Mixing Inequalities

Oktay Günlük

H. Milton Stewart School of Industrial and Systems Engineering, Georgia Tech.

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Padova University

Solving Mixed Integer Linear Programs

Consider a generic MIP:

$$\min\{c^T x : Ax \geq b, x \geq 0,\ x_j \text{ integer for some } j \in \mathcal{I}\}$$

- In practice MIPs are solved via enumeration:
  - The branch-and-bound algorithm, Land and Doig (1960)
  - The branch-and-cut scheme proposed by Padberg and Rinaldi (1987)
- Given an optimization problem $z^* = \min\{f(x) : x \in P\}$,

(i) **Partitioning:** Let $P = \cup_{i=1}^{p} P_i$ (division), then

$$z^* = \min_i \{z_i\} \text{ where } z_i = \min\{f(x) : x \in P_i\},$$

(ii) **Lower bounding:** For $i = 1, \ldots, p$, let $P_i \subseteq P_i^R$ (relaxation), then

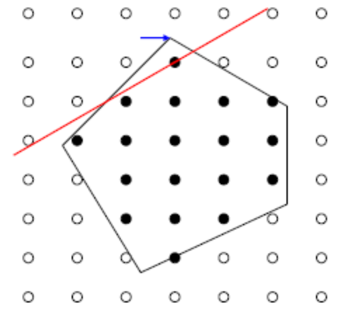
$$z_i \geq z_i^R = \min\{f(x) : x \in P_i^R\}, \quad \text{and} \quad z^* \geq \min_i \{z_i^R\}.$$

(iii) **Upper bounding:** If $\bar{x} \in P_i \subseteq P$ then $f(\bar{x}) \geq z^*$.

[Same framework is used to solve non-convex QPs, for example.]

Strengthening the LP relaxation by cutting planes

- After solving the LP, do not branch right away
- Add valid inequalities to the LP and resolve
- Repeat



Cplex version	Year	Better	Worse	Time	Relative Diff.
11.0	2007	0	0	1.00	91.0%
10.0	2005	201	650	1.91	43.0%
9.0	2003	142	793	2.73	30.4%
8.0	2002	117	856	3.56	28.9%
7.1	2001	63	930	4.59	62.7%
6.5	1999	71	997	7.47	185.1%
6.0	1998	55	1060	21.30	6.0%
5.0	1997	45	1069	22.57	16.5%
4.0	1995	37	1089	26.29	31.7%
3.0	1994	34	1107	34.63	62.3%
2.1	1993	13	1137	56.16	20.9%
1.2	1991	17	1132	67.90	—

* Cplex v. 6.5 uses extensive cutting plane generation.

[Bixby & Achterberg comparison on 1,734 MIP instances]

Main components of a MIP solver

- **Preprocessing** *(before solving the root LP)*
  - *Clean up the model (empty/implied rows, fixed variables, ...)*
  - *Coefficient reduction (ex: p0033, all variables binary)*

$$-230x_{10} - 200x_{16} - 400x_{17} \leq -5 \Longrightarrow x_{10} + x_{16} + x_{17} \geq 1$$

  - . . .
- **Cutting plane generation***: Gomory Mixed Integer cuts, MIR inequalities, cover cuts, flow covers, ...*
- **Primal heuristics***: rounding heuristics, diving heuristics, local search, ...*
- **Branching strategies***: strong branching, pseudo-cost branching, (not most fractional!)*
- **Node selection strategies:** *a combination of best-bound and diving.*

Can MIPs be solved only using cutting planes (without branching)?

A short history of finite cutting plane algorithms:

- *Gomory (1958) developed the first finite cutting plane algorithm for pure IPs using Gomory fractional cut.*
- *Later, (1960) he extended this to MIPs with integer objective using an extended formulation together with Gomory mixed-integer cut (which is a split cut).*
- *Cook/Kannan/Schrijver (1990) gave an example in $\mathcal{Z}^2 \times \mathcal{R}$ which cannot be solved in finite time using split cuts.*
- *Later Dash and Gunluk (2013) generalized this to examples in $\mathcal{Z}^n \times \mathcal{R}$ that cannot be solved in finite time using $(n-1)$-branch split cuts.*
- *For MIPs with bounded feasible region Jörg (2008) gave a finite cutting plane algorithm for MIPs.*
- *For general MIPs Dash et.al. (2014) gave a finite cutting plane algorithm using multi-branch split cuts.*

("All these algorithms are of purely theoretical interest, and are highly impractical".)

A very simple set

Let

$$S = \left\{ y \in Z \, : \, y \; \geq \; b_1, \; y \; \leq \; b_2 \right\}$$

then, the following inequalities:

$$y \geq \lceil b_1 \rceil \quad \textit{and} \quad y \leq \lfloor b_2 \rfloor$$

are valid for S and

$$conv(S) = \left\{ y \in R \, : \, \lfloor b_2 \rfloor \; \geq \; y \; \geq \; \lceil b_1 \rceil \right\}.$$

- *Variable y can be replaced by any integer expression to obtain a valid cut.*
- *This simple idea leads to the Chvatal-Gomory cuts*

Chvátal-Gomory cuts

- Consider a pure integer set:

$$S = \left\{ y \in \mathcal{Z}^n : Ay \le b, \quad y \ge 0 \right\}$$

- Pick any $\lambda \in \mathcal{R}^m$ such that $\lambda \ge 0$. Then

$$\lambda^T A y \le \lambda^T b \qquad \longleftarrow \text{valid inequality for LP relaxation}$$

  is satisfied by all $y \in S$.

- Then, using $y \ge 0$:

$$\underbrace{\sum_{j=1}^{n} \left\lfloor \lambda^T A_j \right\rfloor y_j}_{\text{integral for } y \in S} \le \lambda^T b \qquad \longleftarrow \text{valid for LP relaxation}$$

  and therefore

$$\sum_{j=1}^{n} \left\lfloor \lambda^T A_j \right\rfloor y_j \le \left\lfloor \lambda^T b \right\rfloor \qquad \longleftarrow \text{valid for } S \text{ (integer points)}$$

Note: Remember, $\lfloor -3.4 \rfloor = -4$.

Another simple set

The set

$$S = \left\{ (x, y) \in \mathcal{R} \times \mathcal{Z} \;:\; x + y \geq \beta,\; x \geq 0 \right\}$$

is called the basic mixed-integer rounding set, and the inequality,

$$x + \hat{\beta} y \geq \hat{\beta} \lceil \beta \rceil$$

where $\hat{\beta} = \beta - \lfloor \beta \rfloor$, is called the basic mixed-integer rounding (MIR) inequality.

Example: Let

$$S = \left\{ (x, y) \in \mathcal{R} \times \mathcal{Z} \;:\; x + y \geq 7.3,\; x \geq 0 \right\}$$

then the MIR Inequality is:

$$x + 0.3y \geq 0.3 \cdot 8 = 2.4$$

where $\hat{\beta} = 7.3 - 7 = 0.3$.

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Lemma: *Let* $S = \{(x,y) \in R \times \mathcal{Z} \;:\; x + y \geq \beta,\; x \geq 0\}$ *and*

$$\mathrm{conv}(S) \;=\; \left\{(x,y) \in \mathcal{R} \times \mathcal{R} \;:\; x + y \geq \beta,\; x + \hat{\beta} y \geq \hat{\beta} \lceil \beta \rceil,\; x \geq 0\right\}$$

where $\hat{\beta} = \beta - \lfloor \beta \rfloor \in [0,1)$..

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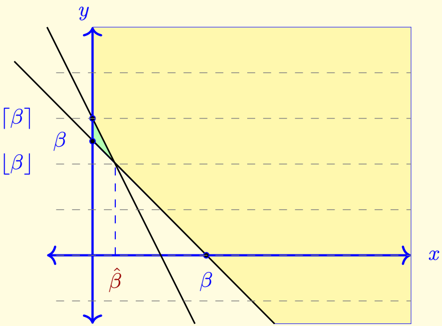

MIR inequalities for single constraint sets

- Let

$$P^1 = \left\{ x \in R^{|C|},\ y \in Z^{|I|}\ :\ \sum_{j \in C} c_j x_j + \sum_{j \in I} a_j y_j \ \geq\ b,\quad x \geq 0,\ y \geq 0 \right\}$$

- Re-write:

$$\sum_{c_j<0} c_j x_j + \sum_{c_j>0} c_j x_j + \sum_{\hat{a}_j<\hat{b}} \hat{a}_j y_j + \sum_{\hat{a}_j \geq \hat{b}} \hat{a}_j y_j + \sum_{j \in I} \lfloor a_j \rfloor y_j \ \geq\ b \ =\ \hat{b} + \lfloor b \rfloor$$

(where $a_j = \hat{a}_j + \lfloor a_j \rfloor$)

- Relax:

$$\underbrace{\sum_{c_j>0} c_j x_j + \sum_{\hat{a}_j<\hat{b}} \hat{a}_j y_j}_{\geq 0} \quad + \quad \underbrace{\sum_{\hat{a}_j \geq \hat{b}} y_j + \sum_{j \in I} \lfloor a_j \rfloor y_j}_{\in Z} \quad \geq\ b$$

- MIR cut:

$$\sum_{c_j>0} c_j x_j + \sum_{\hat{a}_j<\hat{b}} \hat{a}_j y_j + \hat{b} \left(\sum_{\hat{a}_j \geq \hat{b}} y_j + \sum_{j \in I} \lfloor a_j \rfloor y_j \right) \ \geq\ \hat{b} \lceil b \rceil$$

(Applying MIR to the simplex tableau rows gives the Gomory mixed-integer cut)

MIR inequalities for multiple constraint sets

Let

$$P = \Big\{ x \in \mathcal{R}^{n_1},\ y \in \mathcal{Z}^{n_2}\ :\ Cx + Ay\ =\ d,\ \ x \geq 0,\ y \geq 0 \Big\}$$

where $C \in R^{m \times n_1},\ A \in R^{m \times n_2},\ d \in R^m$.

- Obtain a "base" inequality using $\lambda \in R^m$:

$$\lambda C x + \lambda A y\ =\ \lambda d \quad \leftarrow \text{ single row}$$

- Write the corresponding MIR inequality:

$$\sum_{j=1}^{n_1} (\lambda C_j)^+ x_j + \hat{b} \sum_{j=1}^{n_2} \lfloor \lambda A_j \rfloor y_j + \sum_{j=1}^{n_2} \min\{\lambda A_j - \lfloor \lambda A_j \rfloor, \hat{b}\} y_j\ \geq\ \hat{b} \lceil \lambda d \rceil$$

  where $\hat{b} = \lambda d - \lceil \lambda d \rceil$.

- Applied to the simplex tableau gives Gomory mixed-integer cut – Marchand/Wolsey '98

MIR inequalities for multiple constraint sets in $\geq$ form

Let

$$P = \left\{ x \in R^{|C|},\ y \in Z^{|I|}\ :\ Cx + Ay\ \geq\ d,\ \ x,\ y \geq 0 \right\}$$

where $C \in R^{m \times |C|},\ A \in R^{m \times |I|},\ d \in R^m$.

- Obtain a "base" inequality using $\lambda \in R^m,\ \boldsymbol{\lambda \geq 0}$:

$$\lambda Cx + \lambda Ay\ \geq\ \lambda d$$

- Write the corresponding MIR inequality:

$$\sum_{j \in C} (\lambda C_j)^+ x_j + \hat{b} \sum_{j \in I} \lfloor \lambda A_j \rfloor y_j + \sum_{j \in I} \min\{\lambda A_j - \lfloor \lambda A_j \rfloor, \hat{b}\} y_j\ \geq\ \hat{b} \lceil \lambda d \rceil$$

where $\hat{b} = \lambda d - \lceil \lambda d \rceil$.

MIR inequalities for multiple constraint sets in $\geq$ form (take 2)

- *First add (non-negative) slack variables to the defining inequalities:*

$$Cx + Ay - Is \; = \; d$$

- *Obtain a "base" equation using $\lambda \in R^m$ (not necessarily non-negative):*

$$\lambda Cx + \lambda Ay - \lambda Is \; = \; \lambda d$$

- *Write the corresponding MIR inequality:*

$$\sum_{j \in C} (\lambda C_j)^+ x_j + \hat{b} \sum_{j \in I} \lfloor \lambda A_j \rfloor y_j + \sum_{j \in I} \min\{\lambda A_j - \lfloor \lambda A_j \rfloor, \hat{b}\} y_j + \sum_{\lambda_i < 0} |\lambda_i| s_i \; \geq \; \hat{b} \lceil \lambda d \rceil$$

- *Substitute out slacks to obtain*

$$\sum_{j \in C} (\lambda C_j)^+ x_j + \hat{b} \sum_{j \in I} \lfloor \lambda A_j \rfloor y_j + \sum_{j \in I} \min\{\ldots, \hat{b}\} y_j + \sum_{\lambda_i < 0} |\lambda_i| (Cx + Ax - d)_i \; \geq \; \hat{b} \lceil \lambda d \rceil$$

(adding slacks gives a richer family of valid inequalities)

Example

- Consider the set

$$T = \{x \in R,\ y \in Z\ :\ -x - 4y \geq -4,\ -x + 4y \geq 0,\quad x, y \geq 0\}$$

- Any base inequality generated by λ_1, λ_2 has the form

$$(-\lambda_1 - \lambda_2)x + (-4\lambda_1 + 4\lambda_2)y \geq -4\lambda_1$$

- When $\lambda_1, \lambda_2 \geq 0$, x has a negative coefficient and x does not appear in the MIR cut.
- Using multipliers $\lambda = [-1/8, 1/8]$ after slacks added leads to:

$$\begin{array}{rcl} -\frac{1}{8}\Big(-x - 4y - s_1 = -4\Big) & + & \frac{1}{8}\Big(-x + 4y - s_2 = 0\Big) \\ & \Downarrow & (\textit{Base inequality}) \\ y + s_1/8 - s_2/8 & \geq & 1/2 \\ & \Downarrow & (\textit{MIR inequality}) \\ 1/2y + s_1/8 & \geq & 1/2 \ \Rightarrow\ -x/8 \geq 0\ \Rightarrow\ x \leq 0 \end{array}$$

- This inequality defines the only non-trivial facet of T.

Mixing inequalities

Another simple set: The $\{0,1\}$ mixing set

- Now consider the following set where all the constraints share the same continuous variable:

$$M^{01} = \Big\{(x,z) \in \mathcal{R}_+ \times \{0,1\}^n : \quad \begin{aligned} x + Bz_1 &\geq b_1 \\ x + Bz_2 &\geq b_2 \\ &\dots \\ x + Bz_n &\geq b_n \end{aligned} \Big\}$$

- After scaling the x variables and all b_i by B:

$$M^{01} = \Big\{(x,z) \in \mathcal{R}_+ \times \{0,1\}^n : \quad x + z_i \geq b_i, \quad i = 1,\dots,n \Big\}$$

- As $x \geq \max\{b_i\} - 1$, we can shift the x variable, and assume that $0 < b_i \leq 1$ for all i
- And after reordering the z variables, we can also assume that

$$0 < b_1 \leq b_2 \leq \cdots \leq b_n \leq 1$$

- After strengthening the inequalities using MIR

$$M^{01} = \Big\{(x,z) \in \mathcal{R}_+ \times \{0,1\}^n : \quad x + b_i z_i \geq b_i, \quad i = 1,\dots,n \Big\} \quad \leftarrow \quad x \geq b_i(1 - z_i)$$

Mixing inequalities

Let

$$M^{01} = \left\{ (x, z) \in \mathcal{R}_+ \times \{0,1\}^n : x + b_i z_i \geq b_i, \quad i \in N \right\} \qquad \leftarrow \qquad [b_i \leq b_{i+1}]$$

The n*-term mixing inequality is valid for* M^{01}

$$x + \sum_{i=1}^{n} (b_i - b_{i-1}) z_i \geq b_n,$$

where $b_0 := 0$ *.*

Proof.

- *Let* $(\bar{x}, \bar{z}) \in M^{01}$ *. If* $\bar{z} = \mathbf{1}$ *, the inequality is clearly valid as* $\bar{x}$ *must be* ≥ 0 *.*
- *Otherwise let* $t = \max\{i : \bar{z}_i = 0\}$ *. Note that since* $(\bar{x}, \bar{z}) \in M^{01}$*, we have* $\bar{x} \geq b_t$*.*
- *The left-hand side of the inequality for* $(\bar{x}, \bar{z})$ *becomes*

$$\bar{x} + \sum_{i=1}^{n} (b_i - b_{i-1}) \bar{z}_i \quad \geq \quad b_t + \sum_{i=t+1}^{n} (b_i - b_{i-1}) \; = \; b_t + (b_n - b_t) \; = \; b_n.$$

- *Hence the inequality is valid.* □

$\{0,1\}$ *Mixing Set* [1][2]

Let $S \subseteq N = \{1, \ldots, n\}$ *and note that*

$$M^{01} = \left\{ (x,z) \in \mathcal{R}_+ \times \{0,1\}^n : x + b_i z_i \geq b_i, \quad i \in N \right\} \subseteq \left\{ (x,z) \in \mathcal{R}_+ \times \{0,1\}^n : x + b_i z_i \geq b_i, \quad i \in S \right\}$$

where $0 < b_1 \leq \cdots \leq b_n \leq 1$.

Theorem: 0-1 Mixing Convex Hull

For any $I = \{i_{[1]}, \ldots, i_{[k]}\} \subseteq N$ *, the* ***mixing inequalities***

$$x + \sum_{j=1}^{k} (b_{i_{[j]}} - b_{i_{[j-1]}}) z_{i_{[j]}} \geq b_{i_{[k]}},$$

where $b_{i_{[0]}} = 0$ *and* $i_{[1]} < i_{[2]} < \cdots < i_{[k]}$*, are valid for* M^{01}*. Moreover,*

$$\operatorname{conv}(M^{01}) = \left\{ (x,z) \in \mathbb{R}_+ \times [0,1]^n : x + \sum_{j=1}^{k} (b_{i_{[j]}} - b_{i_{[j-1]}}) z_{i_{[j]}} \geq b_{i_{[k]}}, \ \forall I = \{i_{[1]}, \ldots, i_{[k]}\} \subseteq N \right\}$$

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[1] O. Günlük and Y. Pochet. Mixing mixed-integer inequalities. *Math. Program.*, 90:429–457, 2001.

[2] A. Atamtürk, G. L. Nemhauser, and M. W. P. Savelsbergh. The mixed vertex packing problem. *Math. Program.*, 89(1):35–53, 2000.

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where $0 < b_1 \le \cdots \le b_n \le 1$.

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$\{0,1\}$ *Mixing Example*

For $I = \{i_{[1]}, \dots, i_{[k]}\} \subseteq N$ *s.t.* $i_{[1]} < \cdots < i_{[k]}$ *,*

Mixing inequalities $$x + \sum_{j=1}^{k} (b_{i_{[j]}} - b_{i_{[j-1]}}) z_{i_{[j]}} \geq b_{i_{[k]}}$$

Example: *Let* $n = 8$ *and* $b = (0.1, 0.15, 0.25, 0.3, 0.4, 0.6, 0.75, 0.95)$*.*

Then, M^{01} *is defined by* $(x, z) \in \mathbb{R}_+ \times \{0,1\}^8$ *such that*

$$x + z_1 \geq 0.1, \quad x + z_2 \geq 0.15, \quad x + z_3 \geq 0.25, \quad x + z_4 \geq 0.3,$$

$$x + z_5 \geq 0.4, \quad x + z_6 \geq 0.6, \quad x + z_7 \geq 0.75, \quad x + z_8 \geq 0.95.$$

The mixing inequality for $I = \{1, 3, 7\}$ *is*

$$x + (b_1 - b_0) z_1 + (b_3 - b_1) z_3 + (b_7 - b_3) z_7 \geq b_7$$

$$\iff \; x + (0.1 - 0) z_1 + (0.25 - 0.1) z_3 + (0.75 - 0.25) z_7 \geq 0.75$$

$$\iff \; x + 0.1 z_1 + 0.15 z_3 + 0.5 z_7 \geq 0.75.$$

$\{0,1\}$ *Mixing Inequalities: Validity*

For $I = \{i_{[1]}, \dots, i_{[k]}\} \subseteq N$ s.t. $i_1 < \cdots < i_k$, the mixing inequality is:

$$x + \underbrace{\sum_{j=1}^{k} (b_{i_{[j]}} - b_{i_{[j-1]}})}_{\text{add up to the RHS}} z_{i_{[j]}} \geq b_{i_{[k]}} \quad \Longleftrightarrow \quad x \geq \sum_{j=1}^{k} (b_{i_{[j]}} - b_{i_{[j-1]}})(1 - z_{i_{[j]}})$$

Validity proof:

- *Consider an arbitrary point $(\bar{x}, \bar{z}) \in M^{01}$. If $\bar{z} = \mathbf{1}$ then the inequality is valid.*
- *Now assume $\bar{z} \neq \mathbf{1}$ and let $i_{[t]} \in I$ be the largest index such that $\bar{z}_{i_{[t]}} = 0$. Clearly $\bar{x} \geq b_{i_{[t]}}$.*
- *The mixing inequality for $(\bar{x}, \bar{z})$ becomes*

$$\bar{x} \geq \sum_{j=1}^{k} (b_{i_{[j]}} - b_{i_{[j-1]}})(1 - \bar{z}_{i_{[j]}}) = \underbrace{\underbrace{\sum_{j=1}^{t} (b_{i_{[j]}} - b_{i_{[j-1]}})}_{\text{add up to } b_{i_{[t]}}}(1 - \bar{z}_{i_{[j]}})}_{\text{at most } b_{i_{[t]}}} \qquad \square$$

* *After complementing z variables: $\{0,1\}$ mixing ineq. $\equiv$ Star ineq. by Atamtürk–Nemhauser–Savelsbergh.*

$\{0,1\}$ *Mixing and Submodularity*

Recall

$$M^{01} \;=\; \Big\{(x,z) \in \mathcal{R}_+ \times \{0,1\}^n : x \geq b_i(1-z_i), \quad i \in N\Big\}.$$

For a binary vector $z \in \{0,1\}^n$ *, let*

$$S(z) \;=\; \{i \in N :\; z_i = 0\}$$

- *Define the set function* $f : 2^N \to \mathcal{R}$

$$f(S) = \max_{i \in S} b_i, \qquad f(\emptyset) = 0.$$

- *The function* $f : 2^N \to \mathcal{R}$ *is nondecreasing and submodular.*

$$\underbrace{f(A)}_{a} + \underbrace{f(B)}_{b} \;\geq\; \underbrace{f(A \cup B)}_{=\max\{a,b\}} + \underbrace{f(A \cap B)}_{\leq \min\{a,b\}}, \qquad \forall\; A, B \subseteq N.$$

- *Notice that* M^{01} *can also be written as*

$$M^{01} = \Big\{(x,z) \in \mathcal{R}_+ \times \{0,1\}^n : x \geq f(S(z))\Big\}.$$

$\{0,1\}$ *Mixing and extended polymatroid inequalities* [3]

Theorem (Edmonds)

Let $f : 2^N \to \mathcal{R}$ *be a nondecreasing submodular set function and*

$$Q \;=\; \Big\{(x,z) \in \mathcal{R}_+ \times \{0,1\}^n : x \geq f(S(z))\Big\}.$$

Then $\mathrm{conv}(Q)$ *is described by the extended polymatroid inequalities:*

$$x \geq \sum_{j=1}^{n} \Big(f(I_{[j]}) - f(I_{[j-1]})\Big)(1 - z_{i_{[j]}}), \qquad \textit{for every permutation} \quad i_{[1]}, \ldots, i_{[n]} \ \textit{of} \ \ N$$

where $I_{[j]} = \{i_{[1]}, \ldots, i_{[j]}\}, \quad I_{[0]} = \emptyset.$

- $\{0,1\}$ *mixing inequalities are precisely the extended polymatroid inequalities for* $f(S) = \max_{i \in S} b_i$ *:*

$$x + \sum_{j=1}^{k} (b_{i_{[j]}} - b_{i_{[j-1]}}) z_{i_{[j]}} \geq b_{i_{[k]}}, \quad \forall I = \{i_{[1]}, \ldots, i_{[k]}\} \subseteq N.$$

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[3] F. Kılınç-Karzan, S. Küçükyavuz, and D. Lee, *Joint chance-constrained programs and the intersection of mixing sets through a submodularity lens*, Math. Program. 195, 283–326 (2022)

$\{0,1\}$ *Mixing and extended polymatroid inequalities* [3]

Theorem (Edmonds)

Let $f : 2^N \to \mathcal{R}$ *be a nondecreasing submodular set function and*

$$Q \;=\; \Big\{(x,z) \in \mathcal{R}_+ \times \{0,1\}^n : x \geq f(S(z))\Big\}.$$

Then $\mathrm{conv}(Q)$ *is described by the extended polymatroid inequalities:*

$$x \geq \sum_{j=1}^{n} \Big(f(I_{[j]}) - f(I_{[j-1]})\Big)(1 - z_{i_{[j]}}), \qquad \text{for every permutation} \quad i_{[1]}, \ldots, i_{[n]} \text{ of } N$$

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Separating violated of $\{0,1\}$ mixing inequalities

- *Given a fractional point $(\bar{x}, \bar{z})$,*
- *The most violated mixing inequality*

$$x + \sum_{j=1}^{k} (b_{i_{[j]}} - b_{i_{[j-1]}}) z_{i_{[j]}} \geq b_{i_{[k]}},$$

is obtained by sorting $\bar{z}_i$ in nonincreasing order and checking the associated subsets.

Example:

Let

$$b = (.1, .3, .5, .8), \qquad (\bar{x}, \bar{z}) = (0.5, (0.2, 0.9, 0.4, 0.1)).$$

Sorting in nonincreasing order gives: $2,\ 3,\ 1,\ 4.$

Thus it suffices to check the mixing inequalities corresponding to the subsets with increasing indices

$$\{2\}, \qquad \{2,3\}, \qquad \{2,3,\cancel{1}\}, \qquad \{2,3,\cancel{1},4\}.$$

- *This is equivalent to separation of polymatroid inequalities via Edmonds' greedy algorithm.*

Intersection of $\{0,1\}$ mixing sets

For each $q = 1, \ldots, m$, define the $\{0,1\}$ mixing set

$$M^q = \Big\{ (x_q, z) \in \mathcal{R}_+ \times \{0,1\}^n : x_q + b_i^q z_i \geq b_i^q, \quad i \in N \Big\}.$$

The intersection of these sets is

$$M^\cap = \Big\{ (x, z) \in \mathcal{R}_+^m \times \{0,1\}^n : x_q + b_i^q z_i \geq b_i^q, \quad i \in N,\ q = 1, \ldots, m \Big\}.$$

where $x = (x_1, \ldots, x_m)$

- *The sets $M^1, \ldots, M^m$ share the same binary vector z.*
- *Each individual set has a complete convex-hull description via mixing inequalities.*

Theorem (Atamtürk–Nemhauser–Savelsbergh, Edmonds+KKD)

$$\operatorname{conv}(M^\cap) = \bigcap_{q=1}^{m} \operatorname{conv}(M^q). \qquad \leftarrow \quad \textit{in general: } \operatorname{conv}(A \cap B) \subseteq \operatorname{conv}(A) \cap \operatorname{conv}(B)$$

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General mixing

The Mixing Set

We now consider a mixed-integer set with unrestricted general integer variables and arbitrary RHS

$$M = \big\{(x,y) \in \mathbb{R}_+ \times \mathbb{Z}^n : \; x + By_i \geq b_i, \; i \in N = \{1,\ldots,n\}\big\},$$

where $B \in \mathbb{R}_+$, $b_i \in \mathbb{R}$ *for* $i \in N$.

- *As before, we can assume that* $B = 1$ *(after scaling).*
- *Let* $\tau_i = \lceil b_i \rceil$ *and* $\gamma_i = b_i - (\tau_i - 1) \in (0,1]$ *, then for any* $i \in N$*, the simple MIR inequalities*

  $$x \geq \gamma_i(\tau_i - y_i)$$

  are valid for M.
- *Note that for each* $S_i = \{(x,y_i) \in \mathcal{R}_+ \times \mathcal{Z} : x + y_i \geq b_i\}$

$$\operatorname{conv}(S_i) = \big\{(x,y_i) \in \mathcal{R}_+ \times \mathcal{R} : x + y_i \geq b_i, \quad \underbrace{x \geq \gamma_i(\tau_i - y_i)}_{\text{MIR inequality}}\big\}$$

- *However*

$$\operatorname{conv}(M) \; = \; \operatorname{conv}\Big(\bigcap_{i\in N} S_i\Big) \; \neq \; \bigcap_{i\in N} \operatorname{conv}(S_i)$$

Mixing Inequalities

The mixing set

$$M = \left\{(x, y) \in \mathbb{R}_+ \times \mathbb{Z}^n : \; x + y_i \geq b_i, \; i \in N\right\},$$

Simple MIR inequalities
$$x \geq \gamma_i(\tau_i - y_i), \qquad i \in N$$

where $\tau_i = \lceil b_i \rceil \in \mathbb{Z}$ and $\gamma_i = b_i - (\tau_i - 1) \in (0, 1]$. WLOG also assume $\gamma_i \geq \gamma_{i-1}$ for all $i \in N$ and $\gamma_0 = 0$.

The following mixing inequality is valid for M

$$x \;\geq \sum_{i=1}^{n} (\gamma_i - \gamma_{i-1})(\underbrace{\tau_i - y_i}_{\Delta_i}).$$

Example: $M = \left\{(x, y) \in \mathcal{R}_+ \times \mathcal{Z}^3 : x + y_1 \geq -1.6, \quad x + y_2 \geq 0.5, \quad x + y_3 \geq 2.7\right\}.$

Then $\tau = (-1, 1, 3)$, and $\gamma = (0.4, 0.5, 0.7)$.

Mixing inequality is:

$$\begin{aligned} x \quad &\geq \quad 0.4(-1 - y_1) + 0.1(1 - y_2) + 0.2(3 - y_3) \\ &= \quad 0.3 - 0.4y_1 - 0.1y_2 - 0.2y_3. \end{aligned}$$

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The following mixing inequality is valid for M

$$x \ \geq \sum_{i=1}^{n} (\gamma_i - \gamma_{i-1})(\underbrace{\tau_i - y_i}_{\Delta_i}).$$

Validity proof:

- Let $(\bar{x}, \bar{y}) \in M$ and define $\Delta_i = \tau_i - \bar{y}_i, \quad i \in N$. If all $\Delta_i \leq 0$, inequality is valid.
- Let $t \in N$ be such that $\Delta_t \geq \Delta_i$ for all $i < t$ and $\Delta_t > \Delta_i$ for all $i > t$.
- Then

$$RHS \ \rightarrow \ \sum_{i=1}^{n} (\gamma_i - \gamma_{i-1})(\Delta_i) \leq \gamma_t \Delta_t + (1 - \gamma_t)(\Delta_t - 1) = \Delta_t - 1 + \gamma_t$$

- Since $(\bar{x}, \bar{y}) \in M$, we have $\bar{x} + \bar{y}_t \geq b_t = \tau_t - 1 + \gamma_t$, and hence $\bar{x} \geq \Delta_t - 1 + \gamma_t$. □

Convex Hull of the Mixing Set

Recall that: $M = \Big\{(x,y) \in \mathcal{R}_+ \times \mathcal{Z}^n : x + y_i \ge b_i = \tau_i - 1 + \gamma_i, \quad i \in N\Big\}$, , *where* $\gamma_i \in (0,1]$.

Theorem (Günlük–Pochet)

For every $I = \{i_{[1]}, \dots, i_{[k]}\} \subseteq N, \qquad i_{[1]} < \cdots < i_{[k]}$, *the following inequalities are valid for* M*:*

$$x \ge \sum_{j=1}^{k} (\gamma_{i_{[j]}} - \gamma_{i_{[j-1]}})(\tau_{i_{[j]}} - y_{i_{[j]}}), \tag{Type I}$$

$$x \ge \sum_{j=1}^{k} (\gamma_{i_{[j]}} - \gamma_{i_{[j-1]}})(\tau_{i_{[j]}} - y_{i_{[j]}}) + \underbrace{(1 - \gamma_{i_{[k]}})(\tau_{i_{[1]}} - y_{i_{[1]}} - 1)}_{>0 \text{ if } \Delta_{[1]} \ge 2}, \tag{Type II}$$

where $\gamma_{i_{[0]}} = 0$. *Moreover, the Type I and Type II mixing inequalities together with*

$$x + y_i \ge b_i, \qquad i \in N, \qquad x \ge 0, \tag{Formulation}$$

describe $\operatorname{conv}(M)$.

Proof sketch. *Consider an arbitrary (nontrivial) valid inequality* $x + \sum_{i \in N} \alpha_i(\tau_i - y_i) \ge \beta$ *for* M.

- *The coefficients* α *and* β *must satisfy certain properties for validity*
- *Analyzing the properties of these coefficients establishes that they are implied by Type I and II ineq.* ☐

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$$x + y_i \geq b_i, \qquad i \in N, \qquad x \geq 0, \tag{Formulation}$$

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Mixing Inequalities: Example

Consider

$$M = \left\{(x,y) \in \mathcal{R}_+ \times \mathcal{Z}^3 : x + y_1 \geq -1.6, \quad x + y_2 \geq 0.5, \quad x + y_3 \geq 2.7\right\}.$$

Then

$$\tau = (-1, 1, 3), \qquad \gamma = (0.4, 0.5, 0.7).$$

Type I mixing inequality for $I = \{1,2,3\}$, *is*

$$\begin{aligned} x &\geq 0.4(-1 - y_1) + 0.1(1 - y_2) + 0.2(3 - y_3) \\ &= 0.3 - 0.4y_1 - 0.1y_2 - 0.2y_3. \end{aligned}$$

Type II mixing inequality for $I = \{1,2,3\}$, *is*

$$\begin{aligned} x &\geq 0.4(-1 - y_1) + 0.1(1 - y_2) + 0.2(3 - y_3) + \underbrace{0.3(-1 - y_1 - 1)}_{>0 \text{ if } y_1 < -2} \\ &= -0.3 - 0.7y_1 - 0.1y_2 - 0.2y_3. \end{aligned}$$

[Note: For $(\bar{x}, \bar{y}) = (1.4, (-3, 0, 2))$, *Type II is tight but Type I is not.]*

Example: mixing general constraints

- Given valid inequalities where $x \in \mathcal{R}_+^2$ and $y \in \mathcal{Z}_+^3$:

$$\begin{array}{rcrcl} 5.6x_1 + 7.4x_2 & + & 2.4y_1 - 2y_2 & \geq & 11.5 \\ 8.3x_1 + 4.1x_2 & + & 3.3y_1 + y_2 + y_3 & \geq & 18.7 \end{array}$$

- Relax to a common continuous part:

$$\begin{array}{rcrcl} 8.3x_1 + 7.4x_2 + 0.4y_1 & + & 2y_1 - 2y_2 & \geq & 11.5 \\ \underbrace{8.3x_1 + 7.4x_2 + 0.4y_1}_{\text{common nonnegative term}} & + & \underbrace{3y_1 + y_2 + y_3}_{\text{integral term}} & \geq & 18.7 \end{array}$$

- MIR inequalities for the relaxed inequalities:

$$\begin{array}{rcl} 8.3x_1 + 7.4x_2 + 0.4y_1 & \geq & 0.5(12 - 2y_1 + 2y_2) \\ 8.3x_1 + 7.4x_2 + 0.4y_1 & \geq & 0.7(19 - 3y_1 - y_2 - y_3) \end{array}$$

- Type I mixing inequality:

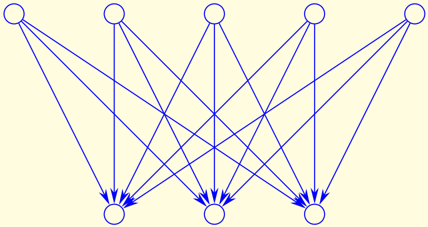
$$8.3x_1 + 7.4x_2 + 0.4y_1 \geq 0.5(12 - 2y_1 + 2y_2) + 0.2(19 - 3y_1 - y_2 - y_3).$$

An Example: Capacitated Facility Location Problem

Given : *A set of potential facilities* F *with capacity* $B \in R_+$, *and customers* D *with demands* $d_k \geq 0$

Decide : *Which facilities to open and find a feasible assignment of demands to open facilities.*

Potential facilities



Customers

A MILP formulation :

$$\min \quad \sum_{i \in F} \sum_{k \in D} c_{ik} x_{ik} + \sum_{i \in F} f_i z_i$$

s.t.

$$\sum_{i \in F} x_{ik} = d_k \qquad \text{for all } k \in D$$

$$\sum_{k \in D} x_{ik} \leq B z_i \qquad \text{for all } i \in F$$

$$x_{ik} \leq d_k \, z_i \qquad \text{for } i \in F \ , \ k \in D$$

$$z \in \{0,1\}_+^{|F|}, x \in R_+^{|F| \times |D|}$$

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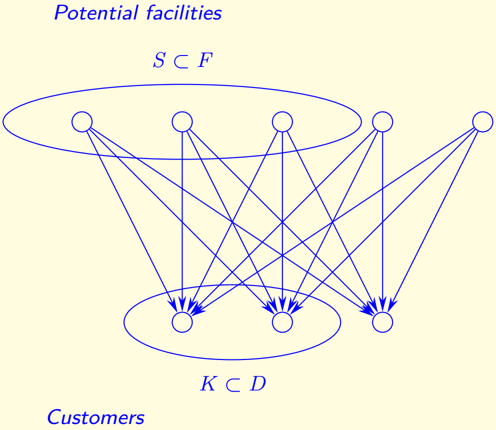

For any subset $S \subseteq F$ and subset $K \subseteq D$

$$\sum_{i \in S} \sum_{k \in K} x_{ik} + B \sum_{i \in F \setminus S} z_i \quad \geq \quad \sum_{k \in K} d_k$$

are valid inequalities (to satisfy the demand of K).

- Choose $S^i \subseteq F$ and $K^i \subseteq D$ for $i = 1, \dots, n$ and mix

  [after "homogenizing" the continuous term]
- If S^i and K^i are nested $\Longrightarrow$ [Aardal, Pochet and Wolsey]
- If they are not $\Longrightarrow$ new mixing inequalities.

How many rounds of MIR are needed?

Recall that Type I and Type II mixing inequalities describe $\mathrm{conv}(M)$.

- Starting from the formulation

$$x + y_i \geq b_i, \qquad i \in N, \qquad x \geq 0,$$

  one may repeatedly apply MIR procedure (equivalently, apply split cuts).

- The MIR closure of M is obtained by adding all possible MIR cuts using the inequality description of M . [this gives is a smaller polyhedron]

- The second MIR closure is the MIR closure of the first MIR closure. And so on...

- The MIR rank (or split rank) of a valid inequality is the minimum number of rounds of MIR closures needed to derive it.

- Since the mixing inequalities describe $\mathrm{conv}(M)$, a natural question is:

How large can the MIR rank of a mixing inequality be?

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MIR Rank of Mixing Inequalities

Dash–Günlük (2008)

Consider the Type I mixing inequality

$$x \geq \sum_{j=1}^{k} (\gamma_{i_{[j]}} - \gamma_{i_{[j-1]}})(\tau_{i_{[j]}} - y_{i_{[j]}})$$

with k *terms. The MIR (split) rank of this inequality is at most* k *.*

The MIR rank of Type II mixing inequality is at most $k-1$ *. Moreover, these bounds are tight for* $k=2$ *.*

Dey (2010)

There exist families of Type I mixing inequalities with k *terms whose MIR (split) rank is at least* $\lceil \log_2(k+1) \rceil$ *.*

- *Therefore the MIR rank of mixing inequalities grows with the number of terms.*
- *The bounds are not known to be tight for* $k \geq 3$*.*

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extensions

Extensions: divisible capacities

Mixing Set with Divisible Capacities *(de Farias and Zhao, 2005) (Conforti, Di Summa and Wolsey, 2005)*

$$M^{DIV} = \left\{(x, y) \in \mathcal{R}_+ \times \mathcal{Z}^n_+ : x + C_t y_t \geq b_t, \quad t \in N\right\}$$

with $C_1 \mid C_2 \mid \cdots \mid C_m$. *(divisible capacities).*

Different divisible capacity levels for different constraint.

Main results:

1. *Characterization of all extreme points and rays.*
2. *A compact extended formulation.*

Divisible Mixing Set *(Constantino, Miller, and Van Vyve , 2010)*

$$M^{DIV2} = \left\{(x, y) \in \mathcal{R}_+ \times \mathcal{Z}^n_+ : x + C_t y_t \geq b_t, \quad t \in N\right\}$$

where $C_t \in \{A, B\}$ *with* $A \mid B$. *(2 divisible capacities).*

Main result: Polyhedral description of $\mathrm{conv}(M^{DIV})$ *when* $m = 2$.

Extensions: continuous variables

Continuous Mixing Set (Van Vyve, 2005)

$$M^C = \left\{ (x, r, y) \in \mathcal{R}_+ \times \mathcal{R}_+^n \times \mathcal{Z}_+^n : x + r_t + y_t \geq b_t, \quad t \in N \right\}.$$

Now each constraint has an additional continuous variable r_t as well as an integer variable y_t.

Main results:

1. Convex hull description using "cycle" inequalities
2. Characterization of all extreme points and rays.
3. A compact extended formulation.

Mixing Set with Flows (Conforti, Di Summa and Wolsey, 2007)

$$M^F = \left\{ (x, r, y) \in \mathcal{R}_+ \times \mathcal{R}_+^n \times \mathcal{Z}_+^n : x + r_t \geq b_t, \ y_t \geq r_t, \quad t \in N \right\}.$$

Now each constraint has an additional continuous variable r_t and the integer variable y_t upper bounds r_t

Main results: Polyhedral description of $\mathrm{conv}(M^F)$, characterization of all extreme points and extreme rays, a polynomial-time optimization algorithm.

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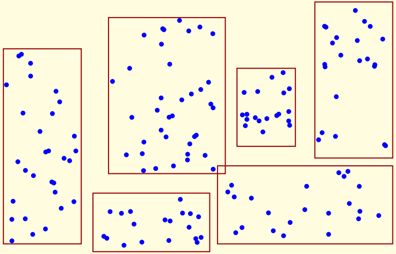
- . . .

Boxing Set

Motivation: Clustering

Consider a (fair) clustering problem:

- *Given a collection of points in the unit box* $[0,1]^n$
- *Partition these points using hyper-rectangles (that might overlap)*
- *Assign each point to a rectangle (a.k.a. box) that contains it*
- *You pay a price for the box depending on its dimensions*
- *There can be other constraints on the collection of points that are assigned to a box (fairness)*



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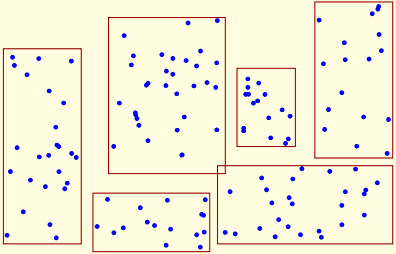
One possible way to solve this is via a covering formulation

- *There is a variable for each possible rectangle*
- *Use column generation to construct the rectangles*
- *The pricing problem becomes finding a good rectangle.*
  (and picking some of the points in it)



Boxing Problem

- *Given a set P of points in $[0,1]^n$ with rewards associated with each one of them*
- *Pick a subset of points to maximize reward*
- *You pay a price for the box (rectangle) depending on its dimensions*

  (there might be other constraints depending on the application)

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Boxing Set

The points are in the unit box: $p^1, p^2, \ldots, p^N \in [0,1]^n$

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- *When $z = \mathbf{0}$ we can have $u_d = 0$ and $l_d = 1$ $\Longrightarrow$ $w_d = -1$ $\longrightarrow$ we need to impose $w_d = u_d - l_d \geq 0$*

The issue

The formulation for lower bounds uses big-Ms:

$$L_d = \left\{(l_d, z) \in \mathbb{R} \times \{0,1\}^n : 1 - l_d \geq (1 - p_d^i) z_i \;\; \forall i\right\} \;\leftarrow\; l_d \leq p_d^i z_i + (1 - z_i)$$

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Let $n = 1$ *and look at the feasible solutions in the* $w = (u - l)$ *and* z *space:*

- *Mixing inequalities give the convex hull of*

$$B = \operatorname{conv}\left\{ (-1, \mathbf{0}),\;\; (w(S), z^S) \text{ for all nonempty } S \subset P \right\} \;+\; \alpha(1, \mathbf{0}) \leftarrow \text{recession direction}$$

- *But we want*

$$B^+ = \operatorname{conv}\left\{ (0, \mathbf{0}),\;\; (w(S), z^S) \text{ for all nonempty } S \subset P \right\} \;+\; \alpha(1, \mathbf{0}) \leftarrow \text{recession direction}$$

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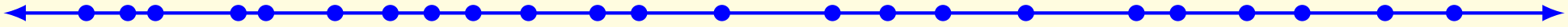
Constructing a valid inequality for B^+ when $n = 1$

The points are now on the line segment: $[0, 1]$



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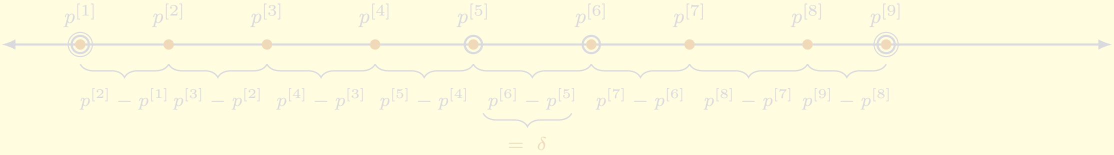
A valid inequality:

- *Pick a subset of the points in P (this is the support of the inequality)*
- *Choose 2 consecutive points and call them the anchor points, call the largest and smallest points leaves.*



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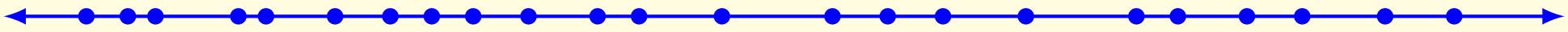

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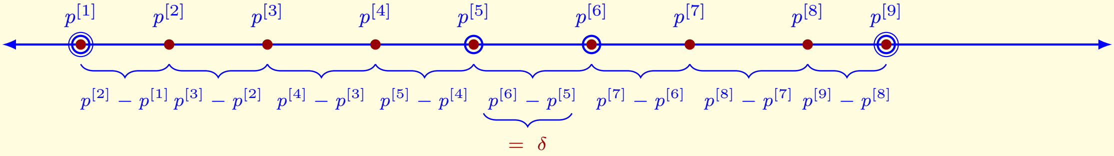
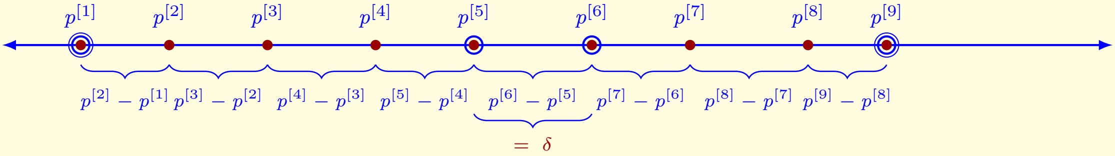


A valid inequality (the seed)



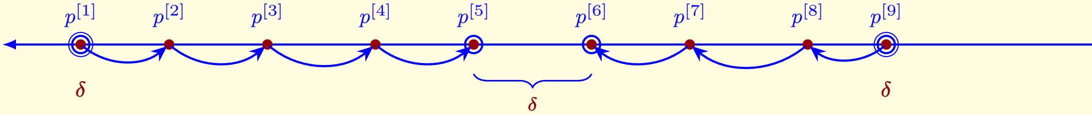
- *The seed inequality:*

$$w \geq \delta z_{[1]} + \delta z_{[9]} - \delta$$

$$+ (p^{[2]} - p^{[1]})z_{[2]} + (p^{[3]} - p^{[2]})z_{[3]} + (p^{[4]} - p^{[3]})z_{[4]} + (p^{[5]} - p^{[4]})z_{[5]} \quad \leftarrow \textit{left side}$$

$$+ (p^{[7]} - p^{[6]})z_{[6]} + (p^{[8]} - p^{[7]})z_{[7]} + (p^{[9]} - p^{[8]})z_{[8]} \quad \leftarrow \textit{right side}$$

- *Coefficients of the seed inequality are computed using distances between consecutive points:*



A facet defining inequality

Given a set of points P on a line segment, we are interested in:

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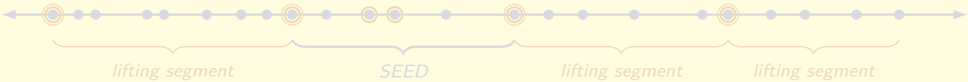
- *Pick a subset of the points $S = \{p^{[1]}, \ldots, p^{[k]}\} \subset P$ together with the 2 anchors and 2 leaves in S*
- *The associated **seed** inequality defines a facet of*

$$B^+ \cap \left\{z_i = 0, \ \ \textit{if } p^i < p^{[1]} \ \ \textit{or} \ \ p^i > p^{[k]}\right\}$$

- *The seed inequality becomes facet defining for B^+ after sequential lifting by picking more leaves*



$$w \ \geq \sum_{[i] \in L} \delta z_{[i]} + \sum_{[i] \notin L} \alpha_{[i]} z_{[i]} - \delta \quad \textit{where} \quad \alpha_{[i]} \ = \begin{cases} p^{[i]} - p^{[i-1]} & \textit{if segment lifted left to right} \\ p^{[i+1]} - p^{[i]} & \textit{if segment lifted right to left} \\ p^{[i+1]} - p^{[i-1]} - \delta & \textit{if } i \textit{ is the chosen "middle" point in segment} \end{cases}$$

- *Lifted seed inequalities can be separated in polynomial time using dynamic programming*

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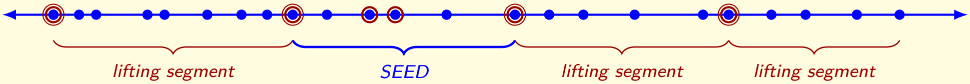

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Convex hull description of B^+

$$B^+ = \operatorname{conv}\left\{(w,z) \in \mathbb{R} \times \{0,1\}^n \ : \ w \geq \max_{i \in P}\{p^i z_i\} - \min_{i \in P}\{p^i z_i\}\right\} \longleftarrow \ [\ z = 0 \Rightarrow w \geq 0\ ]$$

- *B^+ is full-dimensional, and trivial inequalities are facet defining:*

$$1 \geq z_i, \quad z_i \geq 0 \quad \textit{for all } i \in P \quad \textit{and,} \quad w \geq 0$$

- *All other facet defining inequalities involve variable w and have the form*

$$w \geq \alpha^\top z - \delta$$

where $\alpha \geq 0$ and $1 \geq \delta > 0$.

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$$w(S) = \max_{i \in S}\{p^i\} - \min_{i \in S}\{p^i\} \quad \textit{and} \quad S \neq \emptyset$$

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Facet defining inequalities: Structural properties

Let $w \geq \alpha^\top z - \delta$ *be a non-trivial facet and with tight points*

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Crossing Lemma:

Let $A \cap B \neq \emptyset$. *If* $(w(A), z(A)) \in \mathcal{T}$ *and* $(w(B), z(B)) \in \mathcal{T}$, *then*

$$(w(A \cap B), z(A \cap B)) \in \mathcal{T} \quad \textit{and} \quad (w(A \cup B), z(A \cup B)) \in \mathcal{T}$$

Moreover, if $\mathcal{T}'$ *is an inclusion-wise maximal laminar subset of the sets defining* $\mathcal{T}$, *then*

$$\text{affine-hull}(\mathcal{T}) = \text{affine-hull}(\mathcal{T}')$$

Uncrossing Lemma:

Let $w \geq \alpha^\top z - \delta$ *be a non-trivial facet, then there are* $|P| + 1$ *affinely independent tight points*

$$(w(S_1), z(S_1)), \ (w(S_2), z(S_2)), \ \ldots \in B^+$$

such that S_i *and* S_j *do not cross (either inclusion or no intersection). In other words, the sets are laminar.*

[and these points uniquely define the facet coefficients]

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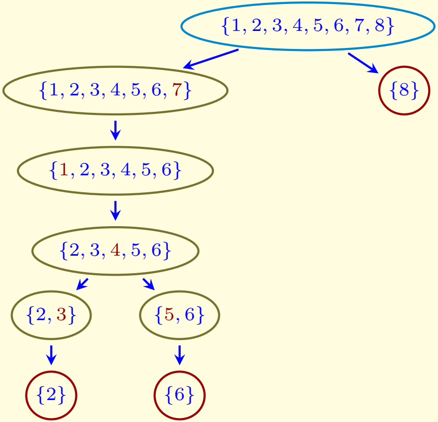
Facet defining inequalities and the laminar tree (forest) of tight points



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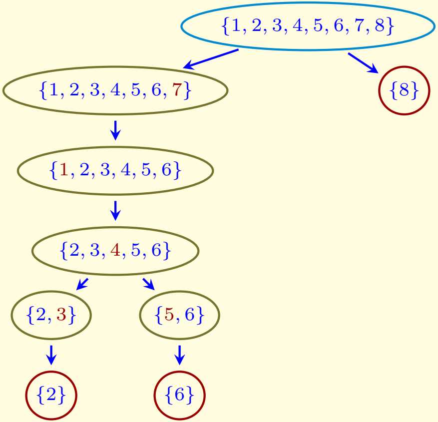

In this tree:

1. *all leaves are singletons*,
2. *all nodes correspond to intervals*,
3. *each parent is obtained by merging at most two intervals and possibly adding one extra point*,
4. *root is the whole interval* $1, \ldots, |P|$

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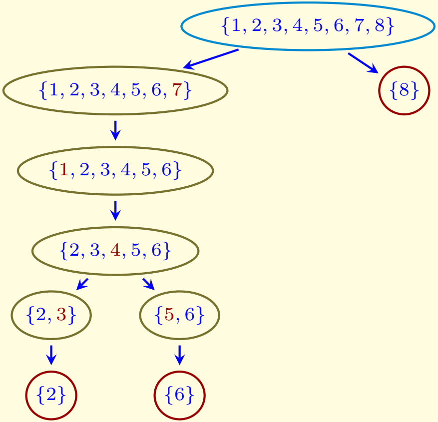

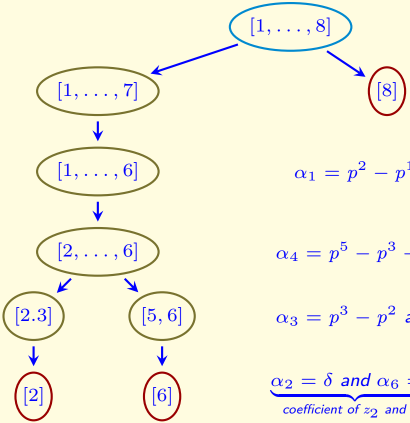

$\delta = p^8 - p^7$

$\alpha_7 = p^7 - p^6$

$\alpha_8 = \delta$

$\alpha_1 = p^2 - p^1$

$\alpha_4 = p^5 - p^3 - \delta$

$\alpha_3 = p^3 - p^2$ *and* $\alpha_5 = p^6 - p^5$

$\underbrace{\alpha_2 = \delta \text{ and } \alpha_6 = \delta}_{\text{coefficient of } z_2 \text{ and } z_6}$

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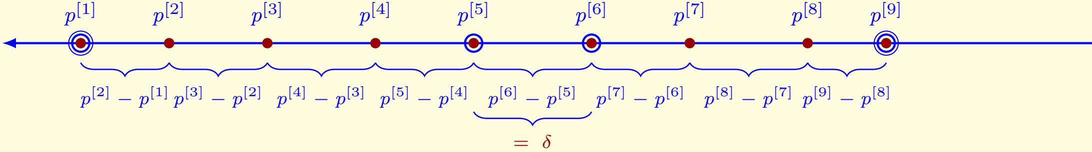
Convex hull description of B^+

Theorem:

Lifted seed inequalities give all non-trivial facet defining inequalities for

$$B^+ = \operatorname{conv}\left\{(w,z) \in \mathbb{R} \times \{0,1\}^n \;:\; w \geq \max_{i \in P}\{p^i z_i\} - \min_{i \in P}\{p^i z_i\}\right\}$$

- *Seed inequality has 2 ankers and 2 leaves at the ends. It defines a lower dimensional face:*



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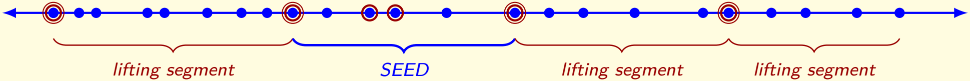


Back to the n-dimensional boxing Problem

- The points in P are now in $[0,1]^n$

$$z^*(IP) = \begin{array}{rl} \max & c^\top z - w \\ \text{s.t.} & w_d \geq \max\limits_{i \in P}\{p^i_d z_i\} - \min\limits_{i \in P}\{p^i_d z_i\} \quad \forall d \\ & w = \sum\limits_{d=1}^{n} w_d \\ & z \in \{0,1\}^n \end{array}$$

$$z^*(LP) = \begin{array}{rl} \max & c^\top z - w \\ \text{s.t.} & w_d \geq \text{all lifted seed inequalities} \quad \forall d \\ & w = \sum\limits_{d=1}^{n} w_d \\ & z \in [0,1]^n \end{array}$$

- For an optimal solution to the relaxed problem (LP) we have:

$$(w_d^*, z^*) \in B_d^+ \;=\; \operatorname{conv}\Big\{\, (0,\mathbf{0}),\;\; (w_d(S), z^S) \text{ for all } S \subseteq P \,\Big\} \;+\; \alpha(1,\mathbf{0}) \leftarrow \text{recession direction}$$

Unfortunately

For any $d \geq 2$ we can construct an example such that

$$z^*(IP) \;\leq\; \frac{1}{2^{d-1}}\; z^*(LP)$$

Back to the n-dimensional boxing Problem

- The points in P are now in $[0,1]^n$

$$z^*(IP) = \quad \max \; c^\top z - w \quad \text{s.t.} \; w_d \geq \max_{i \in P}\{p^i_d z_i\} - \min_{i \in P}\{p^i_d z_i\} \quad \forall d, \quad w = \sum_{d=1}^{n} w_d, \quad z \in \{0,1\}^n$$

$$z^*(LP) = \quad \max \; c^\top z - w \quad \text{s.t.} \; w_d \geq \text{all lifted seed inequalities} \quad \forall d, \quad w = \sum_{d=1}^{n} w_d, \quad z \in [0,1]^n$$

- For an optimal solution to the relaxed problem (LP) we have:

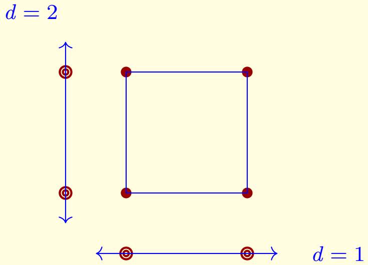
$$(w^*_d, z^*) \in B^+_d \; = \; \operatorname{conv}\Big\{ (0,\mathbf{0}), \;\; (w_d(S), z^S) \text{ for all } S \subseteq P \Big\} \; + \; \alpha(1,\mathbf{0}) \leftarrow \text{recession direction}$$

Unfortunately

For any $d \geq 2$ we can construct an example such that

$$z^*(IP) \; \leq \; \frac{1}{2^{d-1}} \; z^*(LP)$$

The example



LP optimal: $(w^*, z^*) = (\underbrace{0, 0}_{w_1, w_2}, \underbrace{1/2, 1/2, 1/2, 1/2}_{z_1, \ldots, z_4})$

$$(w_1^*, z^*) \;=\; \frac{1}{2}(0, 1, 1, 0, 0) + \frac{1}{2}(0, 0, 0, 1, 1) \;\leftarrow \in B_1^+$$

$$(w_2^*, z^*) \;=\; \frac{1}{2}(0, 1, 0, 1, 0) + \frac{1}{2}(0, 0, 1, 0, 1) \;\leftarrow \in B_2^+$$

- 2^d points are located on the corners of a d-dimensional 0-1 hypercube
- Reward for each point is small: $\epsilon < 1/2^d$
- Optimal solution to IP picks one point with reward ϵ and $w = 0$
  (taking more than 1 point has negative objective value)
- LP optimal takes all points with 1/2. Total reward is $(\frac{1}{2}2^d)\epsilon$ and $w = 0$
- For any d this solution is $= 1/2$ (all points with $p_d = 1$) $+ 1/2($ all points with $p_d = 0)$
  (and therefore in B_d^+)

Work in progress: B^+ with a budget constraint

- *Given a set of points P on the line segment $[0,1]$, we have the linear description of:*

$$B^+ = \operatorname{conv}\left\{(w,z) \in \mathbb{R} \times \{0,1\}^n \;\; : \; w \geq \max_{i \in P}\{p^i z_i\} - \min_{i \in P}\{p^i z_i\}\right\}$$

- *We next consider*

$$B^{W+} = \operatorname{conv}\left\{(w,z) \in \mathbb{R} \times \{0,1\}^n \;\; : \; W \; \geq \; w \geq \max_{i \in P}\{p^i z_i\} - \min_{i \in P}\{p^i z_i\}\right\}$$

- *Trivial facets are: $W \geq w \geq 0$, and $1 \geq z_i \geq 0$ for all $i \in P$*
- *Non-trivial facets are:*
  1. *Clique facets: $z(C) \leq 1$ for maximal incompatible set of points $C \subseteq P$ (poly-time separation by DP)*
  2. *Remaining facets have the form: $w \geq \alpha^T x + \delta$ where $\alpha \geq 0$ and $\delta < 0$*
     - *These facet defining inequalities can be obtained by picking spanning trees on a certain graph on P plus one more edge.*
     - *This can be viewed as a nice representation of the extreme points of the polar.*
     - *We do not have an explicit form for these facets (yet).*

thank you...