Some applications of Mean Field Games

Yves Achdou

LJLL, Université Paris Diderot

Aim and Contents

Aim

Discuss two possible applications of MFG, putting the stress on numerical simulations

Contents

- Crowd motion
  - An example: exit from a hall with obstacles (including a situation with a common risk)
  - A brief discussion on solvers
  - A simple example of a two populations MFG with segregation
- Macro-economics
  - The Aiyagari model
  - A more complex model with financial friction

A. Exit from a hall with obstacles

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\frac{\partial u}{\partial t} + \nu \Delta u - H(x, m, \nabla u) &= -F(m), && \text{in } (0,T)\times\Omega \\
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Congestion: $L(x, m, v) = (\beta - 1)(c_0 + c_1 m)^{\frac{\alpha}{\beta-1}} \left(\frac{|v|}{\beta} \right)^{\frac{\beta}{\beta-1}} - \mathcal{H}(x).$

$$H(x, m, p) = \mathcal{H}(x) + \frac{|p|^\beta}{(c_0 + c_1 m)^\alpha}$$

with $c_0 > 0$, $c_1 \geq 0$, $\beta > 1$ and $0 \leq \alpha < 4(\beta - 1)/\beta$.
Existence and uniqueness were proven by P-L. Lions.

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Existence and uniqueness were proven by P-L. Lions.
The function $\mathcal{H}(x)$ may model the panic in the hall.

Finite differences

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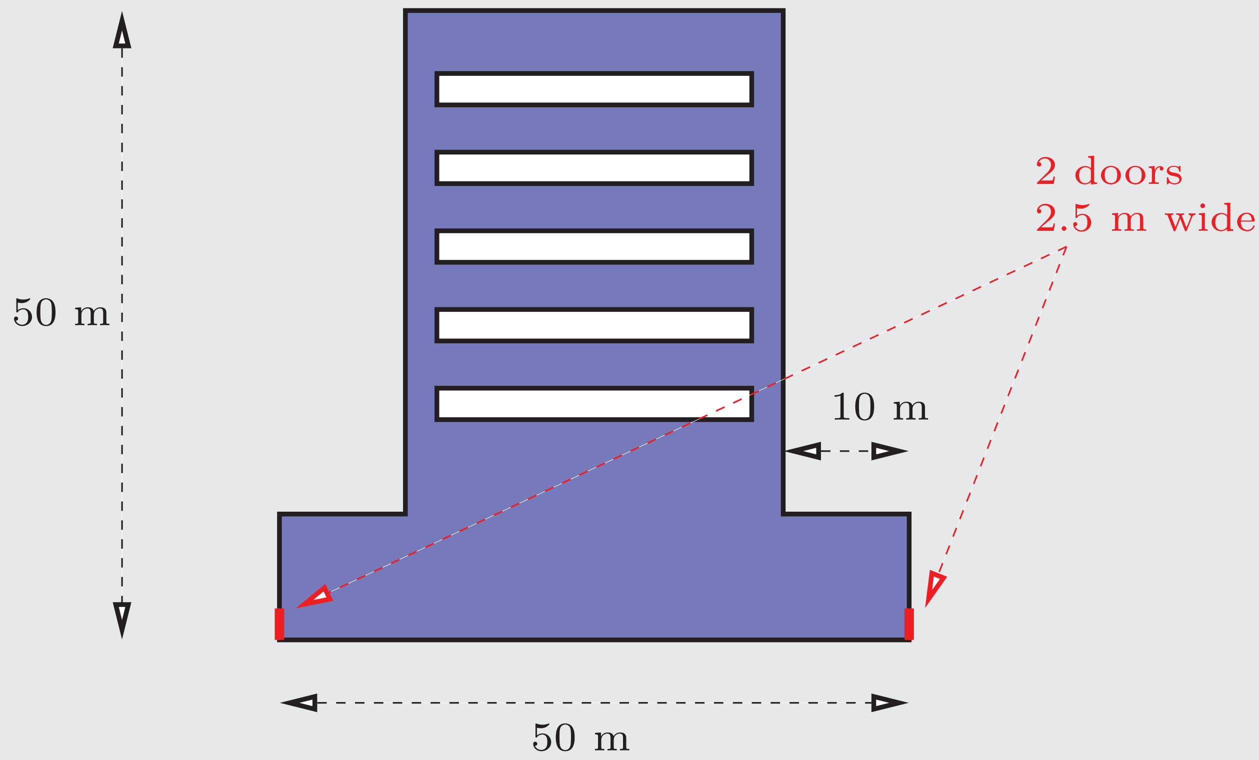
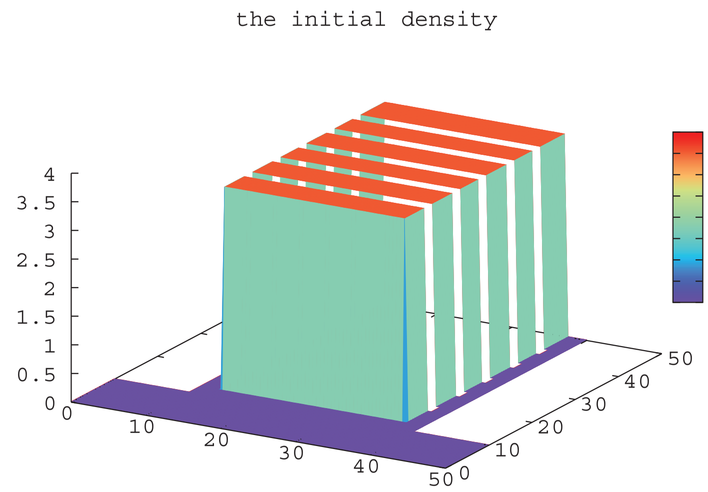
- Theoretical results (with or without congestion)
  - Existence and a priori bounds
  - Uniqueness
  - Convergence as $\Delta t, h \to 0$

The geometry



The data

- The initial density m_0 is piecewise constant and takes two values 0 and 4 people/m^2. There are 2900 people in the hall.



- $\nu = 1/3$
- $H(x,m,p) = |p|^2 \left(\frac{25}{1+6m}\right)^{\frac{3}{2}} - 10^{-3}$
- $F(m) = 5.\,10^{-4} m$

which leads to the following HJB equation

$$\frac{\partial u}{\partial t} + \frac{1}{3}\Delta u - \left(\frac{25}{1+6m}\right)^{\frac{3}{2}} |\nabla u|^2 = -10^{-4}\,(5m+10)$$

The results

The horizon is $T = 20$ min. The two doors stay open from $t = 0$ to $t = T$.

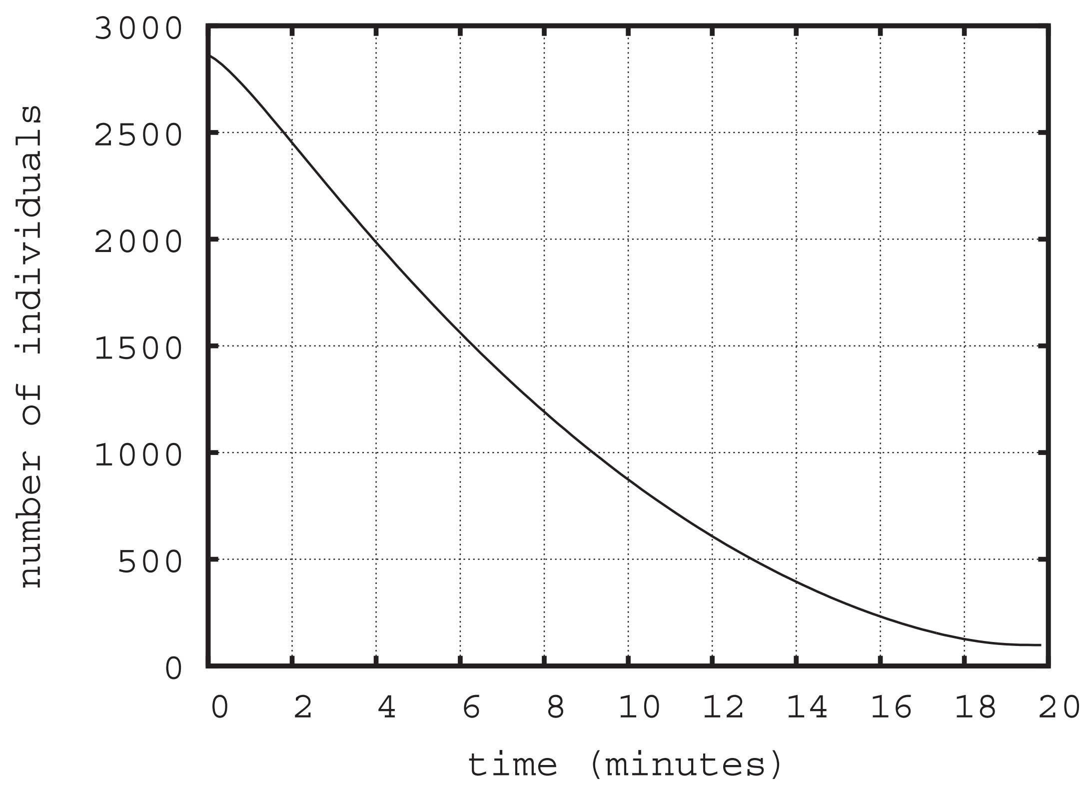


Figure: The number of people in the room vs. time

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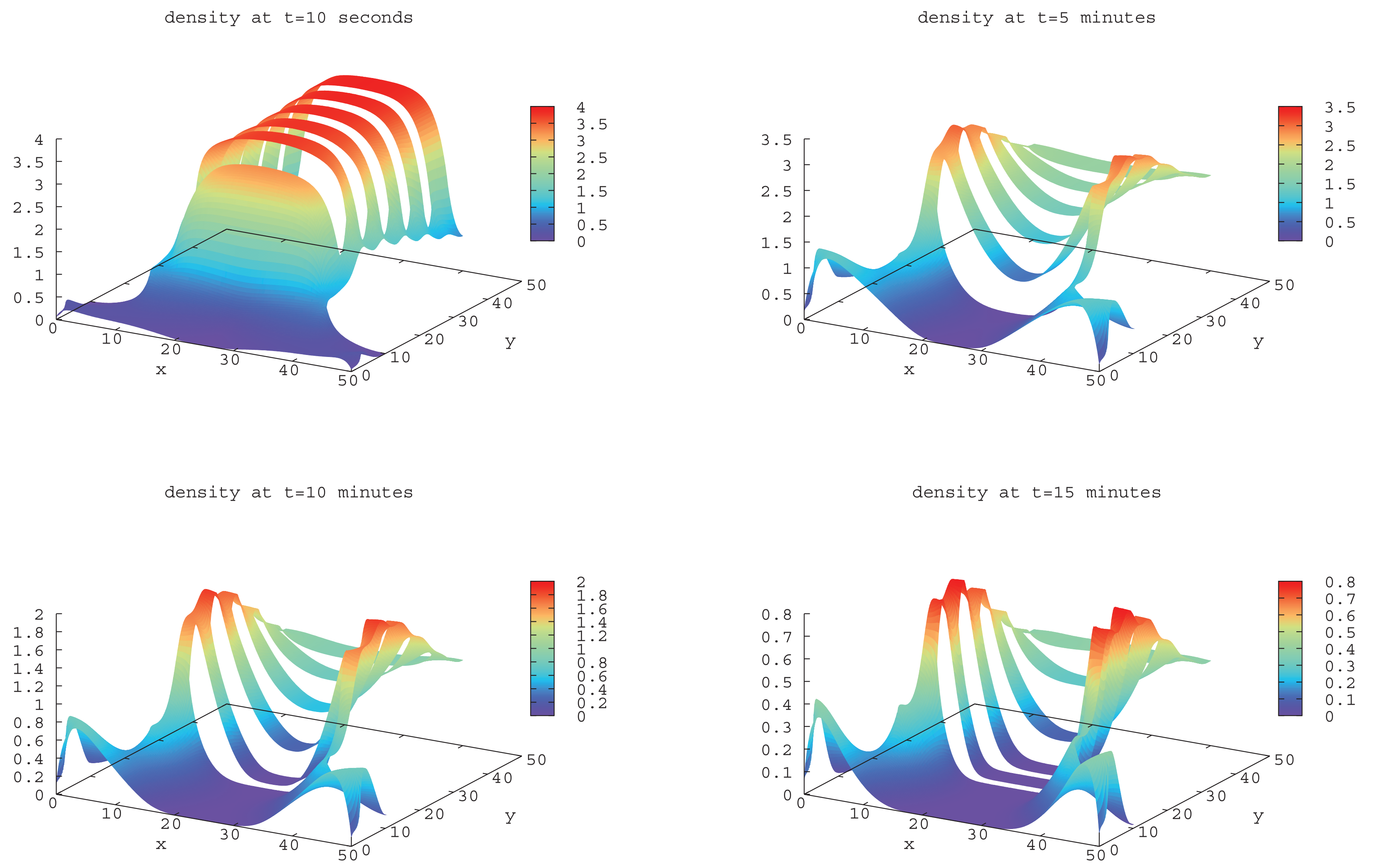
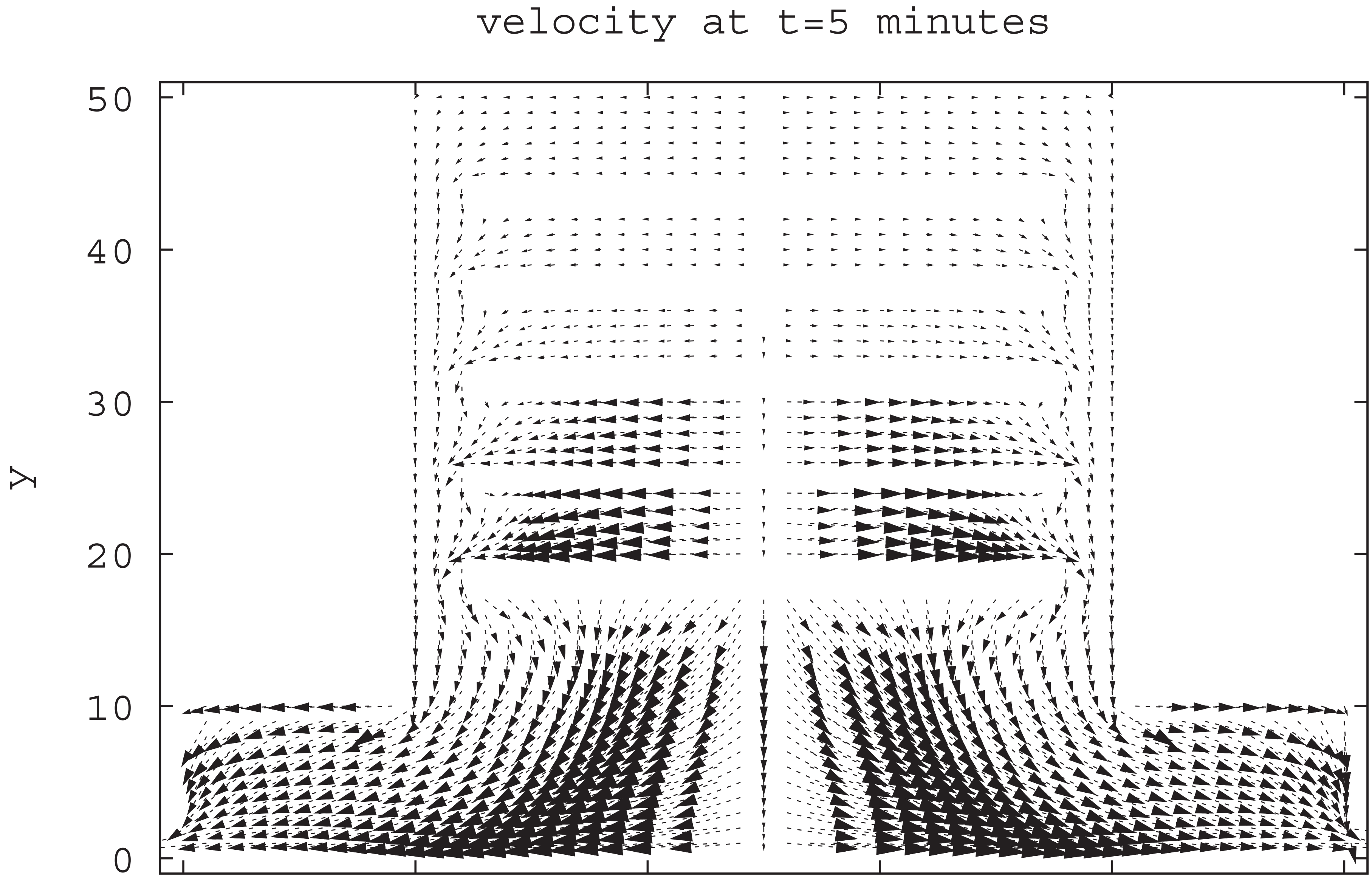
Figure: The evolution of the density

The evolution of the density



(the scale varies w.r.t. t)

The velocity



Exit from a hall with a common uncertainty

Same geometry. The horizon is T.

- Before $t = T/2$, the two doors are closed.

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Hence the model involves three pairs of unknown functions

- (u^C, m^C) is defined on $(0, T/2) \times \Omega$ and corresponds to the situation when the room is closed.
- (u^L, m^L) and (u^R, m^R) are defined on $(T/2, T) \times \Omega$ and resp. correspond to the case when the left (resp. right) door is open.

The boundary value problem

The systems of PDEs: for $j = C, L, R$,

$$\frac{\partial u^j}{\partial t} + \nu \Delta u^j - H(m^j, \nabla u^j) = -F(m^j),$$
$$\frac{\partial m^j}{\partial t} - \nu \Delta m^j - \operatorname{div}\left(m^j \frac{\partial H}{\partial p}(m^j, \nabla u^j) \right) = 0,$$

in $(0, T/2) \times \Omega$ for $j = C$ and in $(T/2, T) \times \Omega$ for $j = L, R$.

The boundary conditions

$$\frac{\partial u^C}{\partial n} = \frac{\partial m^C}{\partial n} = 0 \quad \text{on } (0, \frac{T}{2}) \times \partial\Omega,$$

and for $j = L, R$,
$$\begin{cases} \dfrac{\partial u^j}{\partial n} = \dfrac{\partial m^j}{\partial n} & = 0 \quad \text{on } (\frac{T}{2}, T) \times \Gamma_N^j, \\ u^j = m^j & = 0 \quad \text{on } (\frac{T}{2}, T) \times \Gamma_D^j \end{cases}$$

Transmission conditions at $t = T/2$

$$m^L(\frac{T}{2}, x) = m^R(\frac{T}{2}, x) = m^C(\frac{T}{2}, x) \quad \text{in } \Omega,$$

$$u^C(\frac{T}{2}, x) = \frac{u^L(\frac{T}{2}, x) + u^R(\frac{T}{2}, x)}{2} \quad \text{in } \Omega.$$

Results

$T = 40$ min.

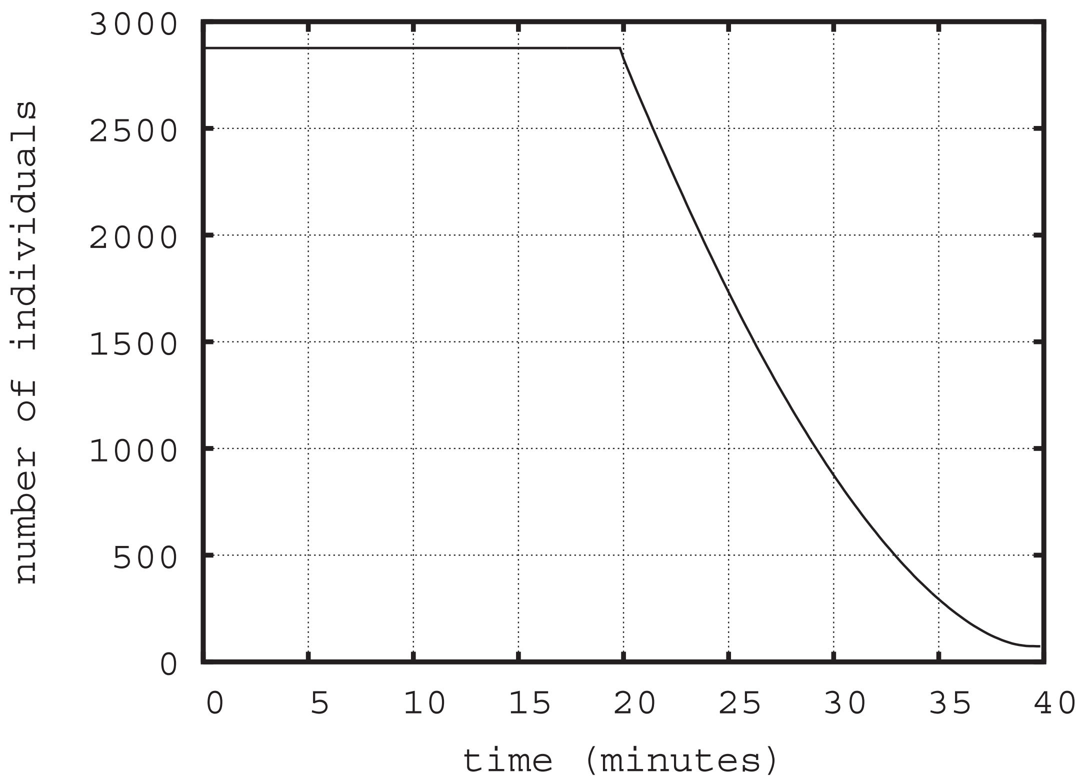

Figure: The number of people in the room vs. time

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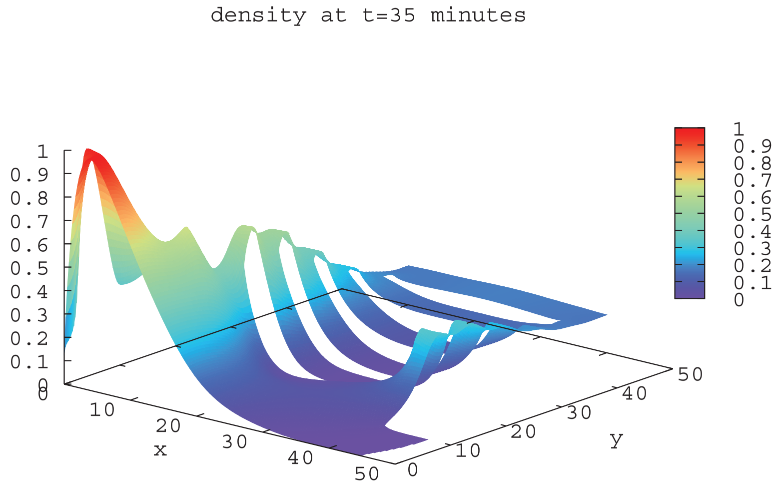
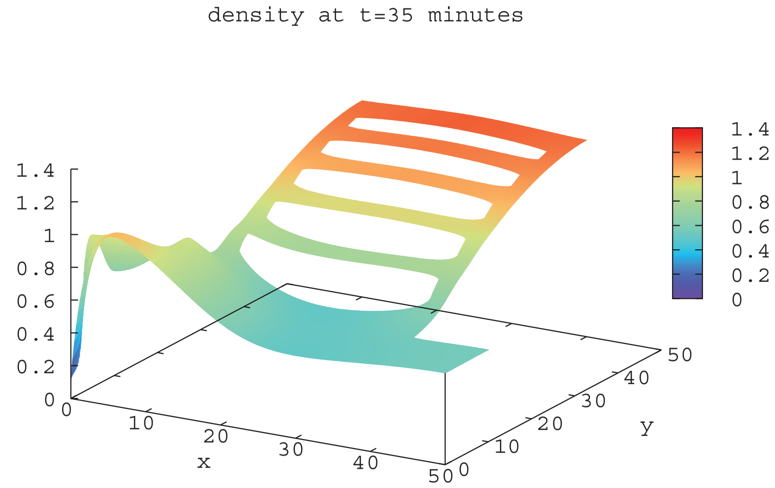
Figure: The evolution of the density

Sensitivity w.r.t. to the model for congestion





- Left : $H(x,m,p) = |p|^2 \left(\frac{25}{1+6m}\right)^{\frac{3}{2}} - 10^{-3}$
- Right : $H(x,m,p) = |p|^2 \left(\frac{25}{1+12m}\right)^{\frac{3}{2}} - 10^{-3}$

B. A strategy for solving the discrete MFG system

Due to the **forward-backward structure, marching in time is not possible.** One has to solve the system for u and m as a whole. This leads to large systems of nonlinear equations with $\sim 2N^{d+1}$ unknowns.

- We first used Newton iterations on the whole system:

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  - Breakdown may occur if the Newton iterations produce a density m with negative values
  - This may happen if the density is small in some regions
- Then we designed an iterative strategy without this drawback

The simplest case (no congestion)

$$(*)\left\{\begin{array}{rcl} \dfrac{\partial u}{\partial t}+\nu\Delta u-H(x,\nabla u) &=& -f(m)\\ \dfrac{\partial m}{\partial t}-\nu\Delta m-\operatorname{div}\left(m\dfrac{\partial H}{\partial p}(x,\nabla u)\right) &=& 0\end{array}\right.$$

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- A fixed point on an auxiliary variable F
  - Given F, one can find $(u(F), m(F))$ by solving
    - a backward Bellman equation for $u(F)$

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  - The resulting equation on F is

    $$F = f(m(F))$$

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- Newton iterations are used for $F = f(m(F))$
  - Needs to solve the linearized versions of the Bellman and Fokker-Planck equations

With congestion

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + \nu \Delta u - \frac{|\nabla u|^\beta}{(c_0 + c_1 m)^\alpha} = \mathcal{H}(x) - f(m), && \text{in } (0,T) \times \Omega \\
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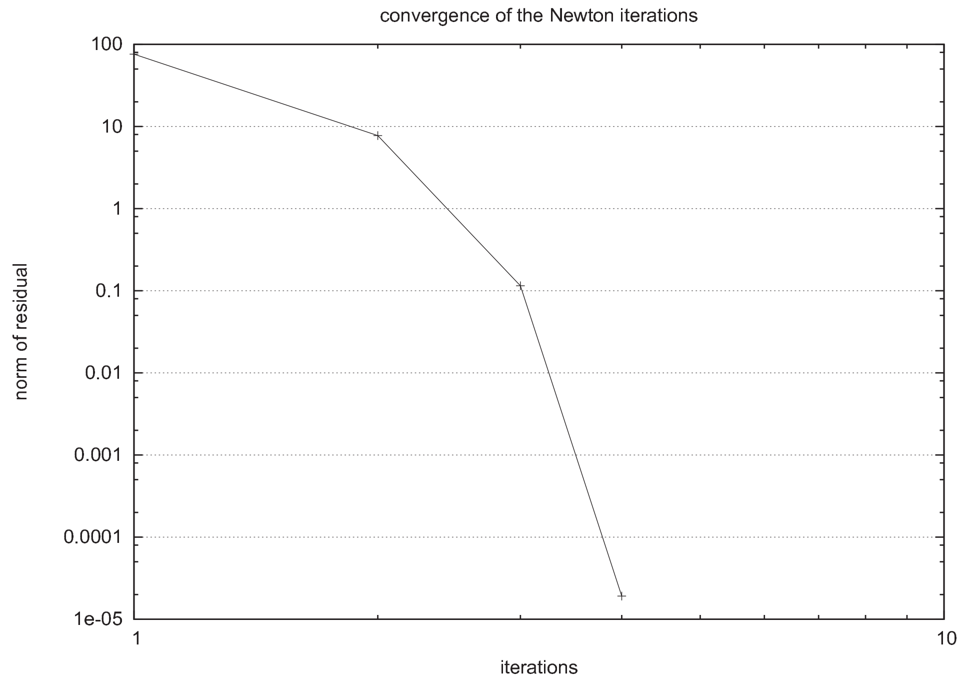
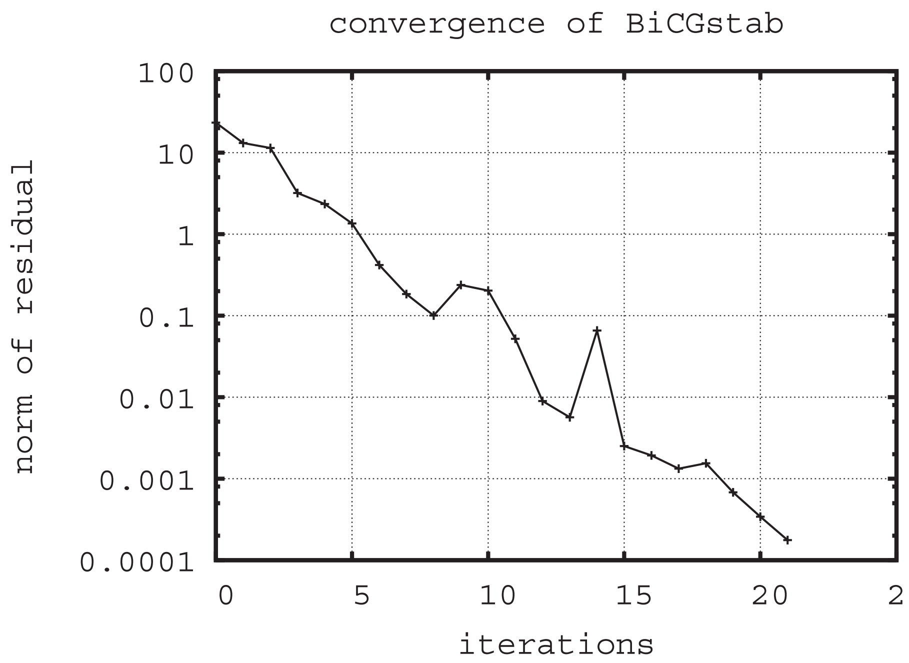
    - The resulting equation on F, G are

$$F = f(m(F,G))$$
$$G = (c_0 + c_1 m(F,G))^\alpha$$

Convergence of Newton iterations and of inner loop for linear problems





A continuation method

- The MFG system is easier with large viscosity ν
- Continuation:
  - start solving the system with a large value of ν
  - decrease ν progressively, using the previously computed F, G, m, u as an initial guess for the Newton method.

C. Two populations

$$
\begin{aligned}
\frac{\partial u_1}{\partial t} + \nu \Delta u_1 - H_1(t, x, m_1 + m_2, \nabla u_1) &= -F_1(m_1, m_2) \\
\frac{\partial m_1}{\partial t} - \nu \Delta m_1 - \operatorname{div}\left(m_1 \frac{\partial H_1}{\partial p}(t, x, m_1 + m_2, \nabla u_1) \right) &= 0 \\
\frac{\partial u_2}{\partial t} + \nu \Delta u_2 - H_2(t, x, m_1 + m_2, \nabla u_2) &= -F_2(m_1, m_2) \\
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A model for segregation proposed by M. Bardi

- The Hamiltonians are uniform in space and the same for the two populations

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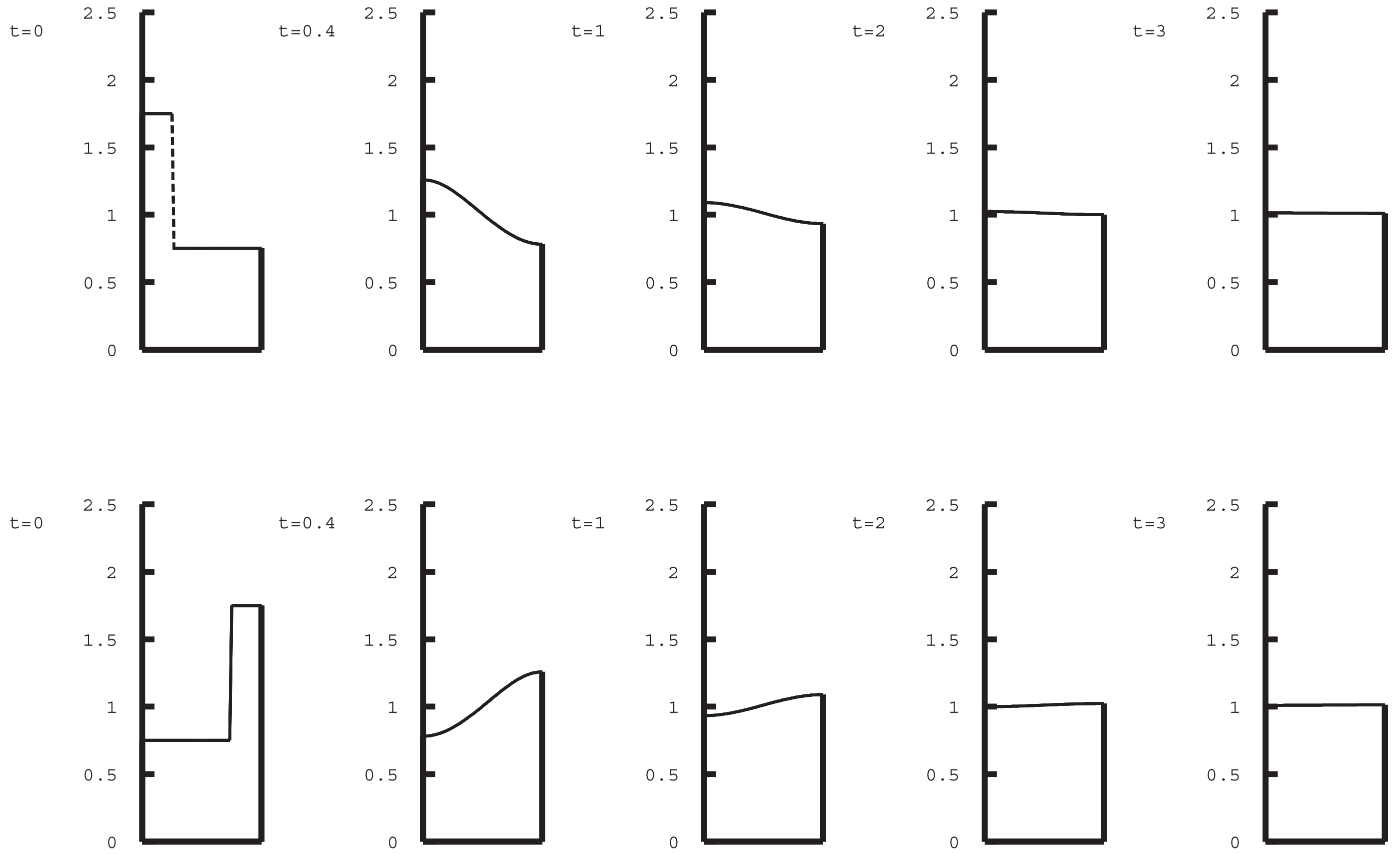
Xenophobia

The cost operators $F_1(m_1, m_2)$ and $F_2(m_1, m_2)$ are given by

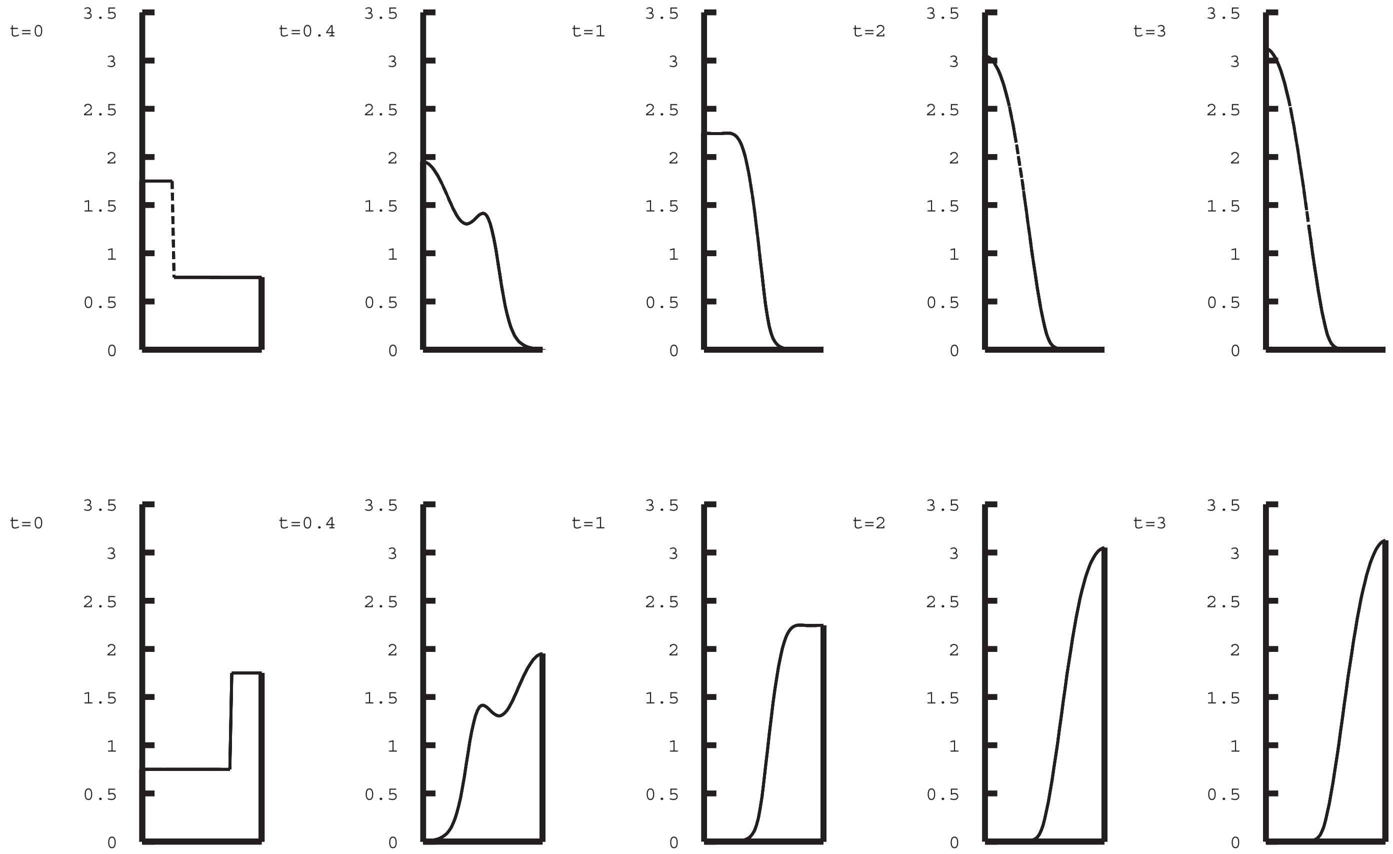
$$F_i(m_i, m_j) = 5\left(\frac{m_i}{m_i + m_j} - 0.45\right)_- + (m_i + m_j - 4)_+$$

$\nu = 0.2$: evolution of the densities



$$m_1(\cdot, t=0) = 0.75 + 1_{[0,0.25]}, \quad m_2(\cdot, t=0) = 0.75 + 1_{[0.75,1]},$$

$\nu = 0.025$: evolution of the densities



$$m_1(\cdot, t=0) = 0.75 + 1_{[0,0.25]}, \quad m_2(\cdot, t=0) = 0.75 + 1_{[0.75,1]},$$

D. Examples of soft MFG in macro-economics

work in progress with J-M. Lasry, P-L. Lions and B. Moll

Heterogeneous agent models in continuous time

1. A prototypical heterogeneous agent model

- Based on Aiyagari model: *Uninsured idiosyncratic risk and aggregate savings*, Quaterly journal of economics 1994
- Here, continuous time instead of discrete time in the original paper
- Two types of agents: households and firms

Households

- The households are heterogeneous in their wealth a and their work ability z. The joint distribution is $g(a, z, t)$
- A given household maximizes

$$\mathbb{E}\left(\int_0^\infty e^{-\rho t} u(c_t)dt\right)$$

subject to

$$\begin{aligned} da_t &= (w_t z_t + r_t a_t - c_t)dt \\ dz_t &= \mu(z_t)dt + \sigma(z_t)dW_t \\ a_t &\geq \underline{a} \quad \text{(borrowing limit)} \end{aligned}$$

  - c_t : consumption
  - u : utility function, nondecreasing and concave
  - ρ : discount rate
  - w_t : wage
  - r_t : interest rate

Firms

- All firms are assumed identical. Their production is

$$Y_t = F(K_t, L_t)$$

where K_t and L_t are the amounts of capital and labor. The firms maximize

$$\max_{K_t, L_t} \Big(F(K_t, L_t) - (r_t + \delta) K_t - w_t L_t \Big),$$

where δ is the rate of depreciation of the capital. Hence

$$r_t = \partial_K F(K_t, L_t) - \delta \qquad w_t = \partial_L F(K_t, L_t)$$

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$$Y_t = F(K_t, L_t)$$

where K_t and L_t are the amounts of capital and labor. The firms maximize

$$\max_{K_t, L_t} \Big(F(K_t, L_t) - (r_t + \delta) K_t - w_t L_t \Big),$$

where δ is the rate of depreciation of the capital. Hence

$$r_t = \partial_K F(K_t, L_t) - \delta \qquad w_t = \partial_L F(K_t, L_t)$$

Equilibrium

$$K_t = \int a g(a, z, t) da dz \qquad L_t = \int z g(a, z, t) da dz$$

The system of PDEs

Hamiltonian

$$H(p) = \max_c \left(u(c) - pc\right)$$

The value function $v(a, z, t)$ and the density $g(a, z, t)$ solve

$$\begin{aligned} 0 = \; & \frac{\partial v}{\partial t} - \rho v + H\left(\frac{\partial v}{\partial a}\right) + \left(w(t)z + r(t)a\right)\frac{\partial v}{\partial a} \\ & + \frac{1}{2}\sigma_z^2(z)\frac{\partial^2 v}{\partial z^2} + \mu_z(z)\frac{\partial v}{\partial z} \end{aligned}$$

$$\begin{aligned} 0 = \; & \frac{\partial g}{\partial t} + \frac{\partial}{\partial a}\left(g\left(w(t)z + r(t)a + H_p\left(\frac{\partial v}{\partial a}\right)\right)\right) \\ & + \frac{\partial}{\partial z}\left(\mu_z(z)g\right) - \frac{1}{2}\frac{\partial^2}{\partial z^2}\left(\sigma_z^2(z)g\right) \end{aligned}$$

in $(\underline{a}, \overline{a}) \times (\underline{z}, \overline{z}) \times (0, \infty)$ or $(\underline{a}, \overline{a}) \times (\underline{z}, \overline{z}) \times (0, T)$.

Boundary conditions

- $g(\cdot, \cdot, t = 0)$ is given
- $v(\cdot, \cdot, t = T)$ is first computed by solving a "long" horizon problem
- at $z = \underline{z}$ and $z = \overline{z}$: Neumann condition on v
- At $a = \underline{a}$: the state is constrained in $a > \underline{a}$, so we use a penalty method for v
- At $a = \overline{a}$: no boundary conditions

Fixed point algorithm for the functions $K(t)$ and $L(t)$

Choose a relaxation parameter $\theta \in (0,1]$. Start with an initial guess $K_0(t)$ and $L_0(t)$, $t \in (0,T)$. Then for $\ell = 0,1,2,\ldots$

1. Given $(K_\ell(t), L_\ell(t))$, calculate $w_\ell(t)$ and $r_\ell(t)$

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2. Given $w_\ell(t)$ and $r_\ell(t)$ and the terminal condition, solve the HJB equation marching backward in time, for $v_\ell(a,z,t)$. Calculate the savings policy function

$$\mu_{a,\ell}(a,z,t) = w_\ell(t)z + r_\ell(t)a + H_p\left(\frac{\partial v_\ell}{\partial a}(a,z,t)\right)$$

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3. Given $\mu_{a,\ell}(a,z,t)$ and the initial condition, solve the Kolmogorov Forward equation, marching forward in time, for $g_\ell(a,z,t)$
4. Generate a new guess by

$$K_{\ell+1}(t) = (1-\theta)K_\ell(t) + \theta \int a g_\ell(a,z,t)\,dadz$$

$$L_{\ell+1}(t) = (1-\theta)L_\ell(t) + \theta \int z g_\ell(a,z,t)\,dadz$$

Results 1/4

We use standard CRRA utility and Cobb-Douglas production functions

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \gamma > 0$$

$$F(K, L) = AK^{\alpha}L^{1-\alpha}, \quad \alpha \in (0, 1)$$

We further assume that the stochastic process for labor productivity is a simple Brownian motion

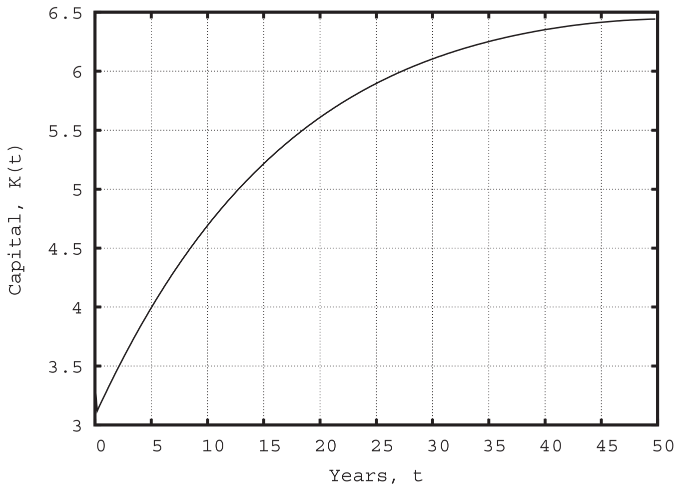
$$dz_t = \bar{\sigma}_z dW_t$$

reflected at $\underline{z}$ and $\bar{z}$.

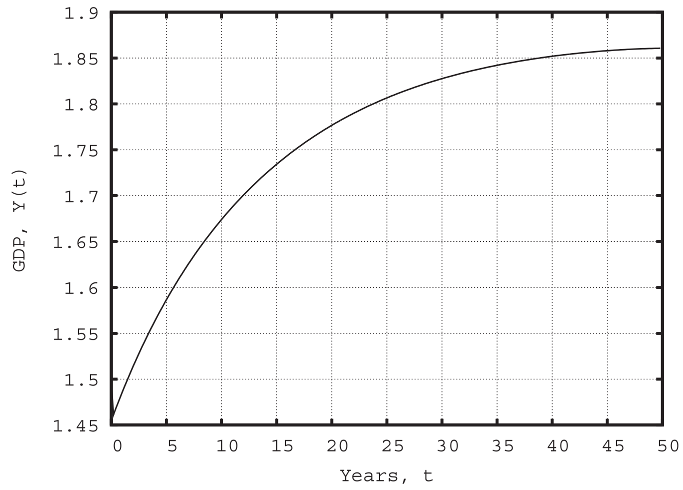
$$\alpha = 1/3, \quad \gamma = 2, \quad \delta = 0.05, \quad \rho = 0.05, \quad \bar{\sigma}_z^2 = 0.02$$

$$\underline{a} = 0, \quad \overline{a} = 22, \quad \underline{z} = 0.5, \quad \overline{z} = 1.5, \quad T = 50$$

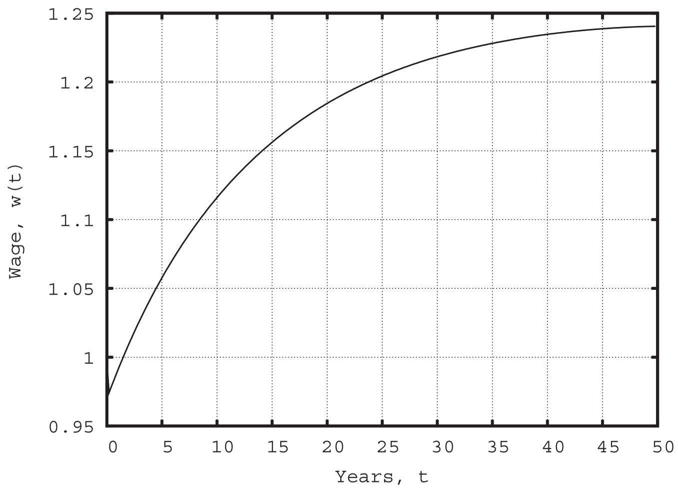
Results 2/4



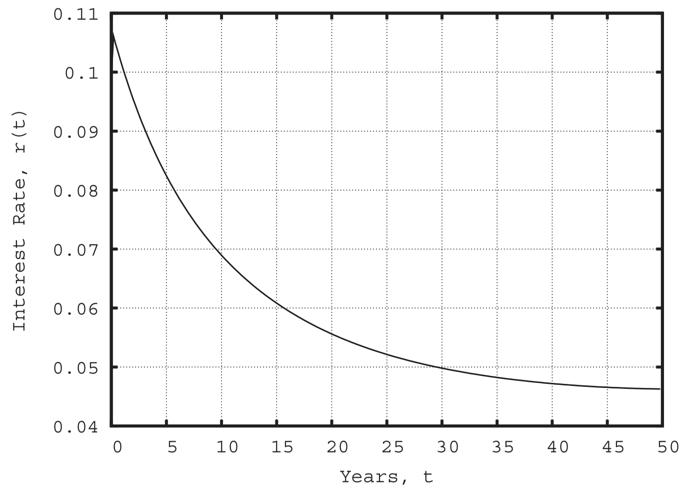
(a) Capital



(b) GDP



(c) Wage



(d) Interest Rate

Results 3/4

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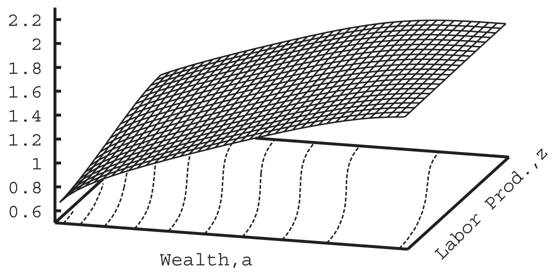
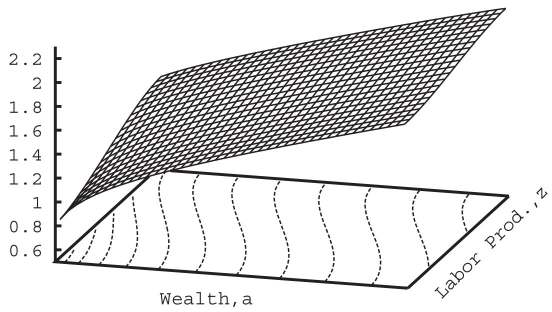
Figure: Evolution of Joint Distribution of Labor Productivity and Wealth

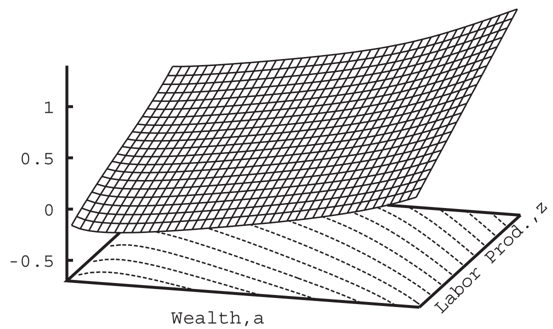
Results 4/4



(a) Consumption Policy Function, $c(a, z, t)$ at $t = 0$



(b) Consumption Policy Function, $c(a, z, t)$ at $t = 50$



(c) Savings Policy Function, $\mu_a(a, z, t)$ at $t = 0$

Aggregate shocks (Krussel-Smith 1998)

The economy may be hit by aggregate productivity shocks, i.e.

$$Y_t = A(t) F(K_t, L_t)$$

$A(t)$ is a random proc. with N shocks at times $\tau_1, \tau_2, \dots, \tau_N$

$$\begin{cases} A(t) = A_0 & \text{for } 0 \le t \le \tau_1 \\ A(t) \text{ is constant with the value } A_1 \text{ or } A_2 & \text{for } \tau_i < t \le \tau_{i+1} \end{cases}$$

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We assume that there exist $p_{ss'} \in [0,1]$ such that

$$\Pr\Big(A(\tau_{i,+}) = A_{s'} | A(\tau_{i,-}) = A_s\Big) = p_{ss'}, \quad \forall i = 2, \dots, N$$

Example: $A_0 = 0.5$, $A_1 = 0.4$ and $A_2 = 0.6$ and the transition probabilities are given by:

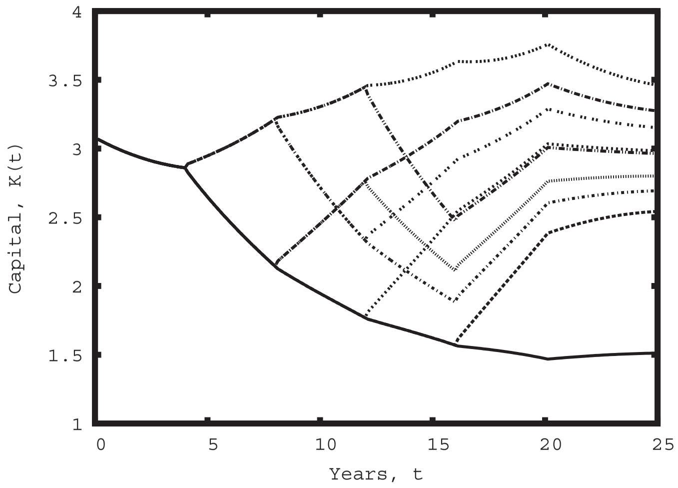
$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}$$

For the system of pdes, the transmission conditions at τ_{i+1} are

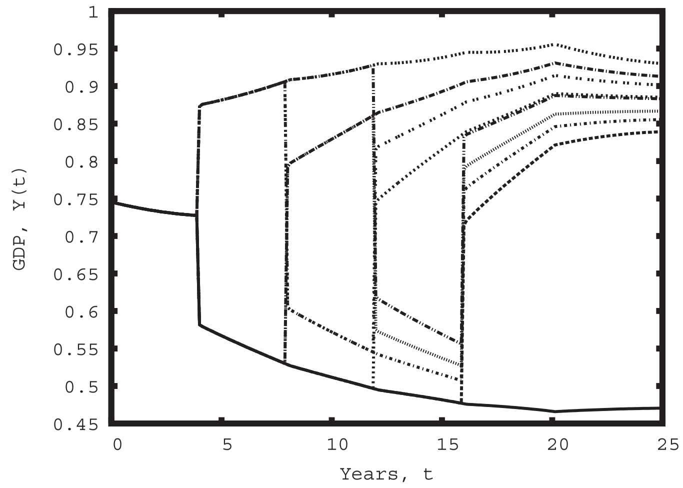
$$v_\sigma\left(a,z,\tau_{i+1,-}\right)=\sum_{s=1,2} p_{\sigma_i s} v_{\sigma^{(i)}|s}\left(a,z,\tau_{i+1,+}\right)$$

$$g_\sigma\left(a,z,\tau_{i+1,+}\right)=g_\sigma\left(a,z,\tau_{i+1,-}\right)$$

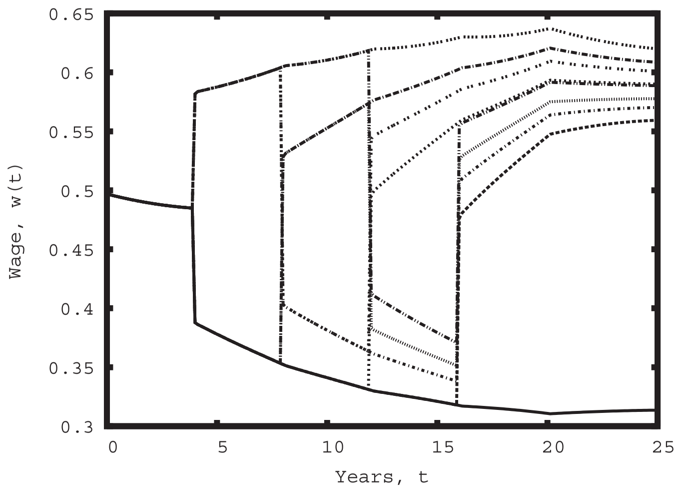
An example with 5 shocks



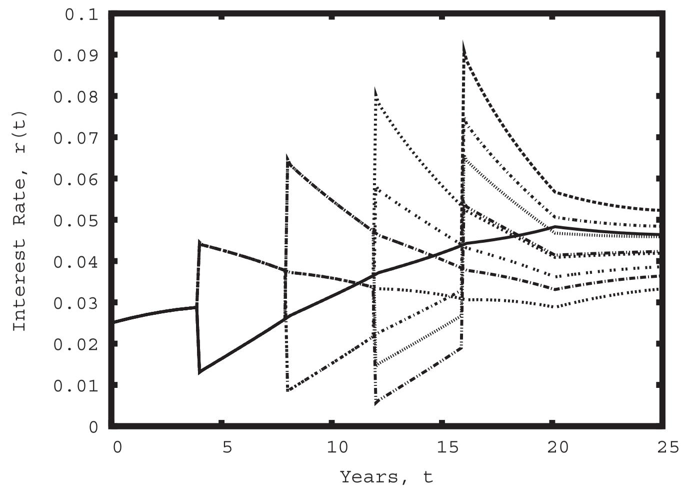
(a) Capital



(b) GDP



(c) Wage



(d) Interest Rate

2. An economy with financial frictions

- A continuum of entrepreneurs, heterogeneous in wealth a and productivity z

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- The production function of an entrepreneur is

$$zf(k) \qquad \text{with} \qquad f(k) = \tilde{f}\left((k-\kappa)^+\right)$$

  - k is the capital used for production and κ is a fixed cost
  - $\tilde{f}(0) = 0$ and $\tilde{f}$ is strictly increasing and strictly concave

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$$\Pi(a,z,r) = \max_{0 \le k \le \lambda a} \big(z f(k) - (r+\delta) k \big) \qquad (\text{possibly} \le 0)$$

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- The wealth and productivity evolve according to

$$da_t = \big(\Pi^+(a_t, z_t, r_t) + r_t a_t - c_t\big)\, dt$$
$$d\log z_t = -\nu \log z_t dt + \bar{\sigma}_z dW_t$$

The entrepreneurs maximize

$$\mathbb{E}\left(\int_0^\infty e^{-\rho t} u(c_t)dt\right)$$

Equilibrium

The interest rate r_t is obtained by solving the equilibrium equation

$$\int k(a,z,r_t) 1_{\{\Pi(a,z,r_t)>0\}} g(a,z,t) dadz = \int a g(a,z,t) dadz$$

Finding the equilibrium is not as easy as in the Aiyagari model because $k(a,z,r)$ is obtained by maximizing

$$\Pi(a,z,r) = \max_{0\le k\le \lambda a} \big(z f(k) - (r+\delta)k \big).$$

PDEs

$$\rho v(a,z,t) = \max_c \Big(u(c) + \partial_a v(a,z,t)[\Pi^+(a,z;r(t)) + r(t)a - c] \Big) \\ + \mu_z(z)\partial_z v(a,z,t) + \frac{1}{2}\partial_{zz} v(a,z,t)\sigma_z^2(z) + \partial_t v(a,z,t)$$

$$\partial_t g(a,z,t) = -\partial_a[\mu_a(a,z,t)g(a,z,t)] - \partial_z[\mu_z(z)g(a,z,t)] \\ + \frac{1}{2}\partial_{zz}[\sigma_z^2(z)g(a,z,t)]$$

where

$$\Pi(a,z,r) = \max_{0\le k\le \lambda a} \big(zf(k) - (r+\delta)k \big) \qquad \text{(possibly} \le 0)$$

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where

$$\Pi(a,z,r) = \max_{0\le k\le \lambda a} \big(zf(k) - (r+\delta)k \big) \qquad \text{(possibly} \le 0)$$

$$\mu_a(a,z,t) = \Pi^+(a,z;r(t)) + r(t)a - c^*(a,z;r(t))$$

is the Savings Policy Function and

$$\mu_z(z) = z\left(\frac{\bar{\sigma}_z^2}{2} - \nu \log(z)\right)$$

Results 1/4

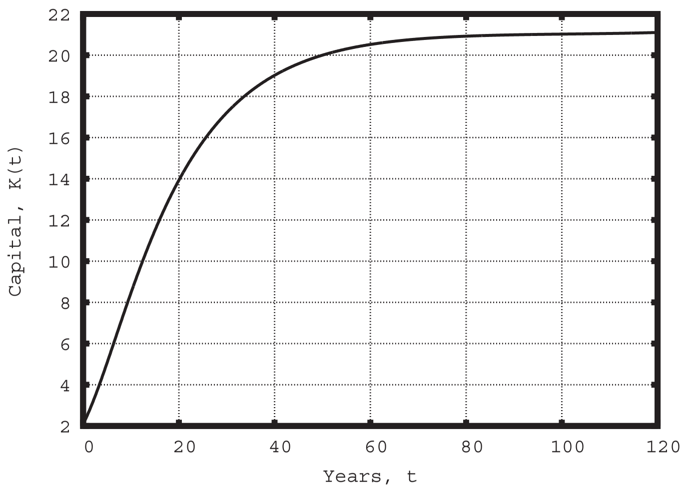
$$f(k) = \left((k-\kappa)^+\right)^\alpha \qquad \alpha = 0.33, \quad \kappa = 5$$

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \gamma = 2$$

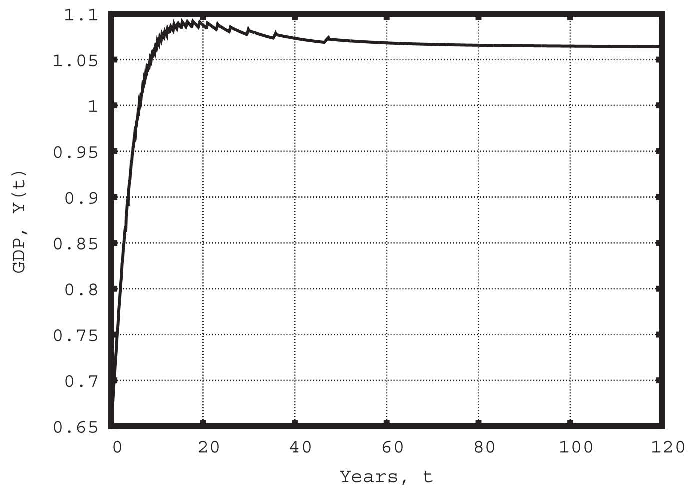
Friction coefficient: $\lambda = 4$

$$\delta = 0.05, \quad \rho = 0.05,$$
$$\nu = 0.05, \quad \bar{\sigma}_z = 0.2,$$
$$\underline{a} = 0, \quad \bar{a} = 100,$$
$$\underline{z} = 0, \quad \bar{z} = 3, \quad T = 150$$

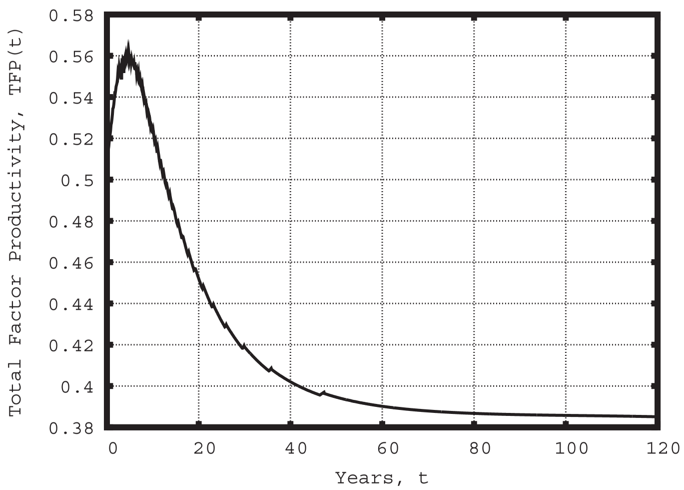
Results 2/4



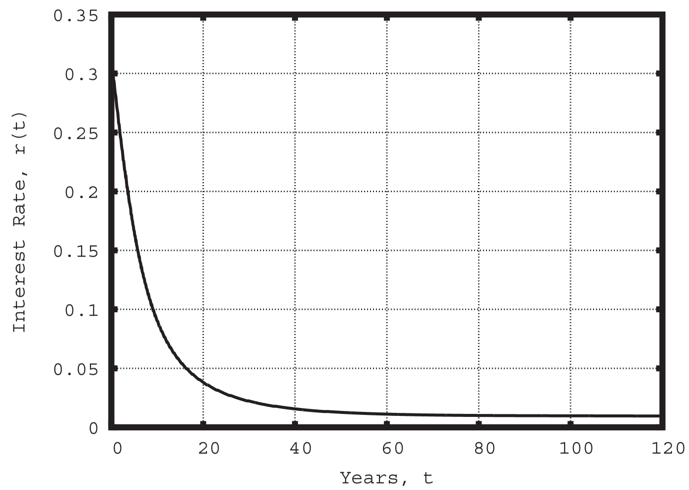
(a) Investments



(b) GDP



(c) Total Factor Productivity= output/investments$^\alpha$



(d) Interest Rate

Results 3/4

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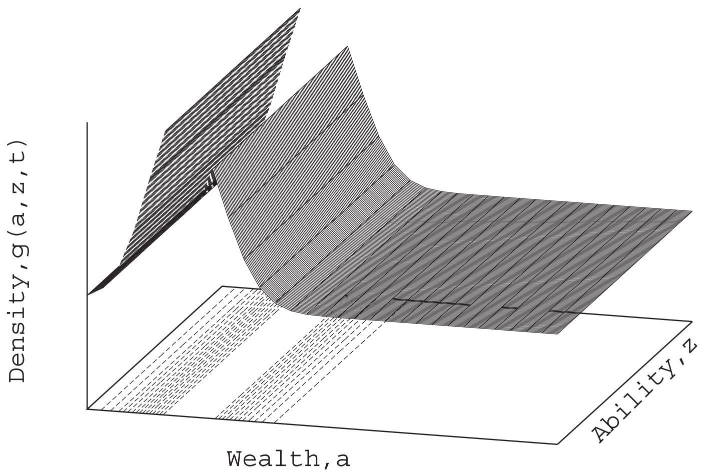
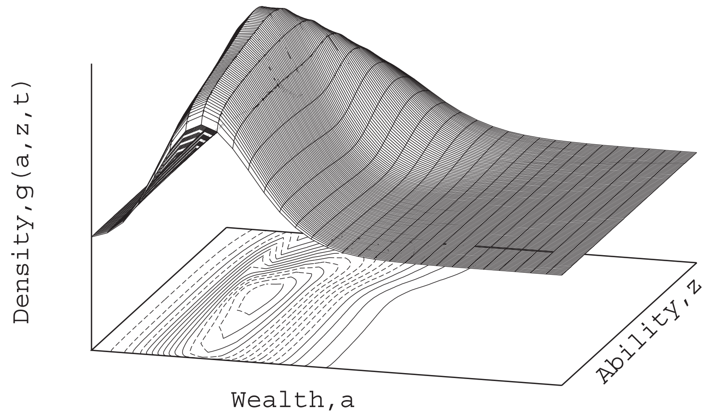
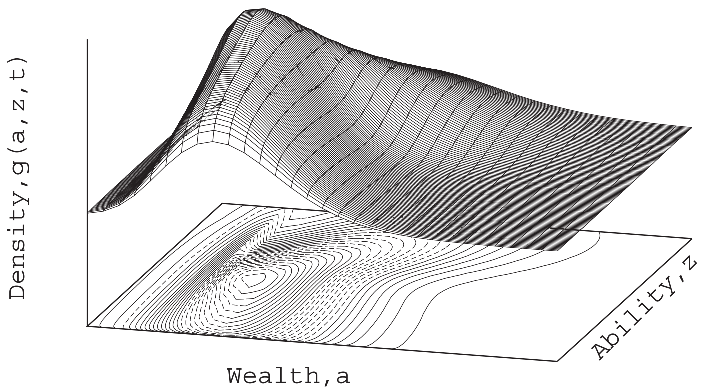
Figure: Evolution of Joint Distribution of Labor Productivity and Wealth



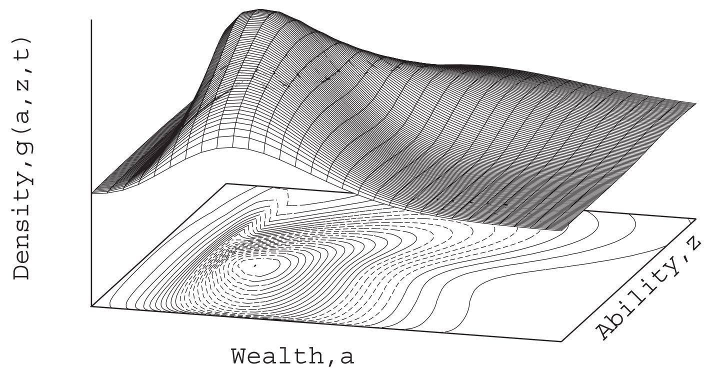
(a) $t = 0$



(b) $t = 10$

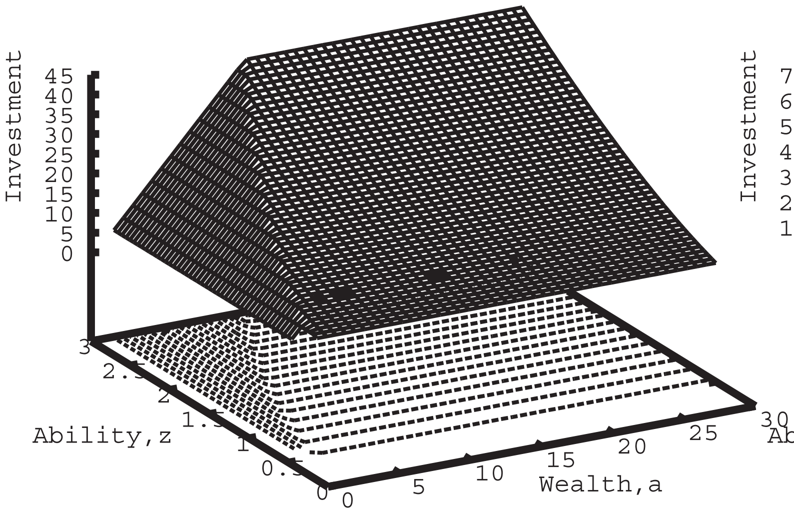


(c) $t = 20$

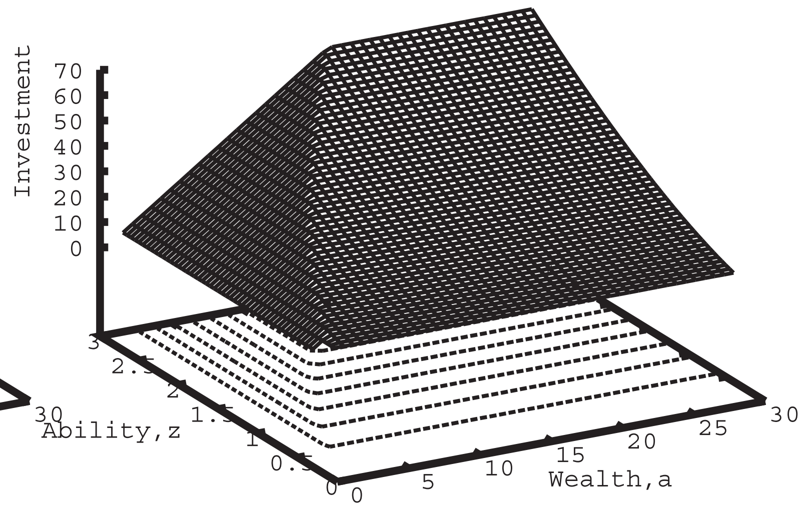


(d) $t = 30$

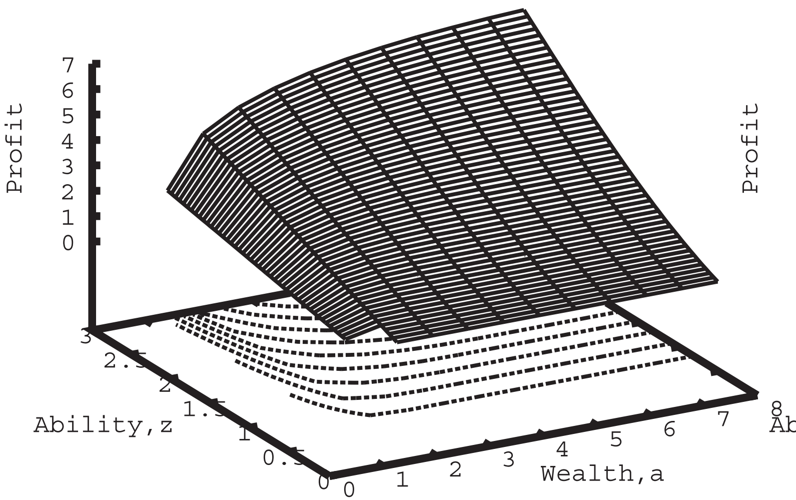
Figure: Evolution of Joint Distribution of Productivity and Wealth



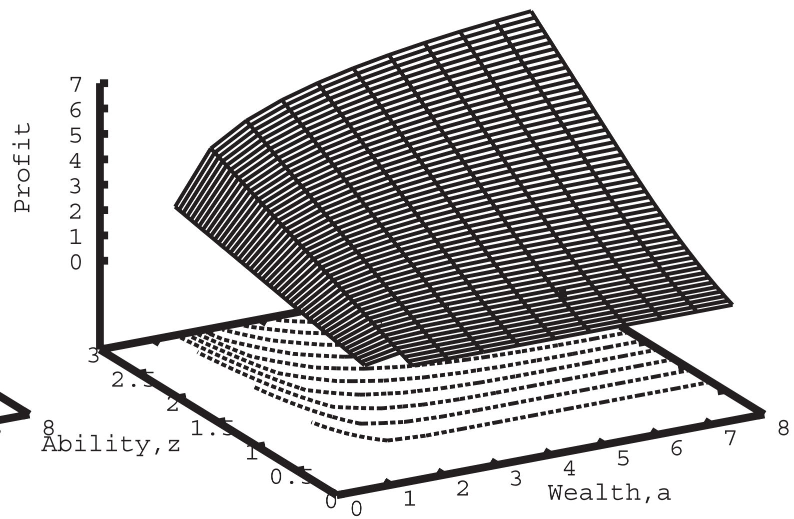
(a) $k(a, z, t)$ at $t = 20$



(b) $k(a, z, t)$ at $t = 40$

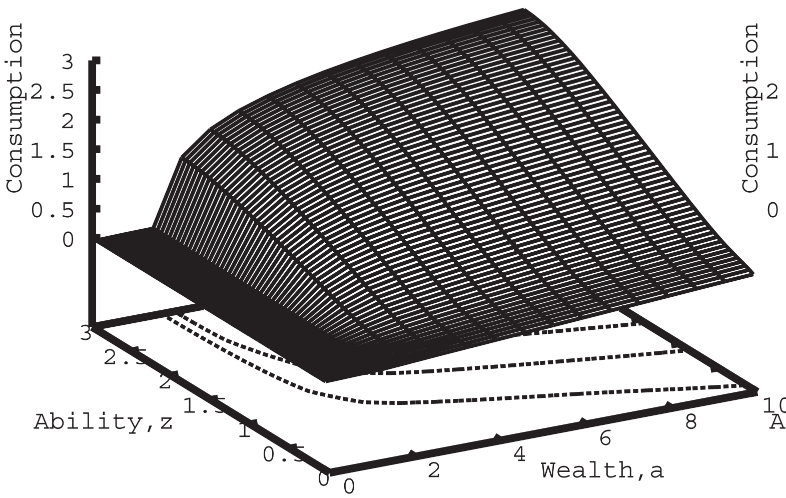


(c) Profits, $\Pi(a, z, t)$ at $t = 20$

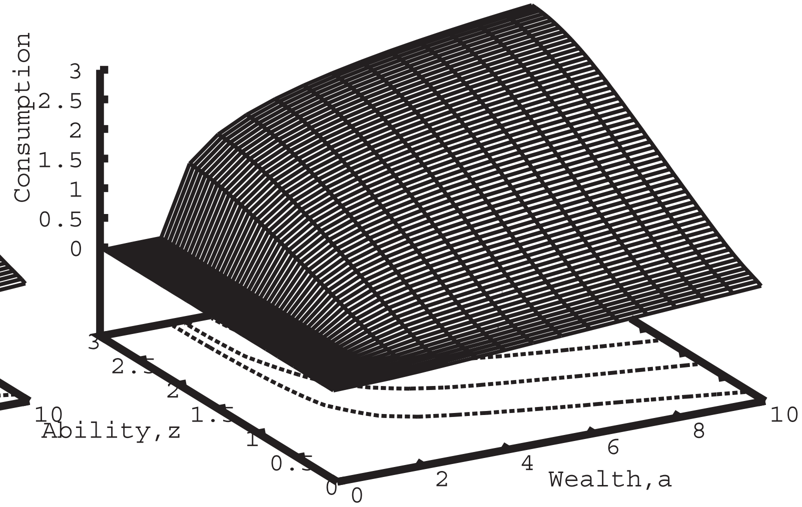


(d) Profits, $\Pi(a, z, t)$ at $t = 40$

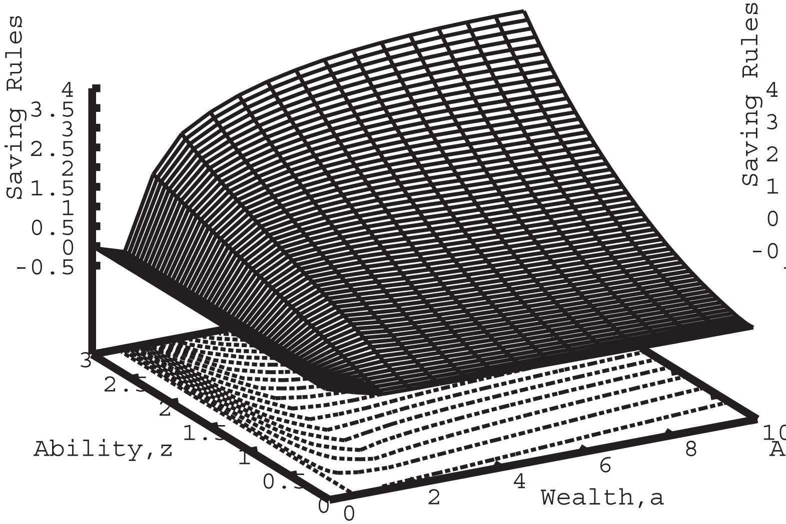
Figure: Capital Use and Profits



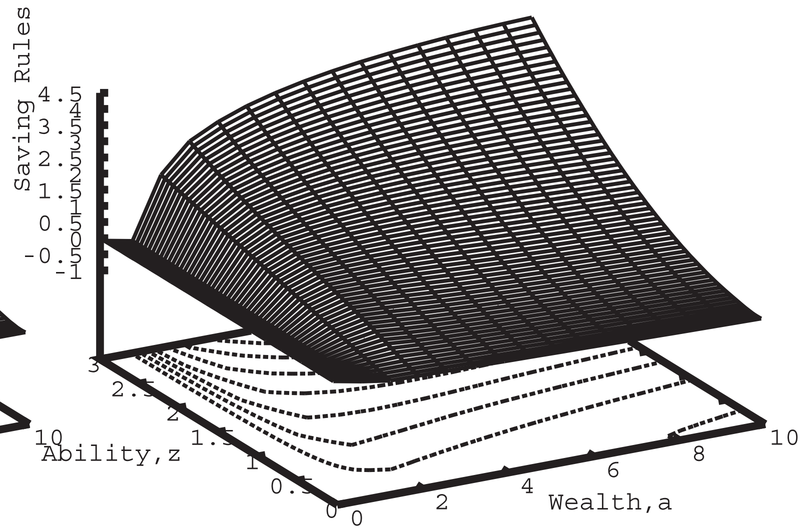
(a) Consumption Policy Function, $c(a, z, t)$ at $t = 20$



(b) Consumption Policy Function, $c(a, z, t)$ at $t = 40$



(c) Savings Policy Function, $\mu(a, z, t)$ at $t = 20$



(d) Savings Policy Function, $\mu(a, z, t)$ at $t = 40$