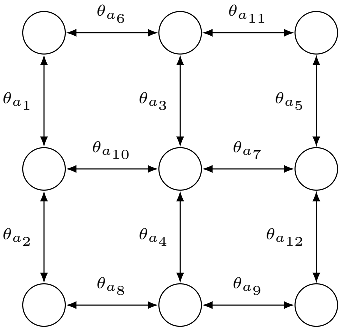

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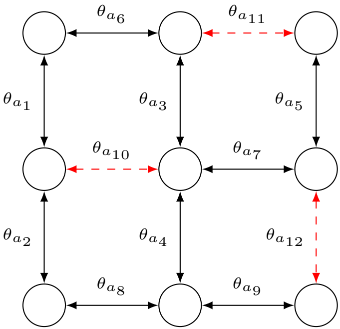

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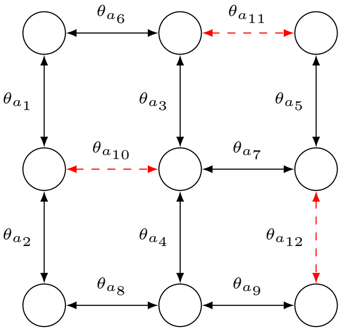

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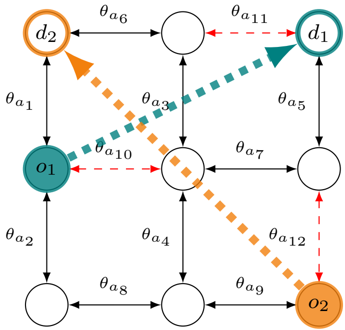

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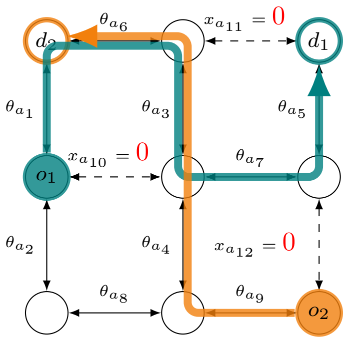

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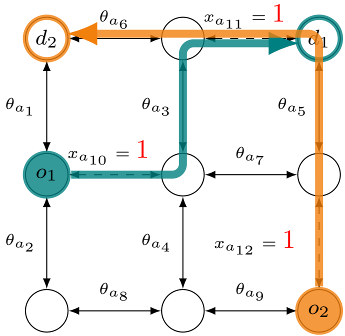

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Bilevel network design problem

$$\min_{x,z} \sum_{a \in A} \left(\sum_{o \in O} z_{oa} \right) \theta_a \left(\sum_{o \in O} z_{oa} \right) \quad \text{(System Optimum Objective)}$$

$$\text{s.t. } x \in X$$

$$z \in Z(x) \quad \text{(User Equilibrium)}$$

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Travel time function

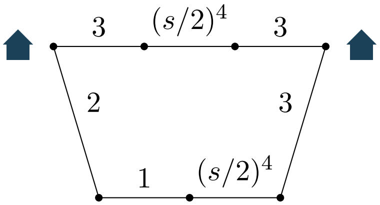
$$\theta_a(s) = f_a + g_a s^p$$

where $f_a \geq 0$ and $g_a \geq 0$ and $p \in \mathbb{Z}_{>0}$.

Contents

Followers

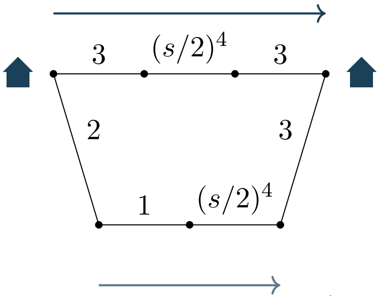
Total flow of 4.



Followers

Total flow of 4. User equilibrium: No incentive to deviate.

flow $= 2$, time $= 3 + (2/2)^4 + 3 = 7$



flow $= 2$, time $= 2 + 1 + (2/2)^4 + 3 = 7$

Followers [Beckmann et al., 1955]

Primal Formulation [Beckmann et al., 1955]

$$\phi(x) = \min_{z} \sum_{a \in A(x)} \int_0^{\sum_{o \in O} z_{oa}} \theta_a(s) ds \tag{1a}$$

$$\text{s.t.} \sum_{a \in A_v^+(x)} z_{oa} - \sum_{a \in A_v^-(x)} z_{oa} = e_{ov}, \quad \forall o \in O, v \in V, \tag{1b}$$

$$z \geq 0. \tag{1c}$$

- $O = \{v \in V : \exists k \in K, o_k = v\}$
- $e_{ov} = \sum_{k \in K: o \in k} e'_{kv}$ where $e'_{ko_k} = -e_k$, $e'_{kd_k} = e_k$, $e'_{kv} = 0$ for $v \notin \{o_k, d_k\}$.
- z_{oa}: Flow of users originating from o traversing arc a.

Followers

Consider for arc a:
Travel time function: $\theta_a(s) = 1 + s^2$
Demand: $o_1 \to 1$ unit, $o_2 \to 1$ unit.
Total flow: $s = 2$.

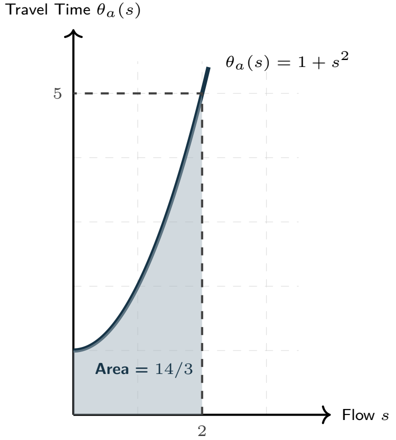
The arc travel time at total flow is:

$$\theta_a(2) = 1 + 4 = 5$$

Contribution in the lower-level objective:

$$\Phi_a(2) = \int_0^2 (1 + s^2)\, ds = \left[s + \frac{s^3}{3} \right]_0^2 = \frac{14}{3}$$

Integrating from 0 to s accumulates the travel time of incremental drivers.



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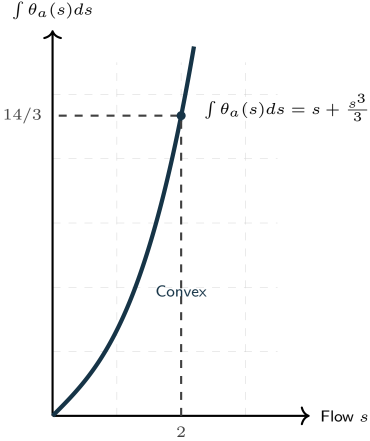

Contents

Followers' dual

To derive KKT conditions, we first introduce dual variables for our constraints:

- π_{ov}: Dual variable for flow conservation.
- $\lambda_{oa} \geq 0$: Dual variable for non-negativity constraints.

The Full Lagrangian $\mathcal{L}(z, \pi, \lambda)$

$$\mathcal{L} = \sum_{a \in A(x)} \int_0^{\sum_{o \in O} z_{oa}} \theta_a(s) ds$$

$$- \sum_{o \in O} \sum_{v \in V} \pi_{ov} \underbrace{\left(\sum_{a \in A_v^+(x)} z_{oa} - \sum_{a \in A_v^-(x)} z_{oa} - e_{ov} \right)}_{\text{Flow Conservation}}$$

$$- \sum_{o \in O} \sum_{a \in A(x)} \lambda_{oa} z_{oa}$$

Optimality conditions

Taking the gradient of $\mathcal{L}$ with respect to flow on arc $a = (t, h)$:

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$$\frac{\partial \mathcal{L}}{\partial z_{oa}} = \theta_a \left(\sum_{o' \in O} z_{o'a} \right) - \pi_{oh} + \pi_{ot} - \lambda_{oa} = 0$$

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Because λ_{oa} is non-negativity, $\lambda_{oa} \geq 0$. We substitute:

$$\theta_a \left(\sum_{o' \in O} z_{o'a} \right) - \pi_{oh} + \pi_{ot} = \lambda_{oa} \geq 0$$

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Applying **complementary slackness** ($\lambda_{oa} \, z_{oa} = 0$):

$$\left(\theta_a \left(\sum_{o' \in O} z_{o'a} \right) - \pi_{oh} + \pi_{ot} \right) z_{oa} = 0$$

We have fully eliminated λ from the system.

Optimality conditions

KKT Conditions

$$\sum_{a \in A_v^+(x)} z_{oa} - \sum_{a \in A_v^-(x)} z_{oa} = e_{ov}, \qquad \forall o \in O, v \in V$$

(Primal Feas.)

$$\theta_a\left(\sum_{o \in O} z_{oa}\right) - \pi_{oh} + \pi_{ot} \geq 0, \qquad \forall o \in O, a = (t,h) \in A(x)$$

(Dual Feas.)

$$\left(\theta_a\left(\sum_{o \in O} z_{oa}\right) - \pi_{oh} + \pi_{ot}\right) z_{oa} = 0, \qquad \forall o \in O, a = (t,h) \in A(x)$$

(Comp. Slack.)

$$z \geq 0. \qquad \text{(Non-negativity)}$$

Optimality conditions

Recall $\theta_a(s) = f_a + g_a s^p$.

Step 1: Substitute the specific travel time function into the integral.

$$L(z,\pi) = \sum_{a \in A(x)} \int_0^{\sum z_{oa}} \theta_a(s) ds - \sum_{o \in O, v \in V} \pi_{ov} \text{(Flow Cons.)}$$

$$=$$

$$=$$

$$=$$

Optimality conditions

Recall $\theta_a(s) = f_a + g_a s^p$.

Step 2: Expand the integral and distribute the node potential duals (π).

$$L(z,\pi) = \sum_{a \in A(x)} \int_0^{\sum z_{oa}} \theta_a(s)ds - \sum_{o \in O, v \in V} \pi_{ov}(\text{Flow Cons.})$$

$$= \sum_a \left(f_a \left(\sum z_{oa}\right) + \frac{g_a}{p+1} \left(\sum z_{oa}\right)^{p+1} \right) - \sum_{o,a=(t,h)} (\pi_{oh} - \pi_{ot}) z_{oa} + \sum_k e_k (\pi_{o_k d_k} - \pi_{o_k o_k})$$

$$=$$

$$=$$

Optimality conditions

Recall $\theta_a(s) = f_a + g_a s^p$.

Step 3: Regroup per arc. Introduce $(\pi_{oh} - \pi_{ot} - f_a)$ and isolate demand.

$$L(z,\pi) = \sum_{a \in A(x)} \int_0^{\sum z_{oa}} \theta_a(s)ds - \sum_{o \in O, v \in V} \pi_{ov}(\text{Flow Cons.})$$

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$$= \sum_a \underbrace{\left(\frac{g_a}{p+1} \left(\sum z_{oa}\right)^{p+1} - \sum_o (\pi_{oh} - \pi_{ot} - f_a) z_{oa} \right)}_{L_a(z_a, \pi_a)} + \sum_k e_k (\pi_{o_k d_k} - \pi_{o_k o_k})$$

$$=$$

Optimality conditions

Recall $\theta_a(s) = f_a + g_a s^p$.

Step 4: Isolate the arc-wise component $L_a(z_a, \pi_a)$. The problem now separates!

$$
\begin{aligned}
L(z,\pi) &= \sum_{a \in A(x)} \int_0^{\sum z_{oa}} \theta_a(s)ds - \sum_{o \in O, v \in V} \pi_{ov} (\text{Flow Cons.}) \\
&= \sum_a \left(f_a \left(\sum z_{oa}\right) + \frac{g_a}{p+1} \left(\sum z_{oa}\right)^{p+1} \right) - \sum_{o, a=(t,h)} (\pi_{oh} - \pi_{ot}) z_{oa} + \sum_k e_k (\pi_{o_k d_k} - \pi_{o_k o_k}) \\
&= \sum_a \underbrace{\left(\frac{g_a}{p+1} \left(\sum z_{oa}\right)^{p+1} - \sum_o (\pi_{oh} - \pi_{ot} - f_a) z_{oa} \right)}_{L_a(z_a, \pi_a)} + \sum_k e_k (\pi_{o_k d_k} - \pi_{o_k o_k}) \\
&= \sum_{a \in A(x)} L_a(z_a, \pi_a) + \sum_{k \in K} e_k (\pi_{o_k d_k} - \pi_{o_k o_k})
\end{aligned}
$$

Optimality conditions

Lagrangian dual problem

$$\max_{\pi} \min_{z \geq 0} L(z, \pi) = \max_{\pi} \left\{ \sum_{a \in A(x)} \min_{z \geq 0} L_a(z_a, \pi_a) + \sum_{k \in K} e_k(\pi_{o_k d_k} - \pi_{o_k o_k}) \right\}$$

Solving $\min_{z \geq 0} L_a(z_a, \pi_a)$ analytically yields a piecewise function dependent on g_a [Sugishita et al., 2025]. To linearize it, we introduce η_a:

$$\eta_a = \max_{o \in O} \{\pi_{oh} - \pi_{ot} - f_a\} \Leftrightarrow \eta_a \geq \pi_{oh} - \pi_{ot} - f_a \quad \forall o \in O$$

Dual Formulation

$$\max_{\pi,\eta} \sum_{k \in K} e_k(\pi_{o_k d_k} - \pi_{o_k o_k}) - \sum_{a \in A(x)} \frac{p}{\bar{g}_a^{1/p}(p+1)} \eta_a^{\frac{p+1}{p}}$$

$$\begin{aligned} \text{s.t. } & \eta_a \geq \pi_{oh} - \pi_{ot} - f_a, && \forall o \in O, a = (t,h) \in A(x), \\ & \eta_a = 0, && \forall a \in A(x) : g_a = 0 \\ & \eta \geq 0 \end{aligned}$$

Reformulation

Constraints are linear and the objective is separable and convex (monotropic programming).

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Constraints are linear and the objective is separable and convex (monotropic programming).

1. Both follower's primal and dual formulations are feasible $\forall x \in X$.
2. Strong duality holds.
3. Primal/Dual variables are optimal $\iff$ they satisfy the KKT conditions.

Primal Problem

$$\min_{z} \sum_{a \in A(x)} \int_0^{\sum_{o \in O} z_{oa}} \theta_a(s) ds$$

$$\text{s.t.} \sum_{a \in A_v^+(x)} z_{oa} - \sum_{a \in A_v^-(x)} z_{oa} = e_{ov}$$

$$z \geq 0.$$

Dual Problem

$$\max_{\pi,\eta} \sum_k e_k(\pi_{o_d d_d} - \pi_{o_d o_d})$$

$$- \sum_a \frac{p}{\overline{g}_a^{1/p}(p+1)} \eta_a^{\frac{p+1}{p}}$$

$$\text{s.t. } \eta_a \geq \pi_{oh} - \pi_{ot} - f_a$$

$$\eta_a = 0 \; a \in A(x) : g_a = 0$$

$$\eta \geq 0$$

Reformulation

The original Bilevel Network Design Problem:

$$\min_{x,z} \sum_{a\in A} \left(\sum_{o\in O} z_{oa} \right) \theta_a \left(\sum_{o\in O} z_{oa} \right) \quad \text{(System Optimum Objective)}$$

$$\text{s.t. } x \in X,\ z \in Z(x) \quad \text{(Follower's Equilibrium)}$$

Reformulation

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Dualize-and-combine Formulation

$$\min_{x,z,\pi,\eta} \sum_{a \in A} \left(\sum_{o \in O} z_{oa} \right) \theta_a \left(\sum_{o \in O} z_{oa} \right)$$

$$\text{s.t. } \underbrace{\sum_{a \in A} \int_0^{\sum_{o \in O} z_{oa}} \theta_a(s) ds}_{\text{convex in } z} \leq \underbrace{\sum_{k \in K} e_k (\pi_{o_k d_k} - \pi_{o_k o_k}) - \sum_{a \in A} \frac{p}{g_a^{1/p}(p+1)} \eta_a^{\frac{p+1}{p}}}_{\text{concave in } (\pi,\eta)} \quad \text{(Strong Duality)}$$

$$\sum_{a \in A_v^+} z_{oa} - \sum_{a \in A_v^-} z_{oa} = e_{ov}, \quad z_{oa} \leq \sum_{k \in K_o} e_k x_a \quad \text{(primal constraints)}$$

$$\eta_a \geq \pi_{oh} - \pi_{ot} - f_a - M_{oa}(1 - x_a) \quad \text{(dual constraints)}$$

$$\eta_a = 0 \text{ (if } g_a = 0), z \geq 0, \eta \geq 0, x \in X$$

Contents

The problem

Combinatorial pricing problems (CPP)

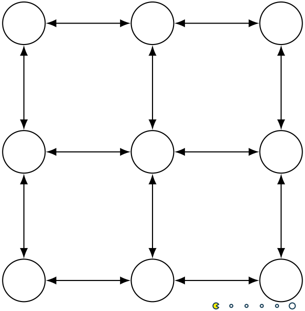
- Leader: sets prices (tolls)
- Follower(s): solves combinatorial problem
  - Shortest path $\rightarrow$ Network pricing problem (NPP)
  - Knapsack $\rightarrow$ Knapsack pricing problem
  - Min set cover pricing, Max stable set pricing, etc.

Natural (non-linear) bilevel formulation

Network pricing problem

$$\max_{t,x} \left\{ \sum_{k \in \mathcal{K}} t^\top x^k \;\middle|\; \begin{array}{ll} t_a \geq 0 & \forall a \in \mathcal{A}_1 \\ t_a = 0 & \forall a \in \mathcal{A}_2 \\ x^k \in \mathbf{R}^k(t) & \forall k \in \mathcal{K} \end{array} \right\}$$

$$\mathbf{R}^k(t) := \operatorname*{argmin}_{x^k} \left\{ (c+t)^\top x^k \;\middle|\; \begin{array}{l} Ax^k = b^k \\ x^k \in \{0,1\}^{|\mathcal{A}|} \end{array} \right\} \qquad \forall k \in \mathcal{K}$$



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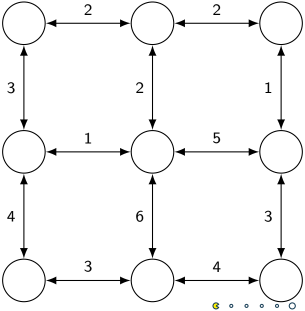

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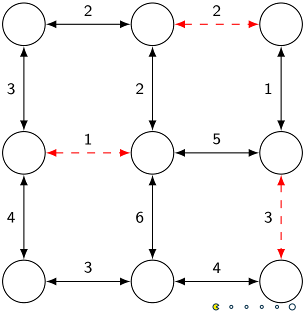

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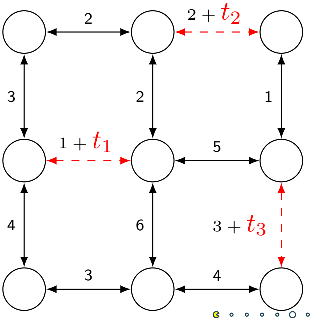

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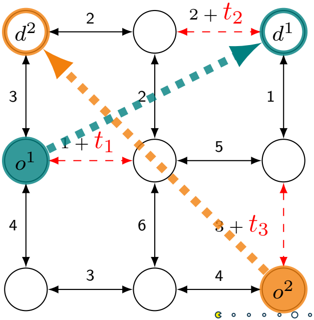

Network pricing problem

$$\max_{t,x} \left\{ \sum_{k \in \mathcal{K}} t^\top x^k \;\middle|\; \begin{array}{ll} t_a \geq 0 & \forall a \in \mathcal{A}_1 \\ t_a = 0 & \forall a \in \mathcal{A}_2 \\ x^k \in \mathbf{R}^k(t) & \forall k \in \mathcal{K} \end{array} \right\}$$

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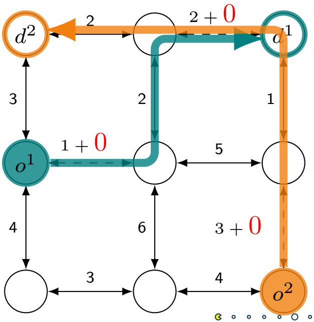

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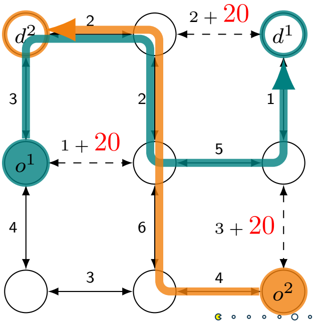

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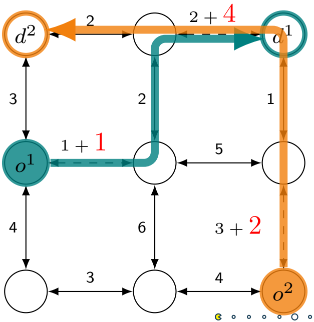

rev $= (1+4) + (4+2) = 11$

Network pricing problem

Solving components [Bui et al., 2022]

- Primal representation (of the lower-level)
- Dual representation (of the lower-level)
- Optimality conditions
- Linearization

Network pricing problem

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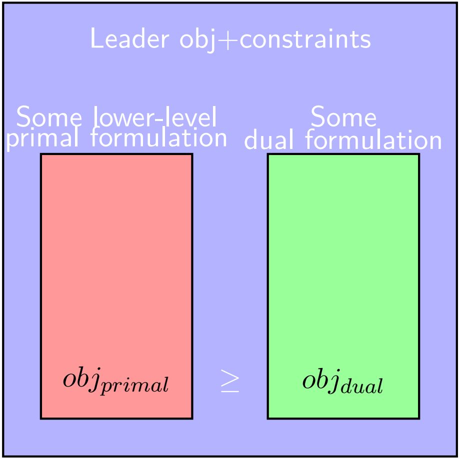
Primal arc, Dual arc, Strong duality

$$
\begin{aligned}
\max \quad & \sum_{k \in \mathcal{K}} t^\top x^k \\
\text{s.t.} \quad & A x^k = b^k && k \in \mathcal{K}, \\
& x^k \geq 0 && k \in \mathcal{K}, \\
& A^\top y^k \leq c + t && k \in \mathcal{K}, \\
& (c+t)^\top x^k = (b^k)^\top y^k && k \in \mathcal{K}, \\
& t \in \mathcal{T}.
\end{aligned}
$$

Key idea



(if primal is a maximization)

Combinatorial pricing problem

The lower level is a combinatorial optimization problem
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How to solve the CPP effectively?

Overview

- Literature:
  - Single-commodity NPP:
    well-studied [Labbé et al., 1998, Brotcorne et al., 2001]
  - Multi-commodity NPP: recently studied through the lens of strong bilevel feasibility [Bui et al., 2024]
  - General CPP: only approximation & complexity results
    - Σ_2^p-hard result [Böhnlein et al., 2023, Grüne et al., 2025]
    - NP-hardness: lower-level is a shortest path [Roch et al., 2005] or a minimum spanning tree [Cardinal et al., 2011]
  - Recent technique to formulate the dual of the lower-level [Lozano et al., 2022, Bui et al., 2025, Vásquez et al., 2025]

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  - Recent technique to formulate the dual of the lower-level [Lozano et al., 2022, Bui et al., 2025, Vásquez et al., 2025]
- Goals:
  - Generalize ideas from the NPP to CPPs
  - Obtain tractable reformulations

Contents

Novel ideas [Bui et al., 2025]

Solve the CPP using embedded dynamic programming models

- Formulation of the CPP using dynamic programming
  - Decision diagram
  - Selection diagram
- Dynamic constraint generation

Contents

Knapsack pricing problem

- Items $\mathcal{I}$
  - Weights w_i for $i \in \mathcal{I}$
  - Base profits p_i for $i \in \mathcal{I}$
- Capacity C
- Tolled items $\mathcal{I}_1 \subseteq \mathcal{I}$
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$$\max_{t,x} \left\{ t^\top x \;\middle|\; \begin{array}{ll} t_i \geq 0 & \forall i \in \mathcal{I}_1 \\ t_i = 0 & \forall i \in \mathcal{I}_2 = \mathcal{I} \setminus \mathcal{I}_1 \\ x \in \mathbf{R}(t) & \end{array} \right\}$$

$$\mathbf{R}(t) := \operatorname*{argmax}_x \left\{ (p-t)^\top x \;\middle|\; \begin{array}{l} w^\top x \leq C \\ x \in \{0,1\}^{\mathcal{I}} \end{array} \right\}$$

Contents

Value function formulation

$$\max_z \left\{ \sum_{\hat{x} \in \mathcal{X}} ((v-t)^\top \hat{x}) z_{\hat{x}} \mid \sum_{\hat{x} \in \mathcal{X}} z_{\hat{x}} = 1,\ z \in \{0,1\}^{|\mathcal{X}|} \right\}$$

$$\min_L \{ L \mid L \geq (v-t)^\top \hat{x},\ \hat{x} \in \mathcal{X} \}.$$

$$\begin{aligned} \max\ & t^\top x \\ \text{s.t. } & w^\top x \leq C, \\ & x \in \{0,1\}^{\mathcal{I}}, \\ & L \geq (p-t)^\top \hat{x}, \qquad \hat{x} \in \mathcal{X}, \\ & (p-t)^\top x = L, \\ & t \in \mathcal{T}. \end{aligned}$$

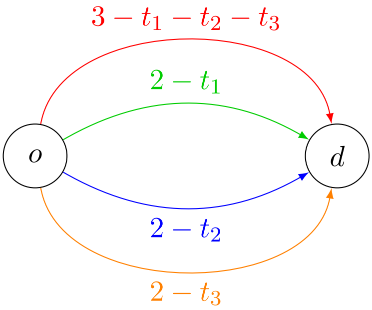
- $L \geq (p-t)^\top \hat{x}$ is generated dynamically
  - Solve master problem for t
  - Solve knapsack problem given t to obtain $\hat{x}$
  - Repeat

Value function example

$\mathcal{I}$	1	2	3	4
w	1	1	1	2
C	3			
p	1	1	1	1
$\hat{x}$	1	1	1	0
	1	0	0	1
	0	1	0	1
	0	0	1	1

$$\begin{aligned} \max\ & t^\top x \\ \text{s.t. } & w^\top x \leq C, \\ & x \in \{0,1\}^{\mathcal{I}}, \\ & L \geq 3 - t_1 - t_2 - t_3, \\ & L \geq 2 - t_1, \\ & L \geq 2 - t_2, \\ & L \geq 2 - t_3, \\ & (p-t)^\top x = L, \\ & t \in \mathcal{T}. \end{aligned}$$

VF as Dynamic programming model



$$L = y_o - y_d$$

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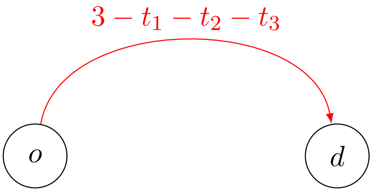
Remark: We keep the primal representation and not the path formulation.

VF as Dynamic programming model



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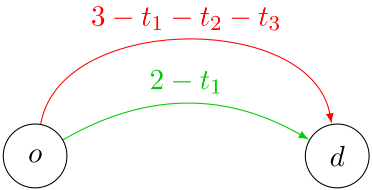

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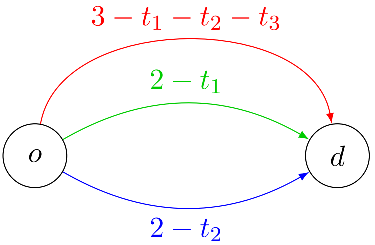

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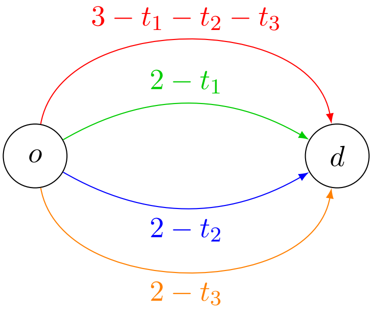

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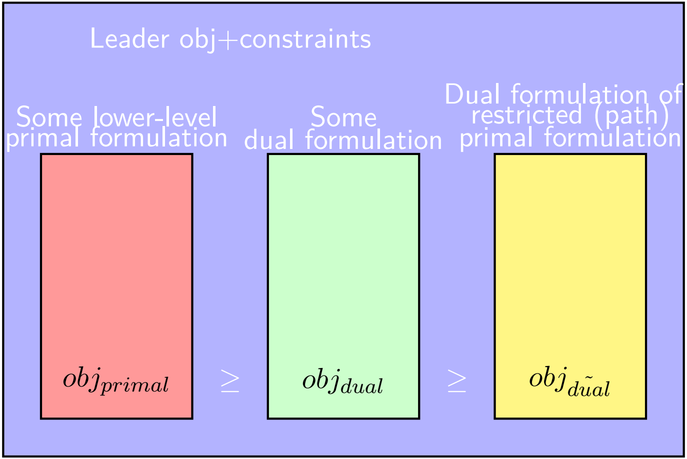
Key idea

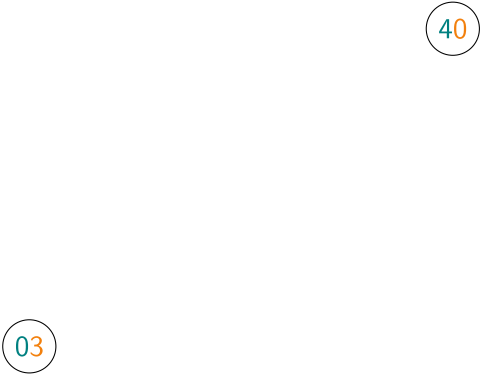


(if primal is a maximization)

Contents

Decision diagram

$$\max\ t^\top x \quad \text{s.t. } w^\top x \le C,\quad x \in \{0,1\}^{\mathcal{I}},$$



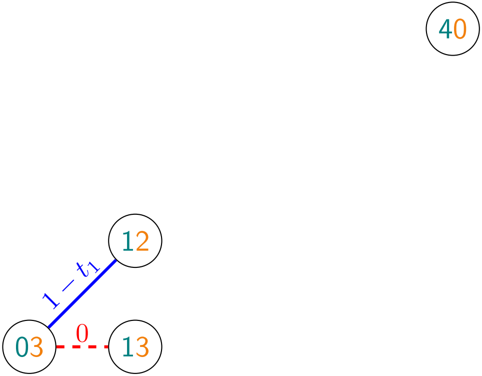
Layer, Remaining capacity

Solid: $x_i = 1$

Dashed: $x_i = 0$

$$(p - t)^\top x = y_{03} - y_{40},$$
$$t \in \mathcal{T}.$$

Decision diagram



$$
\begin{aligned}
\max\ & t^\top x \\
\text{s.t. } & w^\top x \le C, \\
& x \in \{0,1\}^{\mathcal{I}}, \\
& {\color{red} y_{03} - y_{13} \ge 0}, \\
& {\color{blue} y_{03} - y_{12} \ge 1 - t_1}, \\
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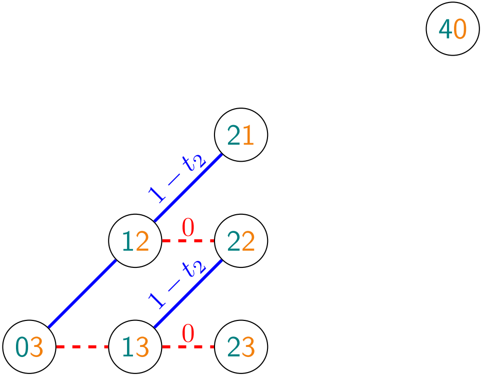

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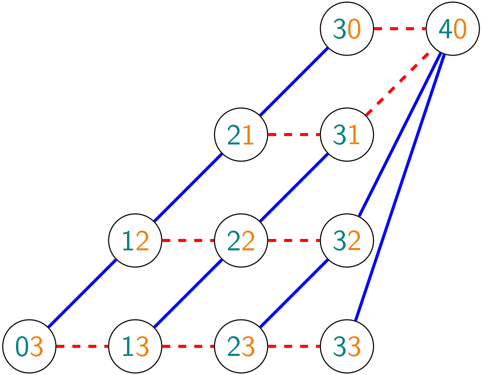

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& \vdots \\
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\end{aligned}
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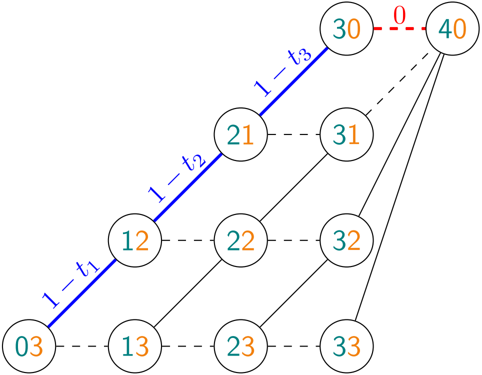
Paths in decision diagram

$\hat{x} \in \mathcal{X} \leftrightarrow$ Paths in decision diagram



$$\hat{x} = (1, 1, 1, 0)$$

$$(p - t)^\top \hat{x} = 3 - t_1 - t_2 - t_3$$

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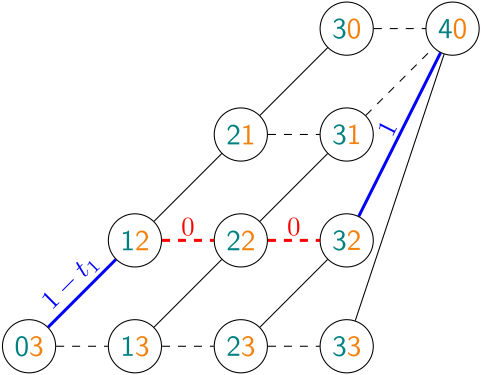

$$\hat{x} = (1, 0, 0, 1)$$

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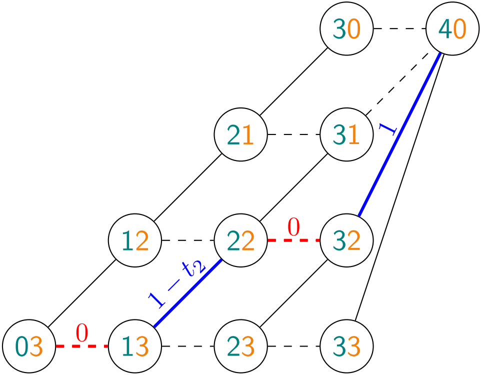

$$\hat{x} = (0, 1, 0, 1)$$

$$(p - t)^\top \hat{x} = 2 - t_2$$

Paths in decision diagram

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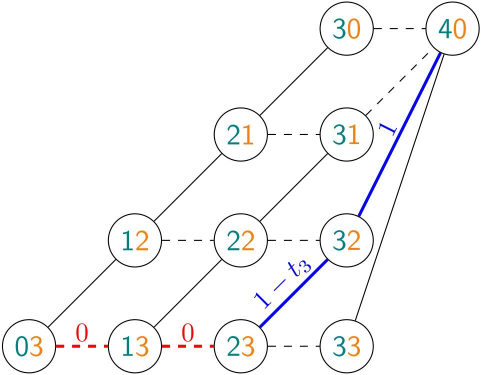

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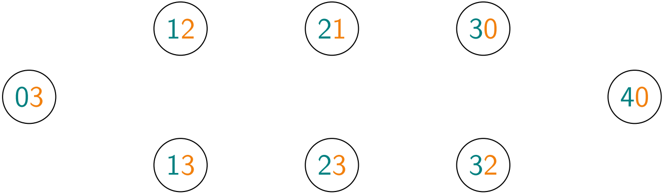
Dynamically-generated constraints in DD

- Start with some nodes
- Given $\hat{x}$, try to fit a path → new constraints
- Worst-case: value function constraint



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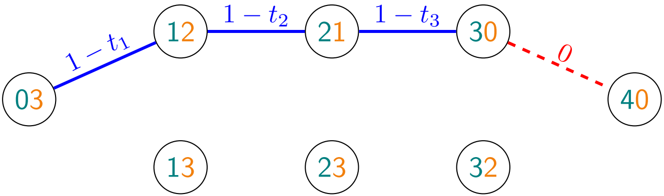

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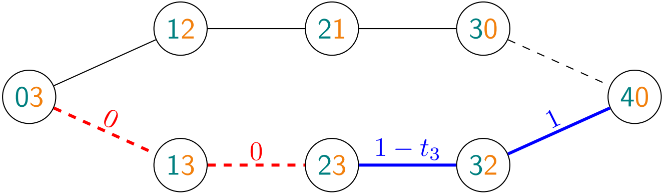

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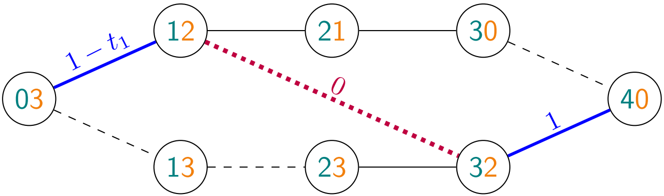

$$\hat{x} = (1,0,0,1)$$
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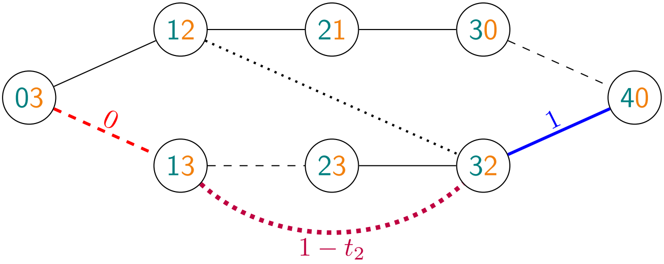


$$\hat{x} = (0, 1, 0, 1)$$

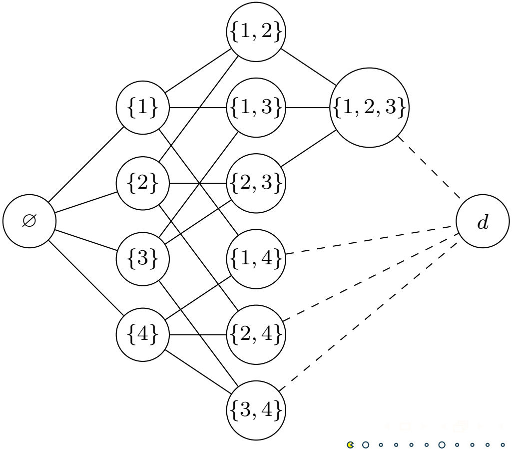
$$(p - t)^\top \hat{x} = 2 - t_2$$

Contents

Selection diagram

At step k:

- Decision diagram: Include or exclude the k-th item
- Selection diagram: Include **exactly 1** item



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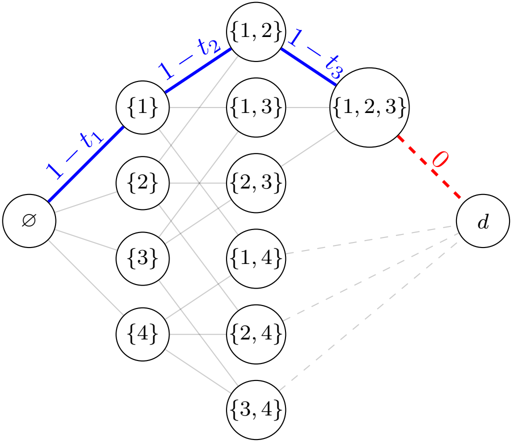

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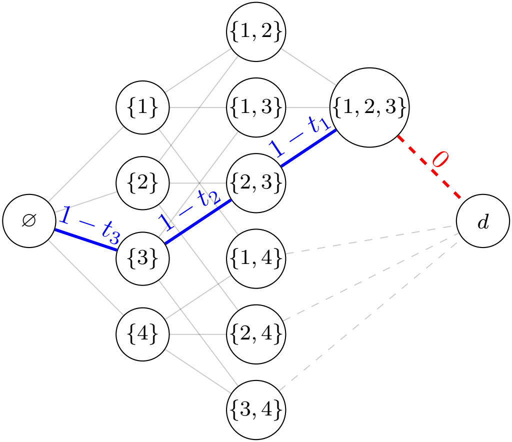

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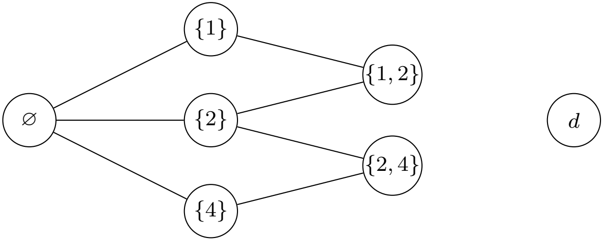
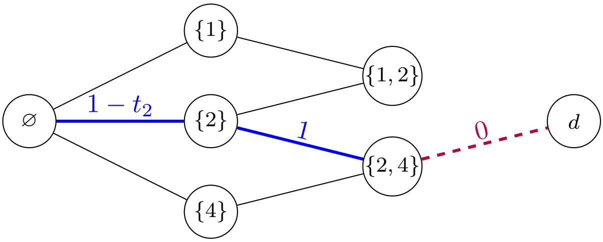
Dynamically-generated constraints in SD



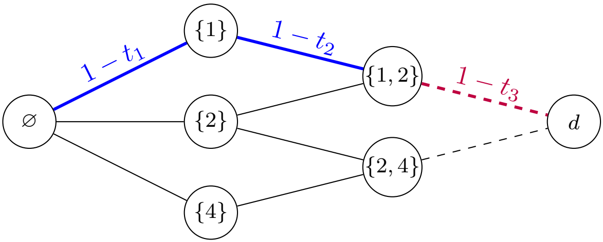
Dynamically-generated constraints in SD



$$\hat{x} = (0, 1, 0, 1)$$

$$(p - t)^\top \hat{x} = 2 - t_2$$

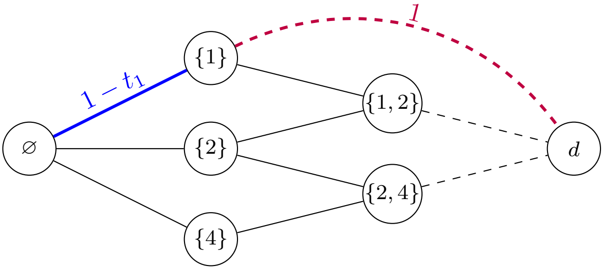
Dynamically-generated constraints in SD



$$\hat{x} = (1,1,1,0)$$

$$(p-t)^\top \hat{x} = 3 - t_1 - t_2 - t_3$$

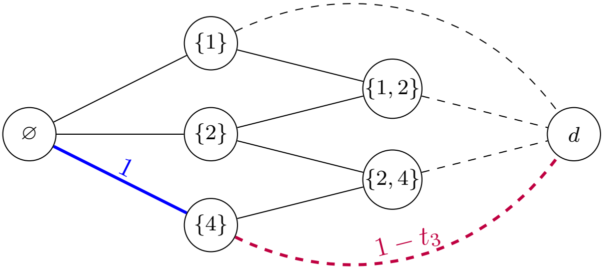
Dynamically-generated constraints in SD



$$\hat{x} = (1, 0, 0, 1)$$

$$(p - t)^\top \hat{x} = 2 - t_1$$

Dynamically-generated constraints in SD



$$\hat{x} = (0, 0, 1, 1)$$

$$(p - t)^\top \hat{x} = 2 - t_3$$

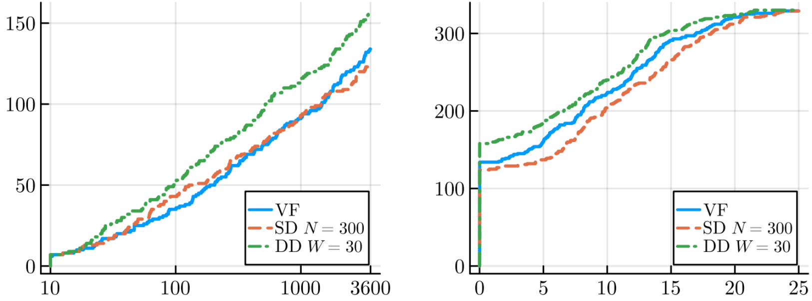
Contents

Experiments

- VF: value function
- SD N: selection diagram with N nodes in layer 3
- DD W: decision diagram with maximum width W and item grouping ($\leq$ 20 layers)
- 1 hour time limit

Experiments: knapsack pricing

- Knapsack pricing (KPP): DD > VF > SD



(a) KPP - Time (s)   (b) KPP - Optimality gap (%)

Cumulative number of instances solved

Maximum stable set pricing problem

$$\mathbf{R}(t) = \arg\max_{x}\{(v-t)^\top x \mid A^\top x \leq \mathbf{1},\ x \in \{0,1\}^{\mathcal{I}}\}$$

where $A \in \{0,1\}^{\mathcal{I}\times\mathcal{E}}$ is the node-edge incidence matrix.

Maximum stable set pricing problem

$$\mathbf{R}(t) = \arg\max_x \{(v-t)^\top x \mid A^\top x \leq \mathbf{1},\ x \in \{0,1\}^{\mathcal{I}}\}$$

where $A \in \{0,1\}^{\mathcal{I}\times\mathcal{E}}$ is the node-edge incidence matrix.

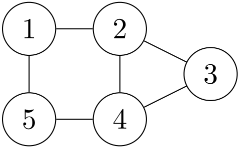
Minimum set cover pricing problem

$$\mathbf{R}(t) = \arg\min_x \{(v+t)^\top x \mid Ax \geq \mathbf{1},\ x \in \{0,1\}^{\mathcal{I}}\}$$

where $A \in \{0,1\}^{\mathcal{E}\times\mathcal{I}}$ is the incidence matrix between $\mathcal{E}$ and $\mathcal{I}$.

MaxSSPP: decision diagram

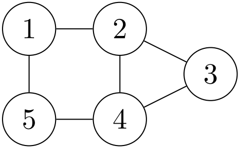
Instance



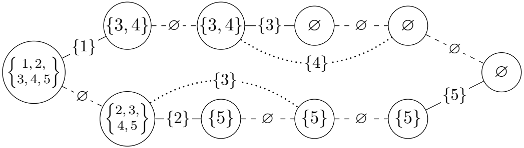
Sampled solutions: $\{1, 3\}, \{2, 5\}$
State: vertices still available

MaxSSPP: decision diagram

Instance



Sampled solutions: $\{1,3\},\{2,5\}$
State: vertices still available



MinSCPP: decision diagram

Instance: $\mathcal{E} = \{a, b, c, d\}$
$\mathcal{I}$ consists of

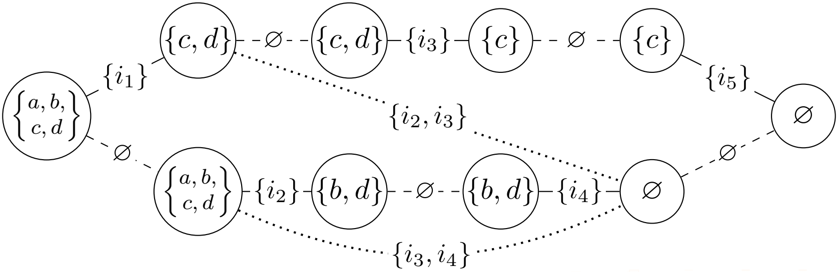
$$i_1 = \{a, b\}, i_2 = \{a, c\}, i_3 = \{a, d\}, i_4 = \{b, c, d\}, i_5 = \{c\}$$

MinSCPP: decision diagram

Instance: $\mathcal{E} = \{a, b, c, d\}$
$\mathcal{I}$ consists of

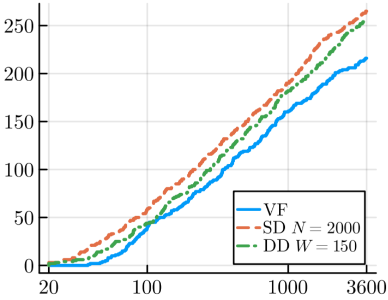
$$i_1 = \{a, b\}, i_2 = \{a, c\}, i_3 = \{a, d\}, i_4 = \{b, c, d\}, i_5 = \{c\}$$

Sampled solutions: $\{i_1, i_3, i_5\}, \{i_2, i_4\}$

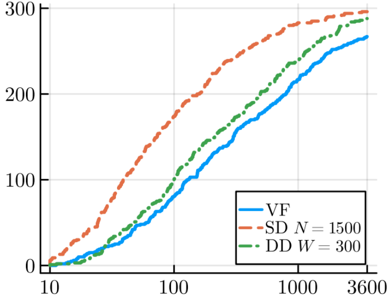


Experiments: Max stable set / Min set cover pricing

- Max stable set pricing (MaxSSPP): SD > DD > VF
- Min set cover pricing (MinSCPP): SD >> DD > VF



(c) MaxSSPP - Time (s)

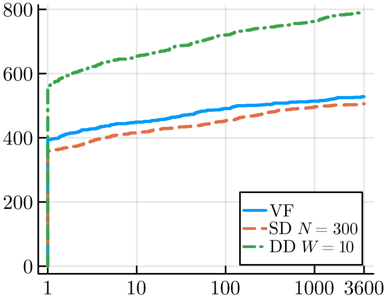


(d) MinSCPP - Time (s)

Cumulative number of instances solved

Experiments: Knapsack interdiction problem

- Knapsack interdiction problem (KIP): DD >> VF > SD

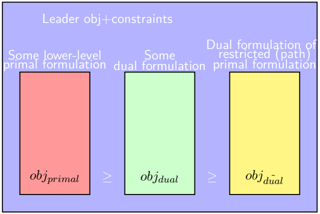


Cumulative number of instances solved

Contents

Summary

- Value function reformulations are oblivious to the follower's primal and dual representations.
- The dual of a follower's primal restriction is a lower bound to the optimal follower's reaction.
- The dual of the restricted lower-level can be dynamically improved to, at the limit, represent the dual of the lower-level.
- Branch-and-cut methodology, essentially improving naive value function representations.



Contents

[Vásquez et al., 2025]: single level reformulations for 0-1 linear bilevel programs.

$$\min_{x,y} \left\{ c_L^\top x + c_F^\top y \;\middle|\; \begin{array}{l} Ax + By \leq a \\ x \in \{0,1\}^{n_L} \\ y \in \mathbf{R}(x) \end{array} \right\}$$

$$\mathbf{R}(x) := \operatorname*{argmin}_{\bar{y}} \left\{ d^\top \bar{y} \;\middle|\; \begin{array}{l} D\bar{y} \leq b - Cx \\ \bar{y} \in \{0,1\}^{n_F} \end{array} \right\}$$

[Vásquez et al., 2025]: Value function reformulation

$$\min_{x,y} \left\{ c_L^\top x + c_F^\top y \;\middle|\; \begin{array}{l} Ax + By \leq a \\ Cx + Dy \leq b \\ (x, y) \in \{0,1\}^{n_L + n_F} \\ d^\top y \leq \phi(x) \end{array} \right\}$$

$$\phi(x) := \min_{\bar{y}} \left\{ d^\top \bar{y} \;\middle|\; \begin{array}{l} D\bar{y} \leq b - Cx \\ \bar{y} \in \{0,1\}^{n_F} \end{array} \right\}$$

[Vásquez et al., 2025]: Value function reformulation

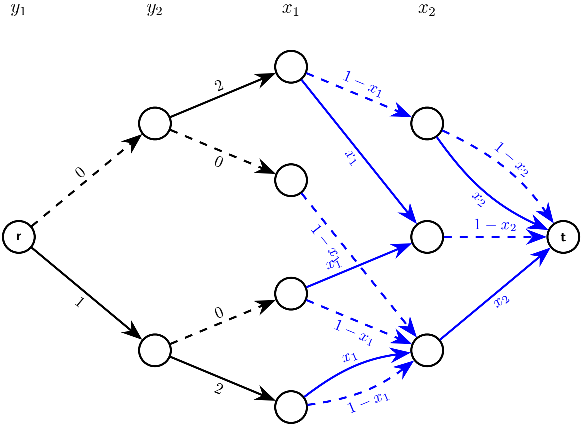
$$\min_{x,y} \left\{ c_L^\top x + c_F^\top y \;\middle|\; \begin{array}{l} Ax + By \leq a \\ Cx + Dy \leq b \\ (x,y) \in \{0,1\}^{n_L+n_F} \\ d^\top y \leq \phi(x) \end{array} \right\}$$

$$\phi(x) := \min_{\bar{y}} \left\{ d^\top \bar{y} \;\middle|\; \begin{array}{l} D\bar{y} \leq b - Cx \\ \bar{y} \in \{0,1\}^{n_F} \end{array} \right\}$$

Transform $\phi(x)$ into a decision diagram.

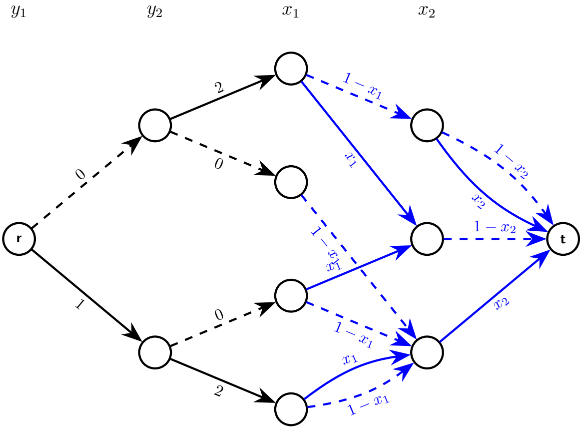
Transform $\phi(x)$ into a decision diagram.

- $n_F + n_L + 1$ layers
- layers for y: from 1 to $n_F + 1$ with value d_i
- layers for x: from $n_F + 2$ to $n_L + 1$ with value 0



[Vásquez et al., 2025]

- Write DD as a capacitated shortest path problem
- Write its dual
- Replace ϕ by the dual



Final thoughts

- Dualize-and-combine is powerful to obtain single-level reformulations or simply to strengthen single-level relaxations.

Final thoughts

- Dualize-and-combine is powerful to obtain single-level reformulations or simply to strengthen single-level relaxations.
- We should focus on "creating more interaction" between primal, dual and upper-level formulations.

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