Mean field games and dynamic demand management in power grids

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- Every agents, at time t is characterized by its temperature $x(t) \in [X_{ON}, X_{OFF}]$, and acts by the control $u \in \{0,1\}$ which stays for OFF/ON respectively.
- The goal is to induce a behavior of the agents in order to stabilize the power network around a reference state, in particular desynchronize (ON/OFF) the agents (Angeli-Kountouriotis, 2012)

$$\begin{cases} x'(s) = \begin{cases} -\alpha(x(s) - X_{ON}) \text{ if } u(s) = 1, \quad t < s < T, \\ -\alpha(x(s) - X_{OFF}) \text{ if } u(s) = 0, \ t < s < T \end{cases} \\ x(t) = x \end{cases}$$

The rate α>0 is given and X_{ON} <X_{OFF} are the steady-state temperatures of the appliances when in state ON or OFF, respectively

We make some hypotheses and passages

$$\sigma = -\alpha(X_{OFF} - X_{ON}), \quad c = \alpha X_{OFF}, \quad X_{OFF} = -X_{ON}$$

$$\begin{cases} x'(s) = -\alpha x(s) + \sigma u(s) + c = f(x(s), u(s)), \; t < s < T, \\ x(t) = x \end{cases}$$

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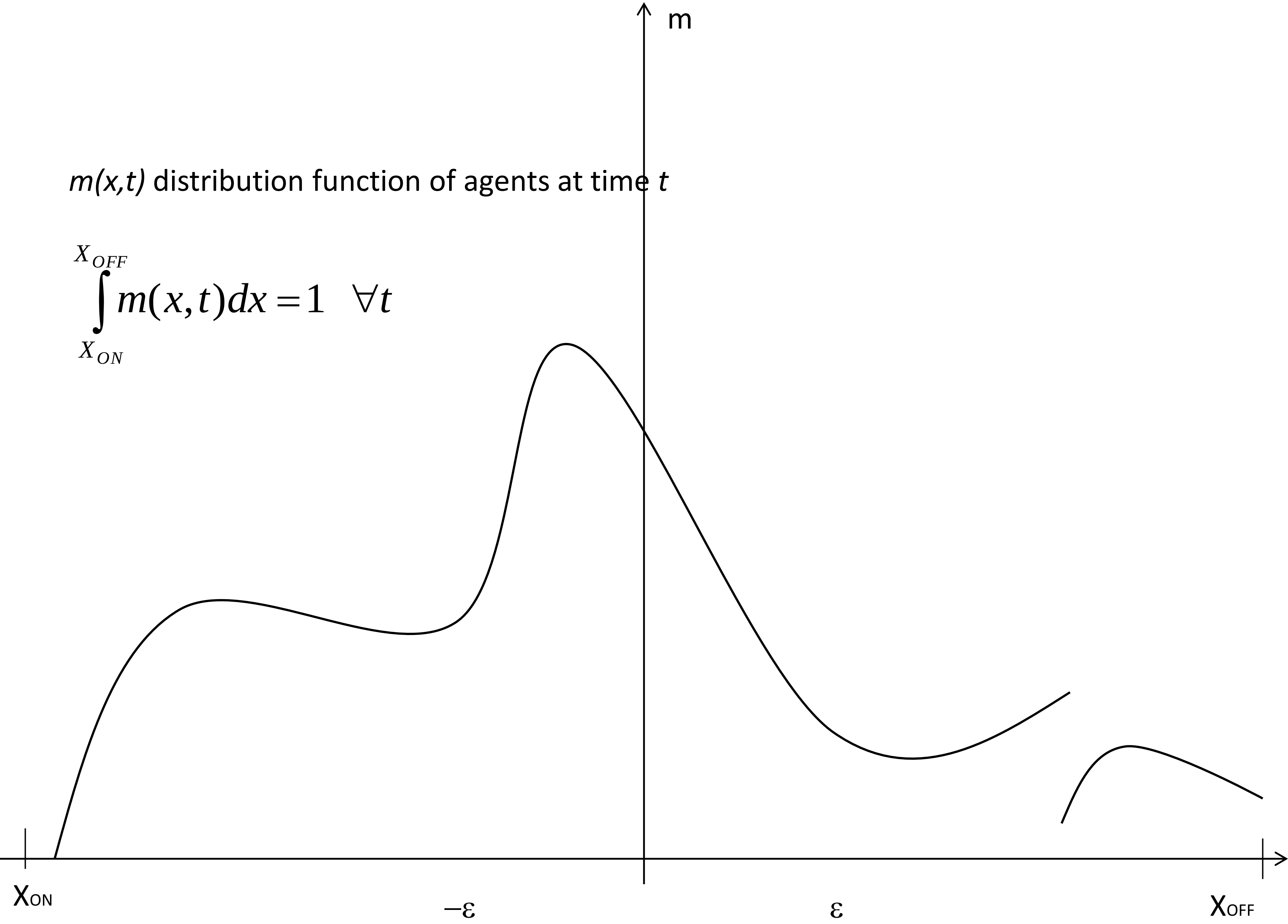
Note that $[X_{ON}, X_{OFF}]$ is invariant for the controlled trajectory, and that the extremes cannot be reached *m(x,t)* distribution function of agents at time *t*

$$\int_{X_{ON}}^{X_{OFF}} m(x,t)\,dx = 1 \quad \forall t$$



m(x,t) distribution function of agents at time *t*

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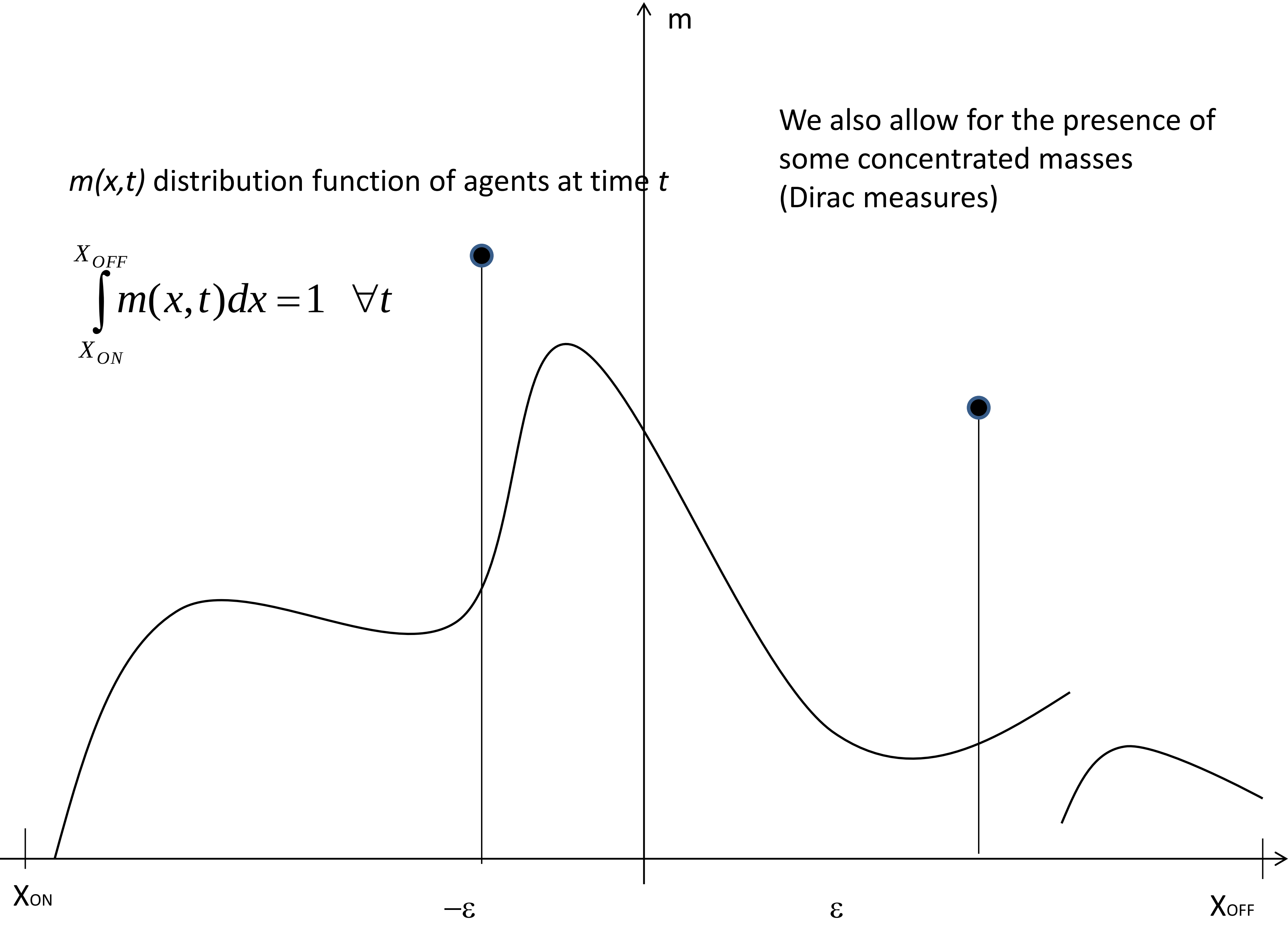
We also allow for the presence of some concentrated masses (Dirac measures)



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- minimization of power: $W_{ON}u + W_{OFF}(1-u)$ where W_{ON} and W_{OFF} are the power consumed when the appliance is ON or OFF respectively.

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- network frequency stabilization: denoting by w and w_{ref} the current frequency and the reference frequency, respectively, frequency stabilization corresponds to a cost of type $u[w-w_{ref}]_+ +(1-u)[w-w_{ref}]_-$ The term $u(s)[w(s)-w_{ref}]_+$ represents a penalty for all those agents that are ON when $w(s) > w_{ref}$; $(1-u(s))[w(s)-w_{ref}]_-$ is a penalty for all those agents that are OFF when $w(s) < w_{ref}$

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- stabilization of the temperature around a comfortable value x_{ref} .
- *Desynchronization: good proportion between ON and OFF agents*

- We make some (simplifying) assumptions

$$w(s) - w_{ref} = -(\overline{m}(s) - \overline{m}_{ref}), \quad \text{where} \quad \overline{m}(s) = \int_{X_{ON}}^{X_{OFF}} x m(x,s) dx$$

$$W_{OFF} = w_{ref} = x_{ref} = \overline{m}_{ref} = 0, \quad r = W_{ON} > 0$$

We consider the following running cost (for *h,k>0* fixed), cost functional (for a given terminal cost Ψ) and value function (depending on the mean temperature)

$$g(x,u,\overline{m}) = ru + qx^2 + h[\overline{m}]_+ u + k[\overline{m}]_-(1-u)$$

$$J(x,t,u(\cdot)) = \int_t^T g(x(s),u(s),\overline{m}(s))ds + \Psi(x(T))$$

$$v(x,t) = \inf_{u(\cdot)} J(x,t,u(\cdot))$$

Every agent wants to minimize J, where the mean $\overline{m}$ is the mean of the actual distribution of temperatures, supposing that all agents optimally behave.

The network manager wants to induce a behavior of the agents such that the mean temperature is as close as possible to the reference one $\overline{m}_{ref} = 0$.

Given the running cost g we want to design the final cost Ψ such that the desired behavior is obtained.

Let $u^*(x,t)$ be the optimal feedback, then the actual distribution m "satisfies" $m_t(x,t) + (f(x, u^*(x,t))m(x,t))_x = 0.$

Denoting the optimal mean control by

$$\bar{u}(t) = \int_{X_{ON}}^{X_{OFF}} u^*(x,t)m(x,t)dx$$

then the mean temperature $\bar{m}$ "satisfies" the equation

$$\bar{m}' = -\alpha\bar{m} + \sigma\bar{u} + c = f(\bar{m}, \bar{u}).$$

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- The network manager is interested in controlling the mean temperature, hence we regard the mean temperature as the solution of the following mean field system which formally results in

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-v_t(x,t) + \sup\limits_{u\in[0,1]}\left\{- f(x,u)v_x(x,t) - g(x,u,\overline{m}(t))\right\} = 0, \text{ in } [X_{ON}, X_{OFF}]\times]0,T], \\
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\end{cases} \\
u^*(x,t) = \arg\max\limits_{u\in[0,1]}\left\{- f(x,u)v_x(x,t) - g(x,u,\overline{m}(t))\right\}, \\
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m(X_{ON},t) = m(X_{OFF},t) = 0, \text{ in } t\in[0,T], \\
m(x,0) = m_0(x), \text{ in } x\in[X_{ON}, X_{OFF}], \\
\int_{X_{ON}}^{X_{OFF}} m(x,t)dx = 1, \text{ in } [0,T],
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u^*(x,t) = \gamma(t)x, \quad \overline{u}(t) = \int_{X_{ON}}^{X_{OFF}} \gamma(t)xm(x,t)dx = \gamma(t)\overline{m}(t)
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g(x,u,\overline{m})=ru+qx^2+h[\overline{m}]_+u+k[\overline{m}]_-(1-u)
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m not separated, non-monotone in m;

bounded controls and states

Let $\wp$ be the set of positive probability measures on $[X_{ON}, X_{OFF}]$ endowed with the weak-star topobgy. A weak solution of

$$\begin{cases} m_t(x,t) + (f(x,u^*(x,t))m(x,t))_x = 0, \text{ in }]X_{ON}, X_{OFF}[\times]0,T[, \\ m(X_{ON},t) = m(X_{OFF},t) = 0, \text{ in } t \in [0,T], \\ m(x,0) = m_0(x), \text{ in } x \in [X_{ON}, X_{OFF}], \\ \int_{X_{ON}}^{X_{OFF}} m(x,t)dx = 1, \text{ in } [0,T], \end{cases}$$

is a continuous function $m:[0,T] \to \wp,\ t \mapsto m[t],$ such that

$$\int_{X_{ON}}^{X_{OFF}} \varphi(x,0)dm_0 + \int_0^T \int_{X_{ON}}^{X_{OFF}} \left[\varphi_t(x,t) + f(x,u^*(x,t))\varphi_x(x,t)\right] dm[t]dt = 0,$$

$$\forall \varphi \in C_c^1([X_{ON}, X_{OFF}] \times [0,T[)$$

We expect solutions of the form

$$m[t] = \tilde{m}(\cdot, t) + \sum_{i=1}^{\ell} \gamma_i(t)\delta_{y_i(t)}$$

with $\tilde{m}, \gamma_i \in L^1$, y_i continuous. Hence, we have to give a meaning to the following duality-integral, when the optimal feedback u^* is dicontinuous

$$\int_0^T \int_{X_{ON}}^{X_{OFF}} \left[\varphi_t(x,t) + f(x, u^*(x,t))\varphi_x(x,t)\right] dm[t]dt$$

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We require that u^* is defined almost everywhere by

$$u^*(x,t) = \arg\max_{u\in[0,1]} \left\{- f(x,u)v_x(x,t) - g(x,u,\overline{m}(t)\right\}$$

and that, where the formula does not define,

it can be anyway defined in a uniquely manner

in such a way that the optimal trajectory exists for all time.

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A solution is a continuous function $\overline{m} : [0,T] \to [0,+\infty[$

which is a fixed point of the procedure

$\overline{m} \to v \to u^* \to m \to \overline{u} \to \overline{m}$

where u^* is as required.

Note that the last ODE is also solved in a distributional sense.

- Now, we want to construct a suitable terminal cost Ψ such that, at least starting from some initial data, there is a solution $\overline{m}$ constantly equal to zero.

Take $\overline{m} \equiv 0$ and consider the corresponding Bellman equation

$$-v_t + \alpha v_x x - c v_x - q x^2 + [-\sigma v_x - r]_+ = 0, \quad v(x,T) = \Psi(x)$$

consider the stationary equation

$$\alpha \Psi_x x - c \Psi_x - q x^2 + [-\sigma \Psi_x - r]_+ = 0$$

$$-\sigma \Psi_x - r \leq 0 \Rightarrow \Psi_x = \frac{q x^2}{\alpha x - c} \quad \text{in } [X_{ON}, X_{OFF}[,$$

$$-\sigma \frac{q x^2}{\alpha x - c} - r \leq 0. \quad \text{Take } \Psi^0 \text{ a primitive.}$$

$$-\sigma \Psi_x - r > 0 \Rightarrow \Psi_x = \frac{q x^2 + r}{\alpha x + c} \quad \text{in }]X_{ON}, X_{OFF}],$$

$$-\sigma \frac{q x^2 + r}{\alpha x + c} - r > 0. \quad \text{Take } \Psi^1 \text{ a primitive.}$$

It can be then seen that Ψ^i is the value function of the control problem in $[X_{ON}, X_{OFF}]$ with cost

$$\int_t^T g(x(s),u(s),0)ds + \Psi^i(x(T))$$

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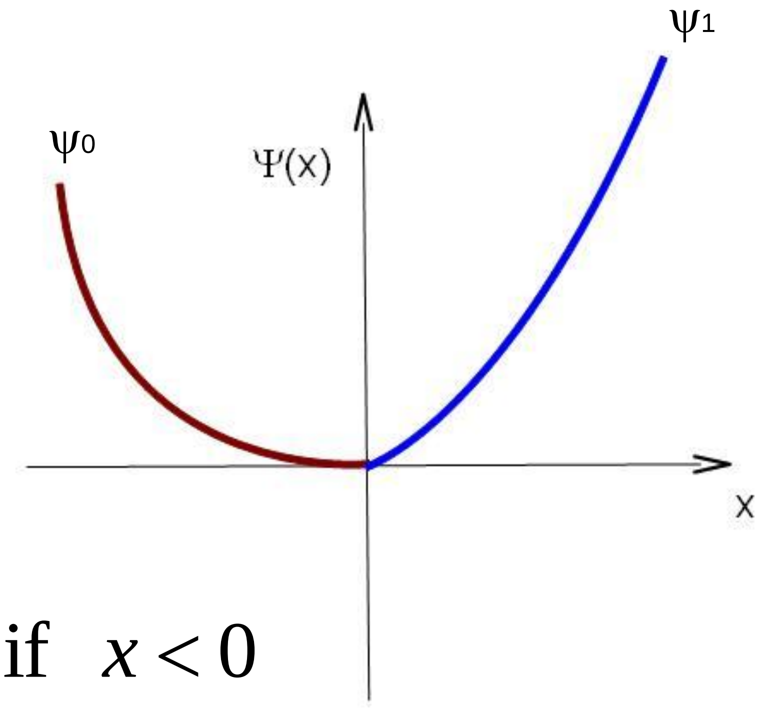
$$\Psi^0 \Rightarrow u^* \equiv 0,\ \Psi^1 \Rightarrow u^* \equiv 1.$$



$$\Psi(x) = \begin{cases} \Psi^0(x) \quad \text{if} \quad x < 0 \\ \Psi^1(x) \quad \text{if} \quad x > 0 \\ \Psi^0(0) = \Psi^1(0) = 0 \end{cases}$$

We use this as terminal cost in our originary problem.

The value function is

$$v(x,t) = \begin{cases} \Psi(x) & \text{if } \; T - t < t^*(x), \\ \Psi(x) + \dfrac{r}{2}(T - t^*(x) - t) & \text{otherwise} \end{cases}$$

where $t^*(x)$ is the arrival time at $x = 0$,

under the optimal feedback control

$$u^*(x,t) = \begin{cases} 1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x < 0 \\ \dfrac{1}{2} & \text{if} \quad x = 0 \end{cases}$$

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$$x>0 \Rightarrow f(x,1)<0,\ \ x<0 \Rightarrow f(x,0)>0,\ \ f(0,\tfrac{1}{2})=0$$

For every intial state x the optimal trajectory exists for all time and coverges to zero, remaining there, when reached;

If the initial distribution of temperatures m_0 is symmetric with respect to $x=0$ (and hence zero-mean $\overline{m}_0=0$) then it remains symmetric and the mean optimal control $\overline{u}$ is constantly equal to $\frac{1}{2}$;

If the initial distribution m_0 is symmetric then the solution of

$$\overline{m}'=-\alpha\overline{m}+\frac{\sigma}{2}+c,\ \ \overline{m}(0)=0,$$

is $\overline{m}\equiv 0.$

Let m_0 be symmetric and absolutely continuous.

Then agents accumulate at $x = 0$.

Let $\tilde{m}(\cdot,\cdot)$ be the solution of

$$\begin{cases} \tilde{m}_t(x,t) + (f(x,0)\tilde{m}(x,t))_x = 0 & \text{in } [X_{ON,} 0[\times]0,T[, \\ \tilde{m}_t(x,t) + (f(x,1)\tilde{m}(x,t))_x = 0 & \text{in }]0, X_{OFF}]\times]0,T[\\ \tilde{m}(x,0) = m_0(x) \end{cases}$$

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The weak solution of the Kolomogorov equation is the zero-mean function

$$m[t] = \tilde{m}(\cdot,t) + \gamma(t)\delta_0, \quad \text{where } \gamma(t) = 1 - \int_{X_{ON}}^{X_{OFF}} \tilde{m}(x,t)dx,$$

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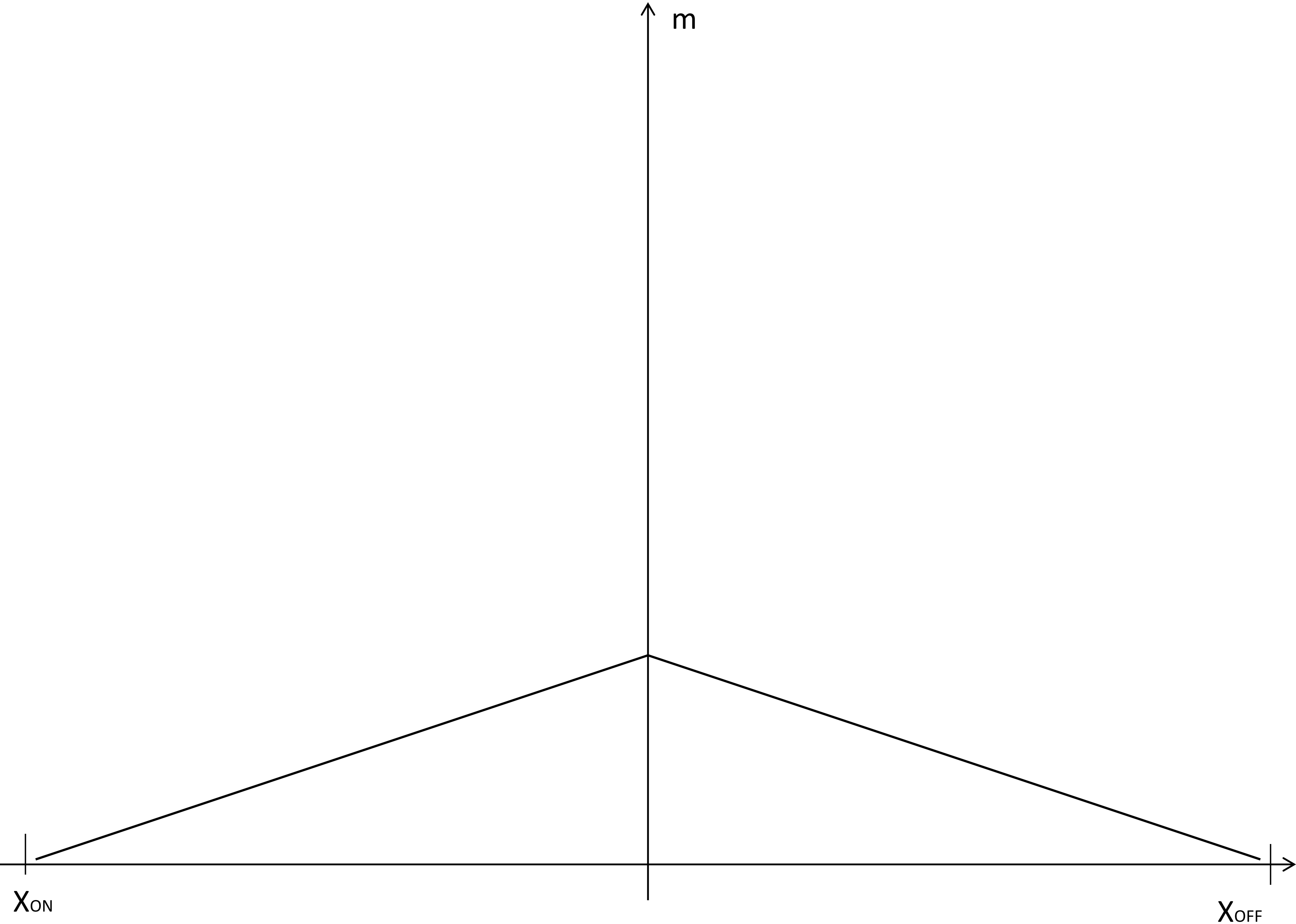
Let $\tilde{m}(\cdot,\cdot)$ be the solution of

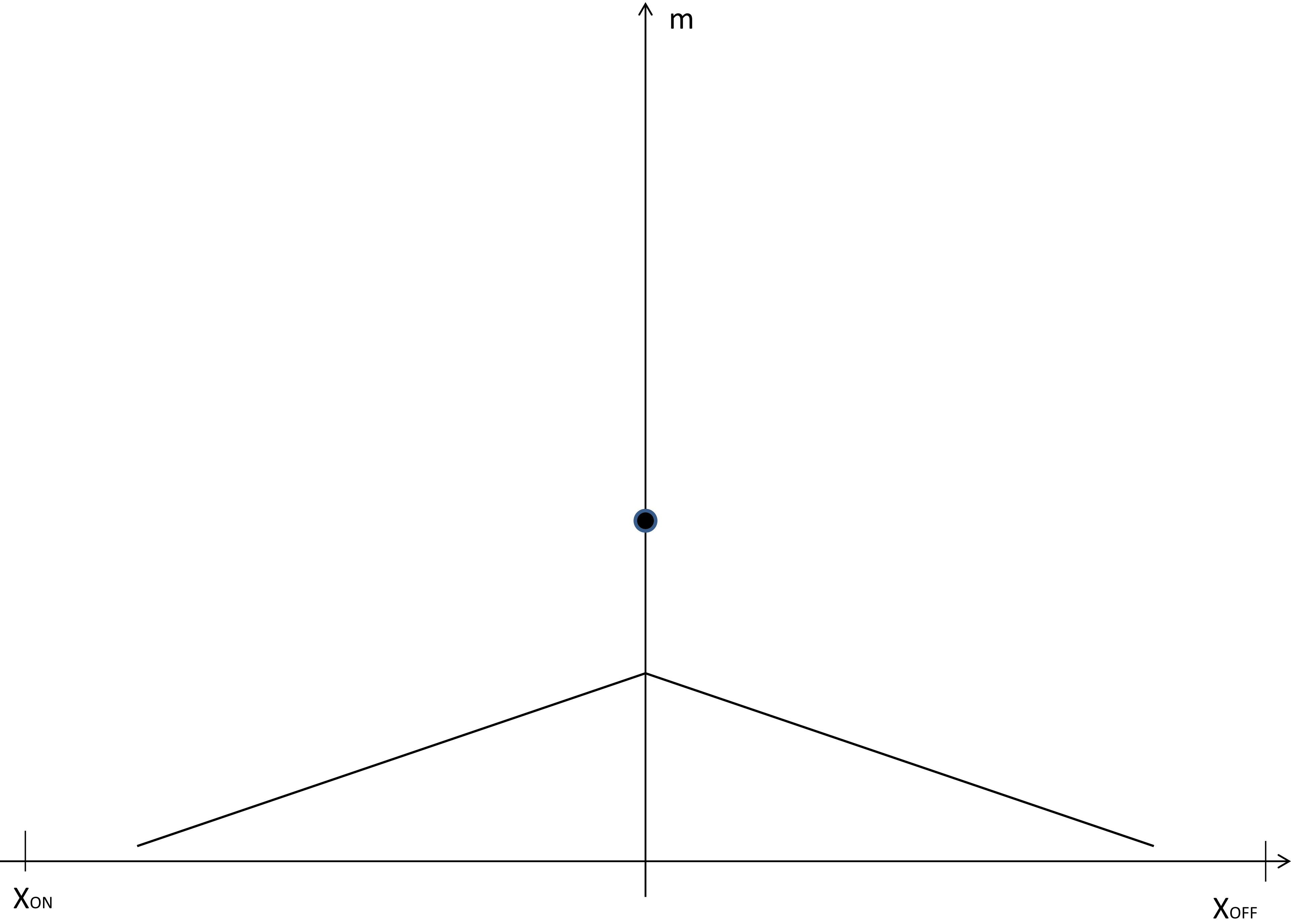
$$\begin{cases} \tilde{m}_t(x,t) + (f(x,0)\tilde{m}(x,t))_x = 0 & \text{in } [X_{ON,}0[\times]0,T[, \\ \tilde{m}_t(x,t) + (f(x,1)\tilde{m}(x,t))_x = 0 & \text{in }]0,X_{OFF}]\times]0,T[\\ \tilde{m}(x,0) = m_0(x) \end{cases}$$

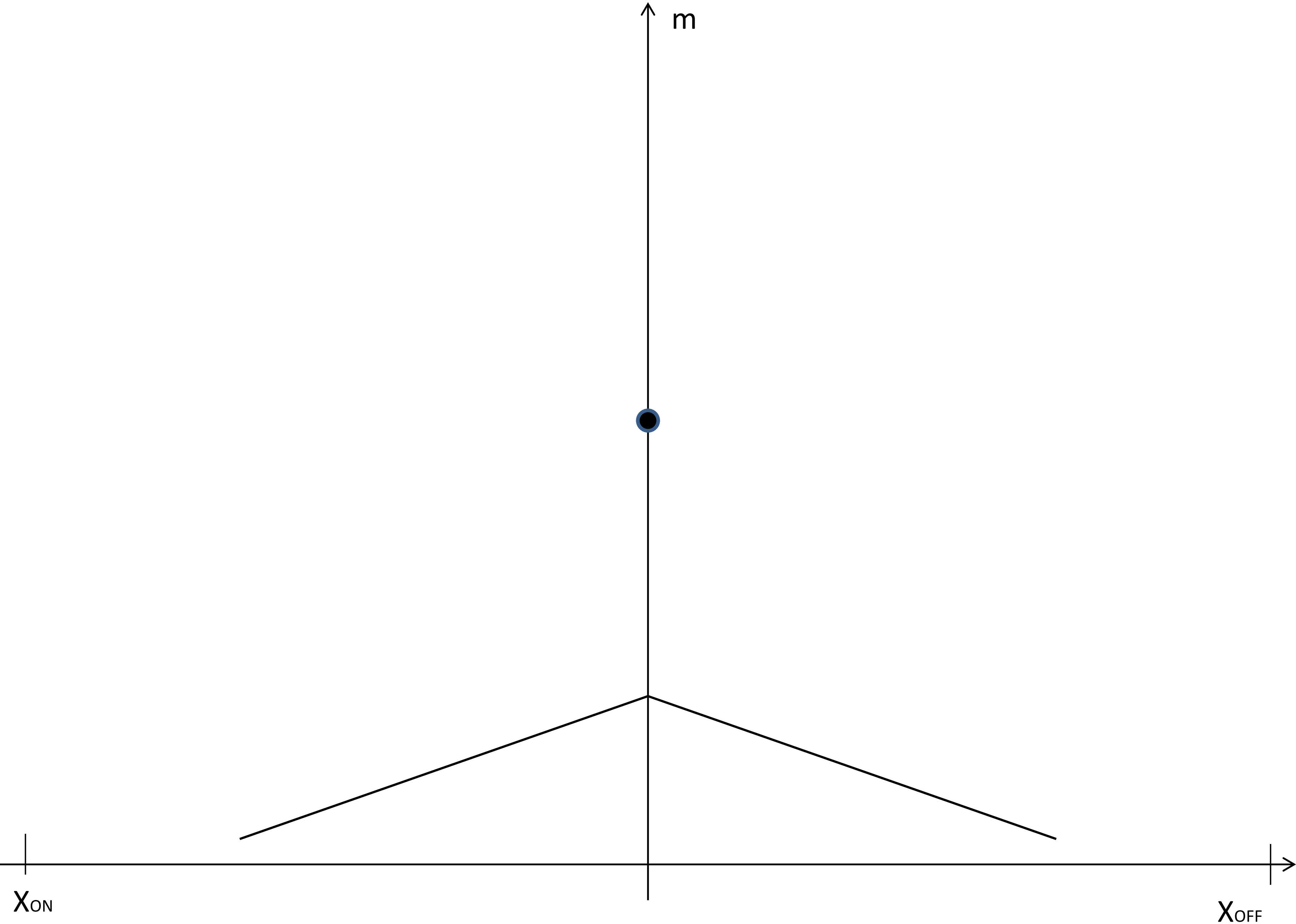
The weak solution of the Kolomogorov equation is the zero-mean function

$$m[t] = \tilde{m}(\cdot,t) + \gamma(t)\delta_0, \quad \text{where } \gamma(t) = 1 - \int_{X_{ON}}^{X_{OFF}} \tilde{m}(x,t)dx,$$

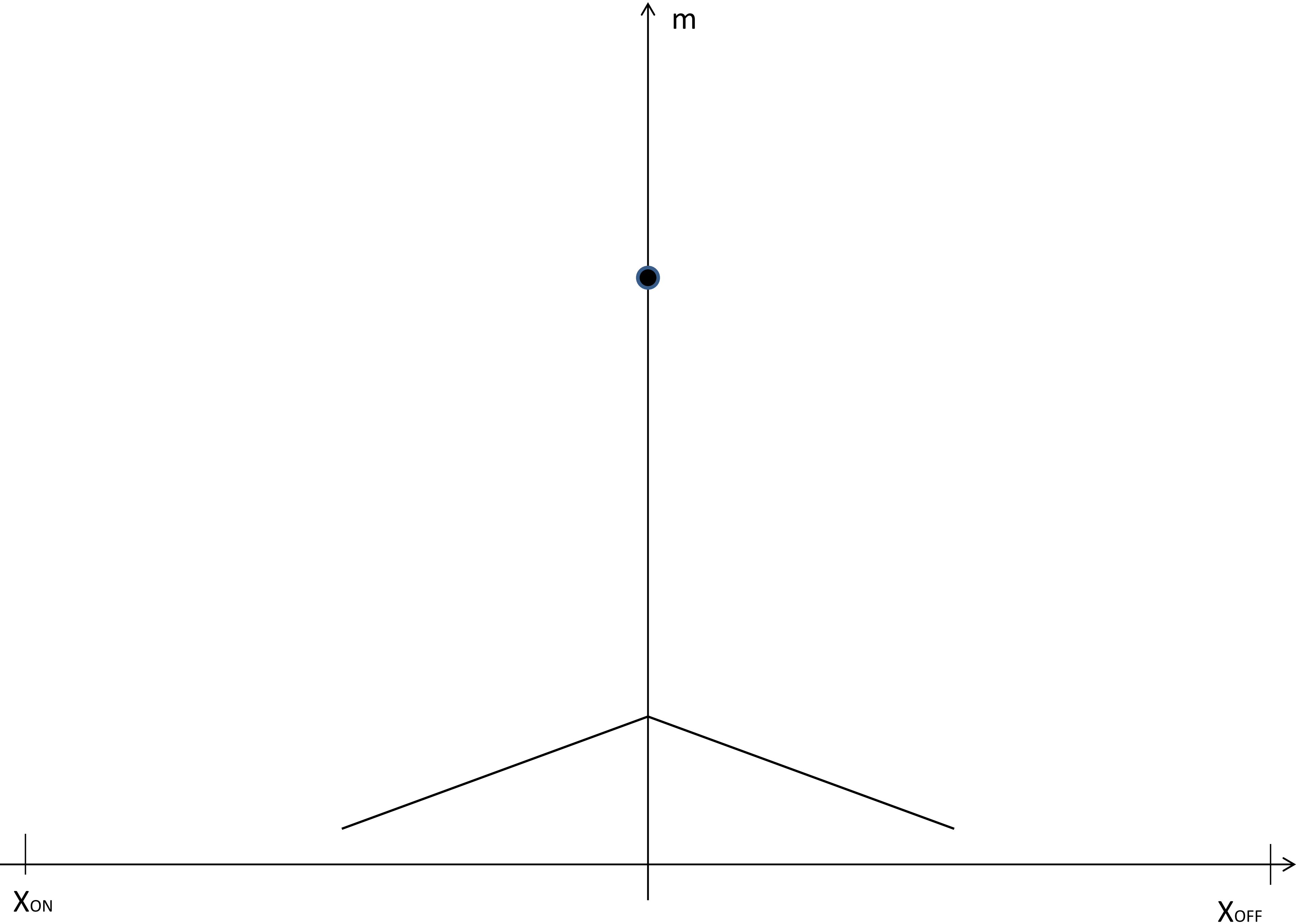
$$f(x,0) = -f(x,1), \qquad f(0,u^*(0,t)) = f(0,1/2) = 0$$

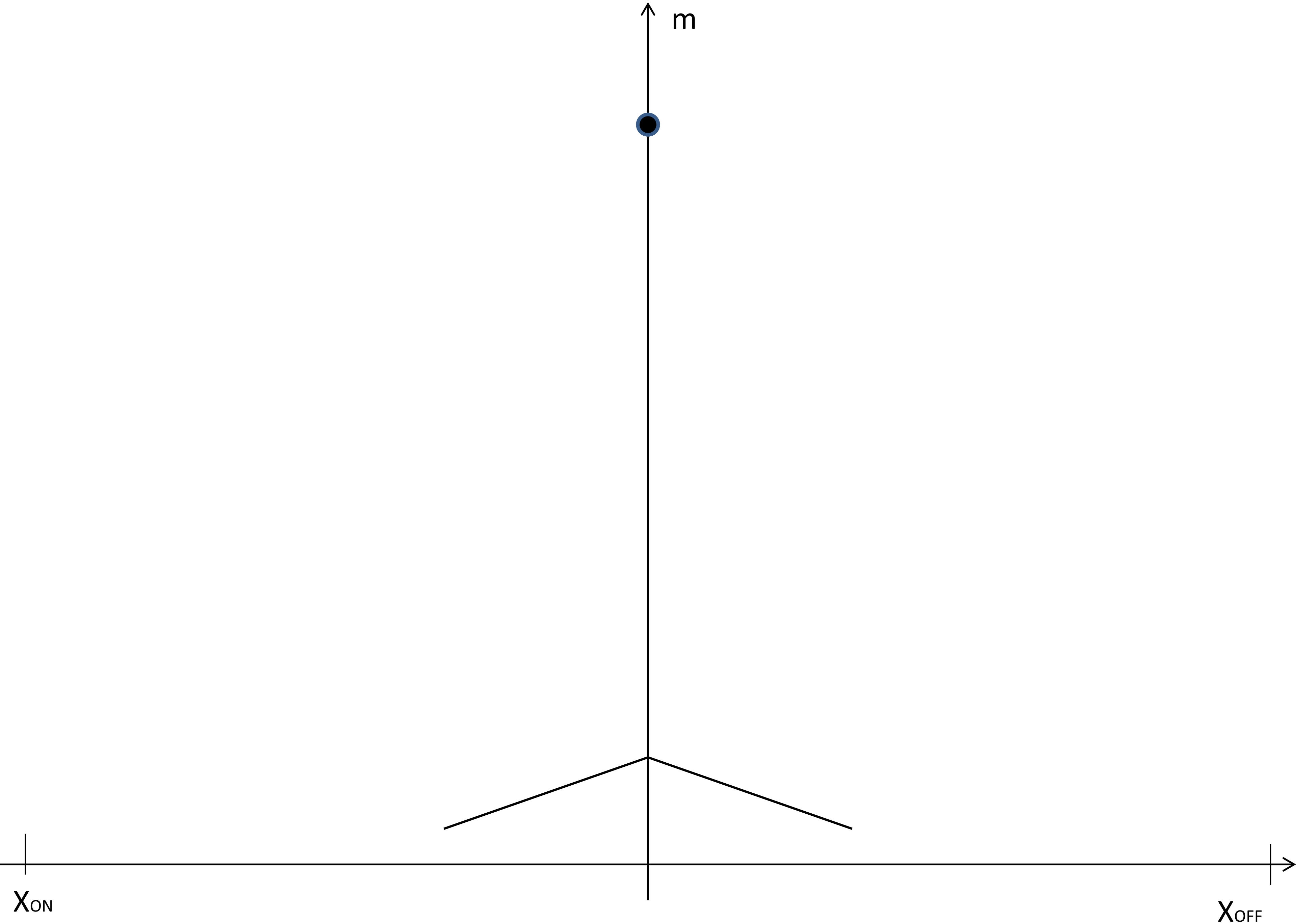

m
X ON
X OFF

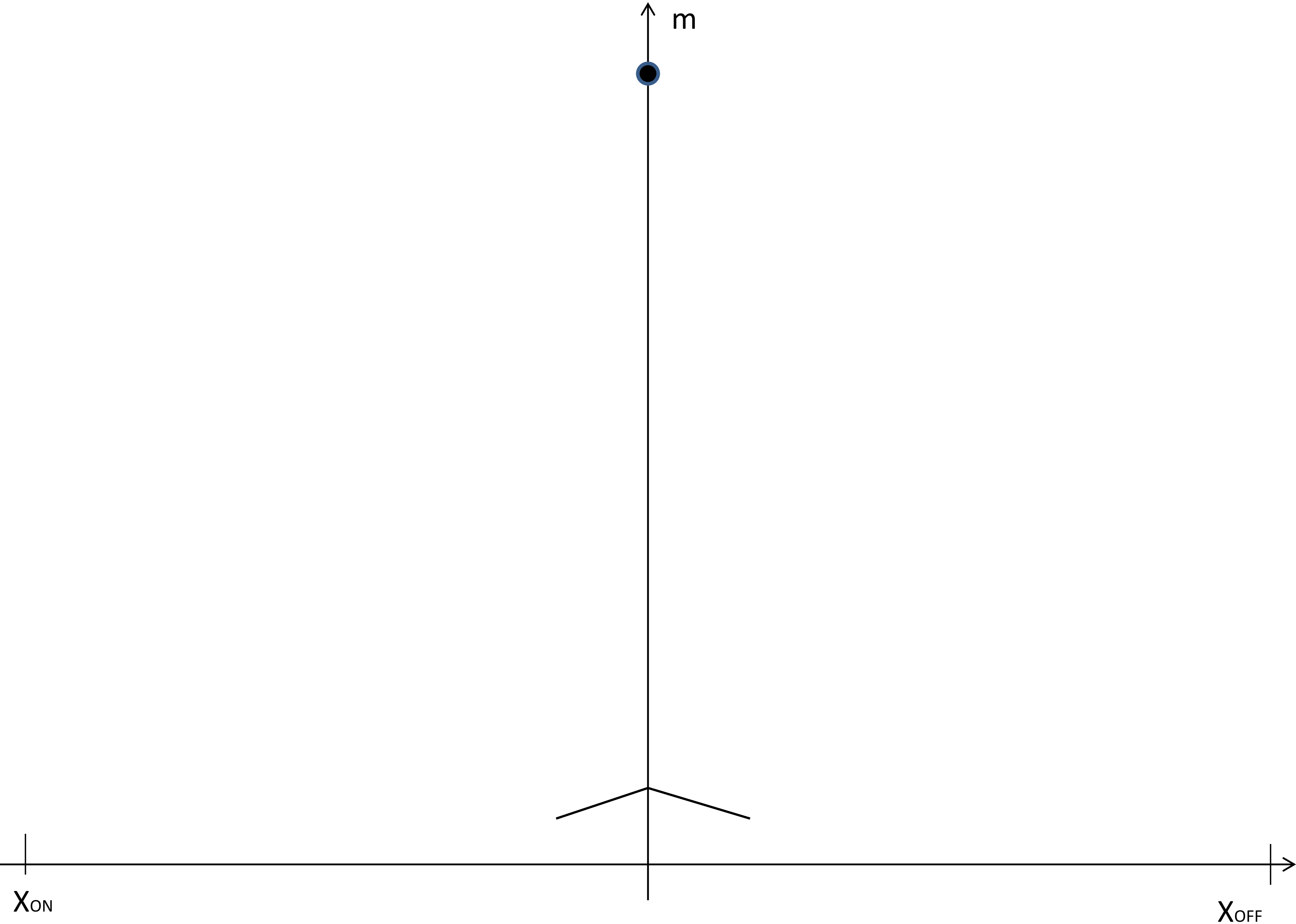

m
XON
XOFF

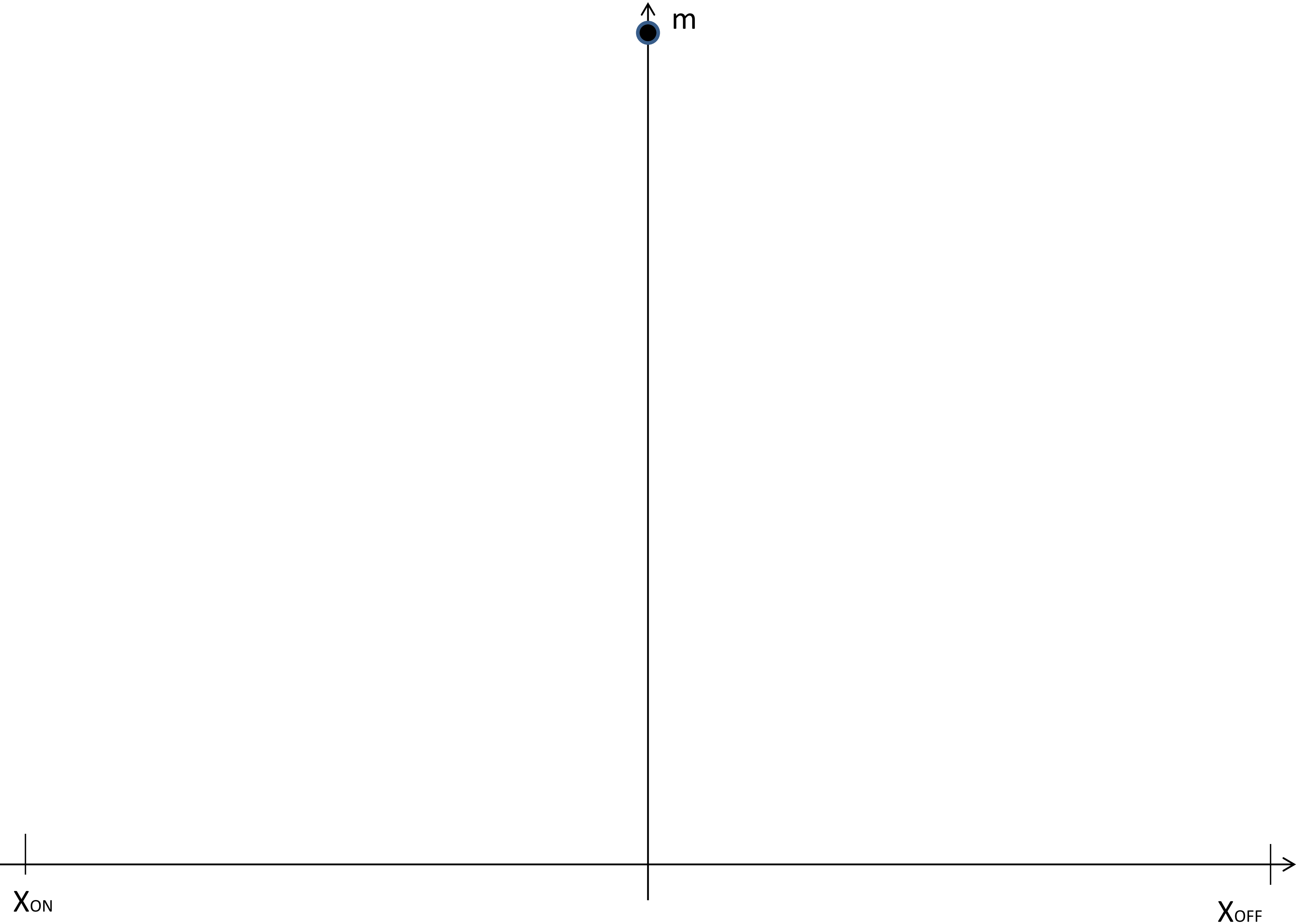


m
XON
XOFF


m
X_{ON}
X_{OFF}


m
XON
XOFF

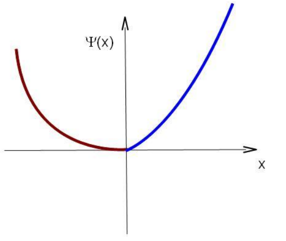

m
X ON
X OFF

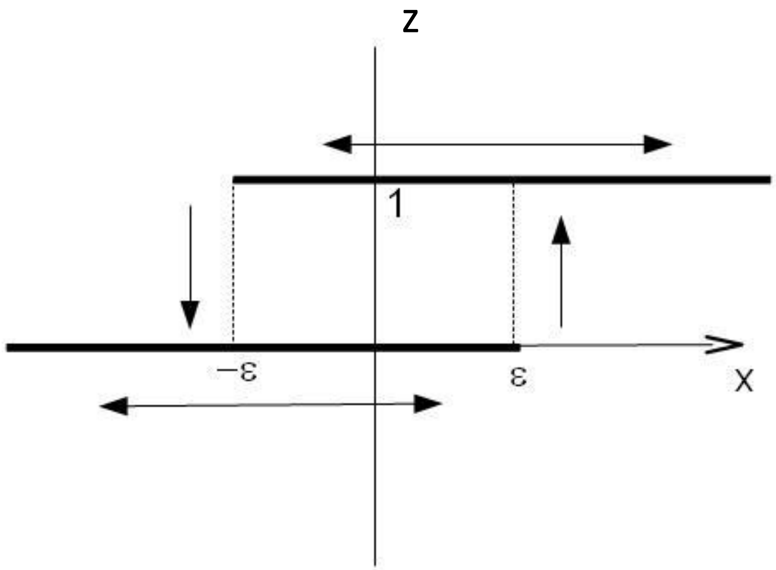

m
X_{ON}
X_{OFF}

All agents tend to the reference temperature $x = 0$. For $x = 0$, the optimal feedback $u=1/2$ stabilizes the optimal trajectories and the mean in 0 and means that the agents at $x = 0$ are in the state ON with probability $1/2$.

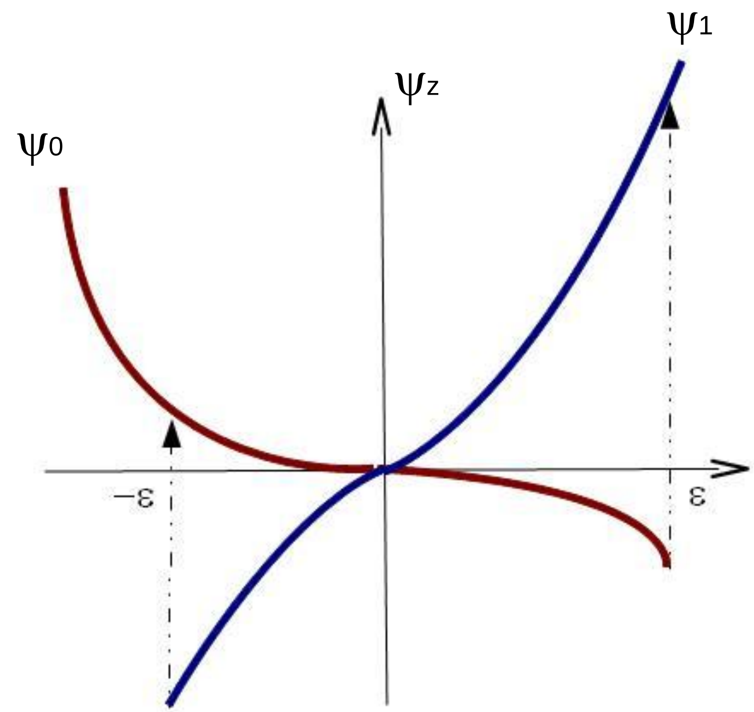
- At a macroscopic level the agents are not all in the ON or OFF state at the same time (desynchronized). At a microscopic level, looking at every single agent, this induces a fast switching ON/OFF infinitely many times. Such a behavior is undesirable as well as unrealizable in reality. We then change the terminal cost in order to force the agents to avoid fast switching while maintaining the desynchronization.

- The fast switching behavior is due to the fact that in the terminal cost Ψ we have only one threshold, $x = 0$, where the agents switch from *0* to *1* and back. Hence we split such threshold in two different thresholds, one determining the switches from 0 to 1 and the other one for the switches in the opposite direction. That is we insert a hysteretic thermostatic rule in the mathematical model



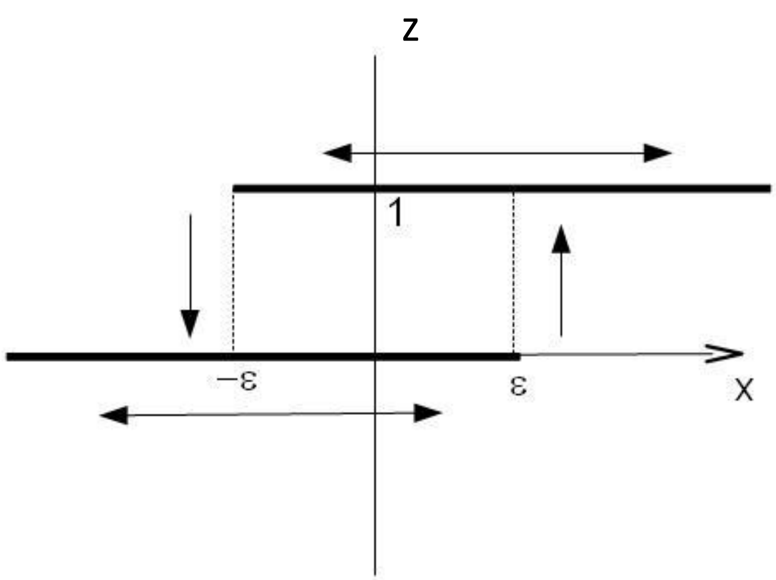


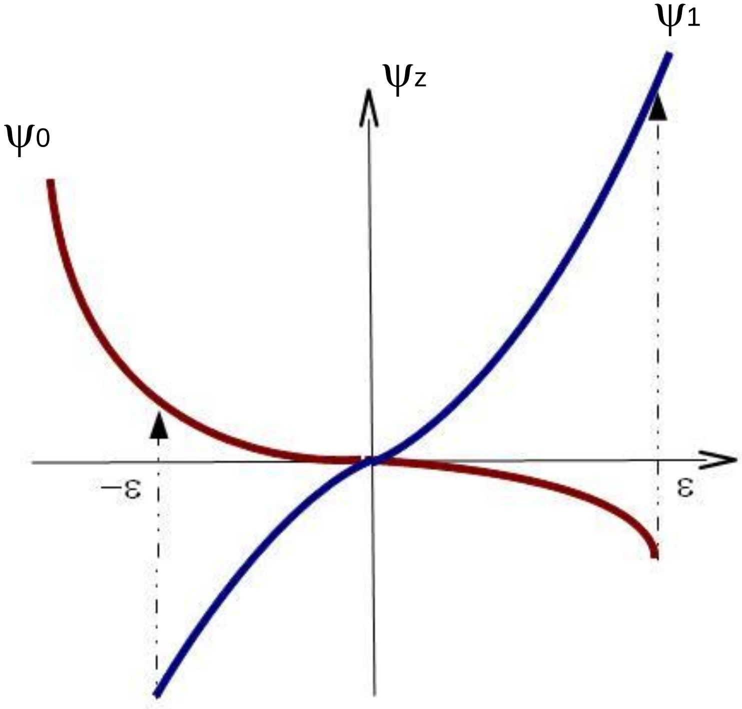
z
1
−ε
ε
x



ψ1
ψz
ψ0
−ε
ε

The new state variable is $(x, z, \eta_{01}, \eta_{10})$
where η_{01} is the number of switches from 0 to 1,
η_{10} the number of switches from 1 to 0.





The new state variable is $(x, z, \eta_{01}, \eta_{10})$

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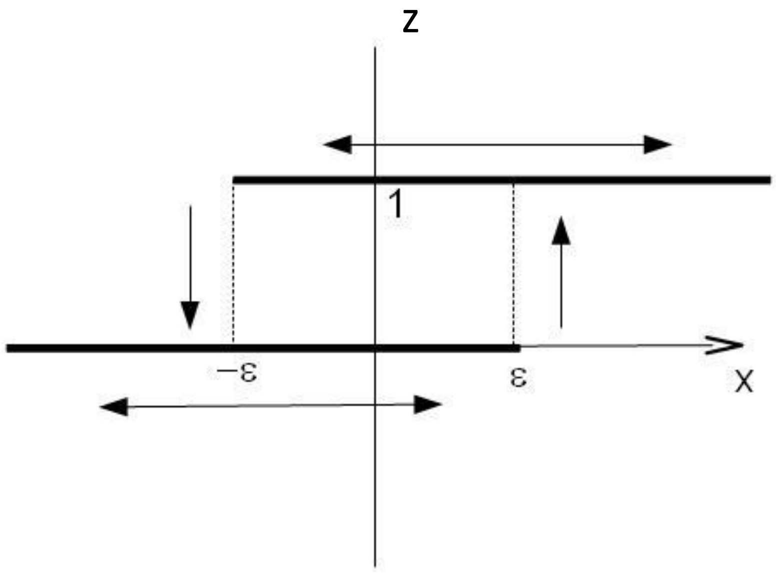
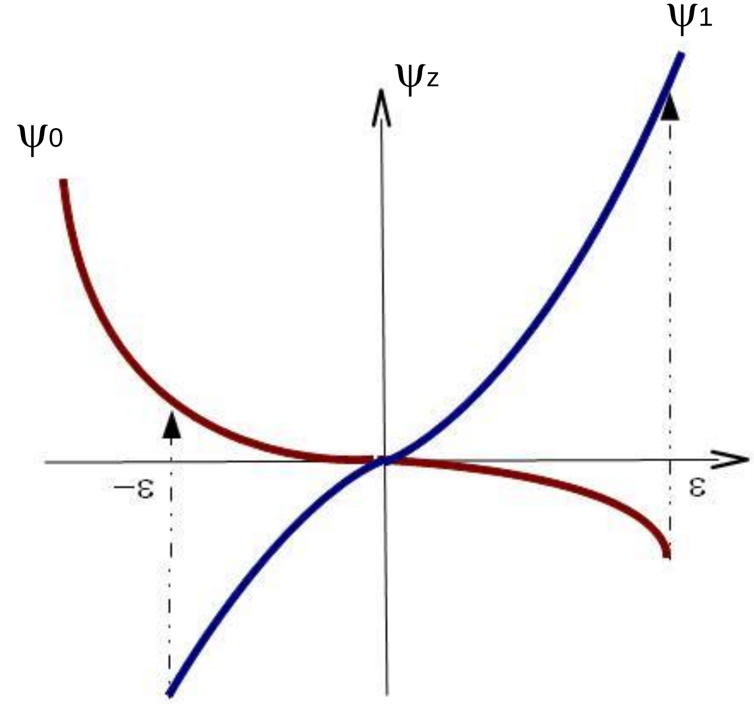
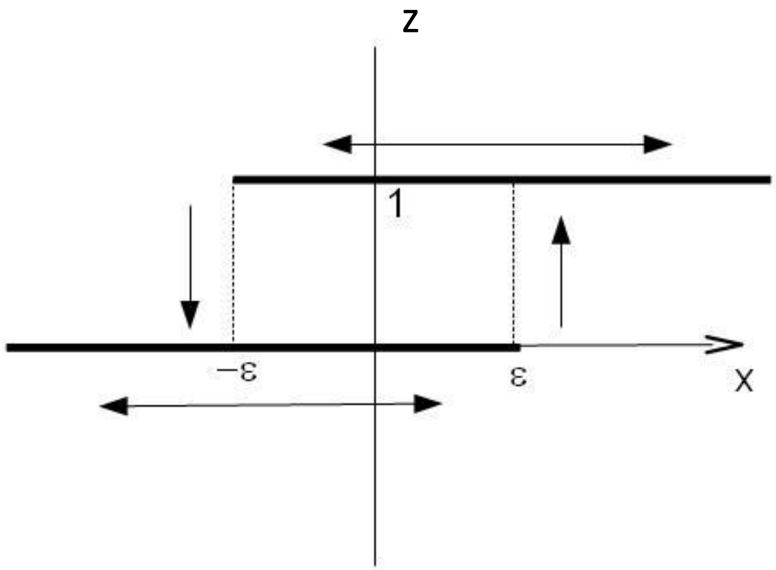
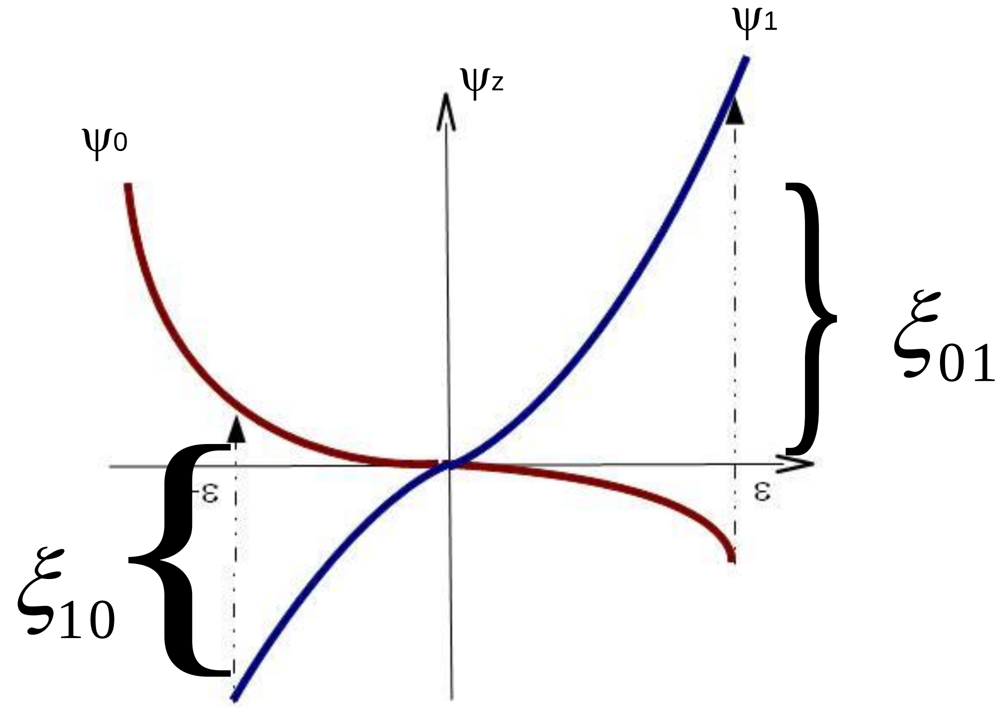




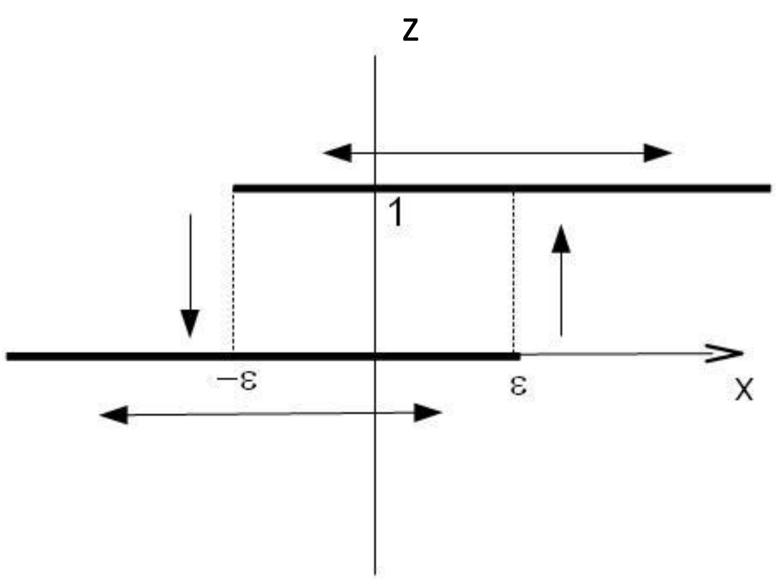
$$\tilde{\Psi}(x(T), z(T), \eta_{01}(T), \eta_{10}(T)) = \Psi^{z(T)}(x(T)) - \xi_{01}\eta_{01}(T) - \xi_{10}\eta_{10}(T)$$

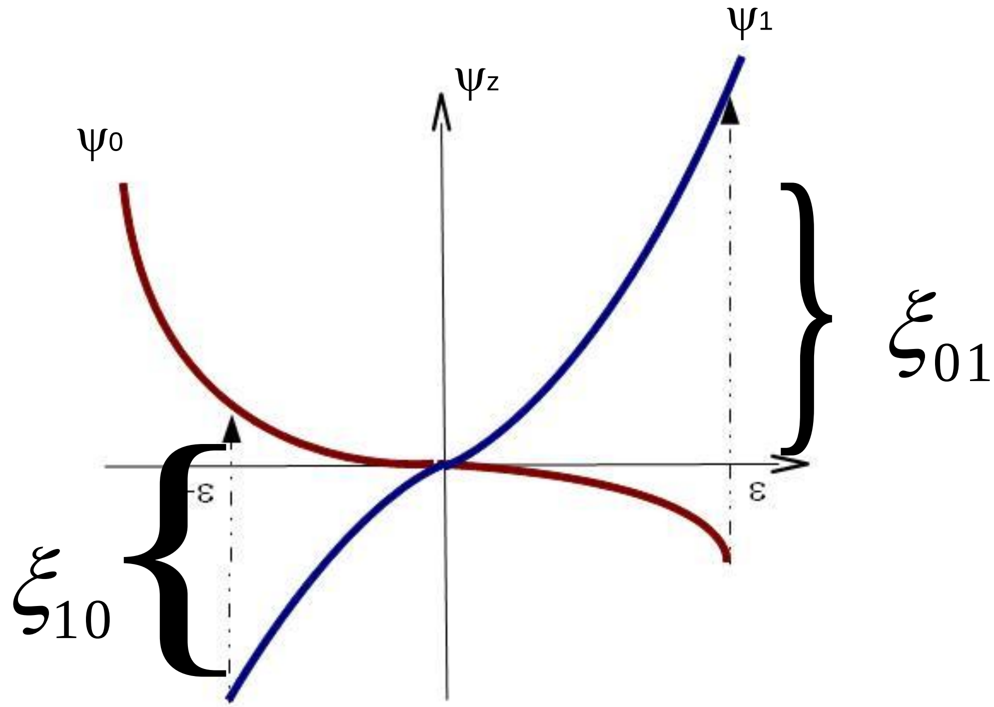




$$\widetilde{\Psi}(x(T), z(T), \eta_{01}(T), \eta_{10}(T)) = \Psi^{z(T)}(x(T)) - \xi_{01}\eta_{01}(T) - \xi_{10}\eta_{10}(T)$$

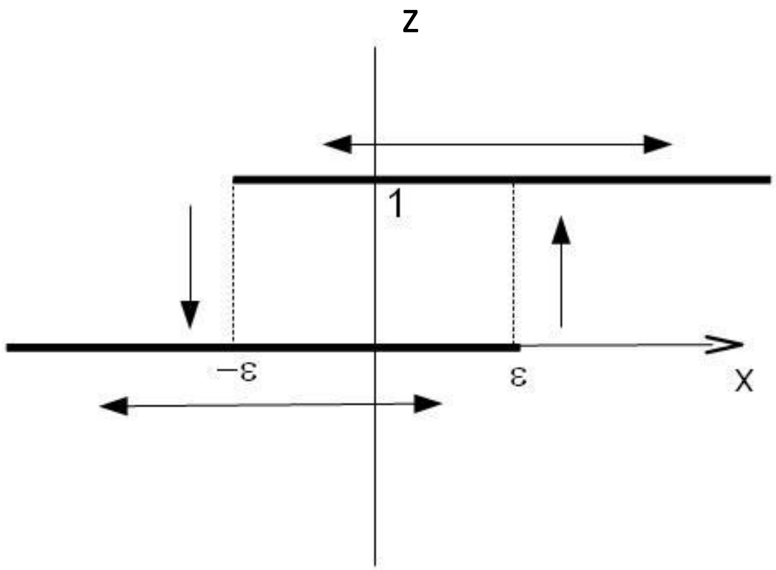
The feedback law $u(x, z, \eta_{01}, \eta_{10}, t) = z$ is optimal.

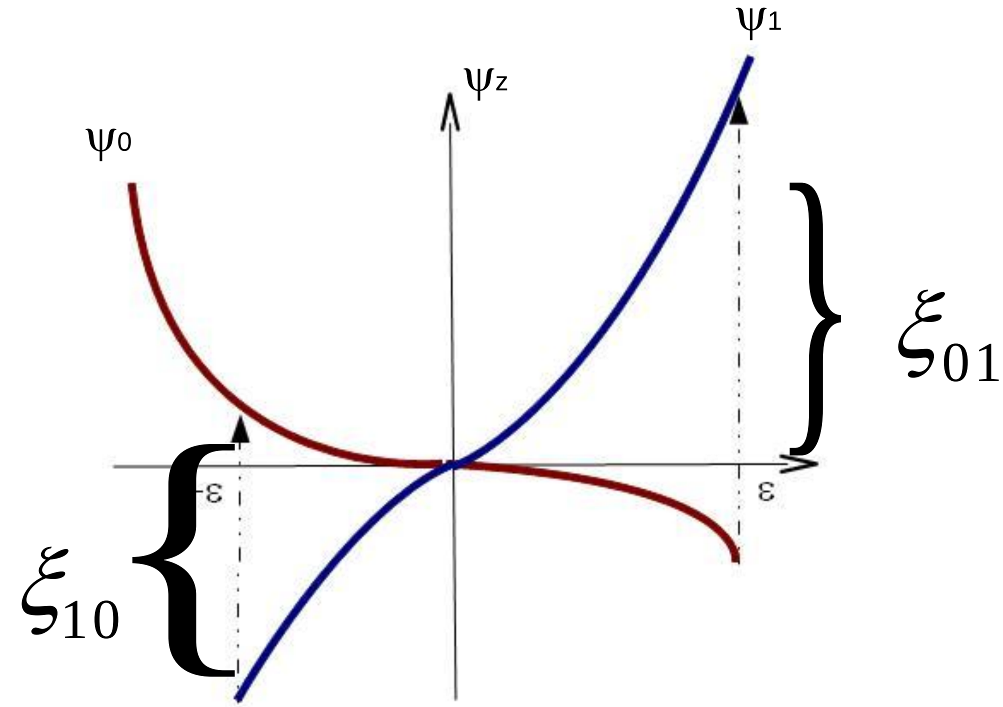




$$\tilde{\Psi}(x(T), z(T), \eta_{01}(T), \eta_{10}(T)) = \Psi^{z(T)}(x(T)) - \xi_{01}\eta_{01}(T) - \xi_{10}\eta_{10}(T)$$

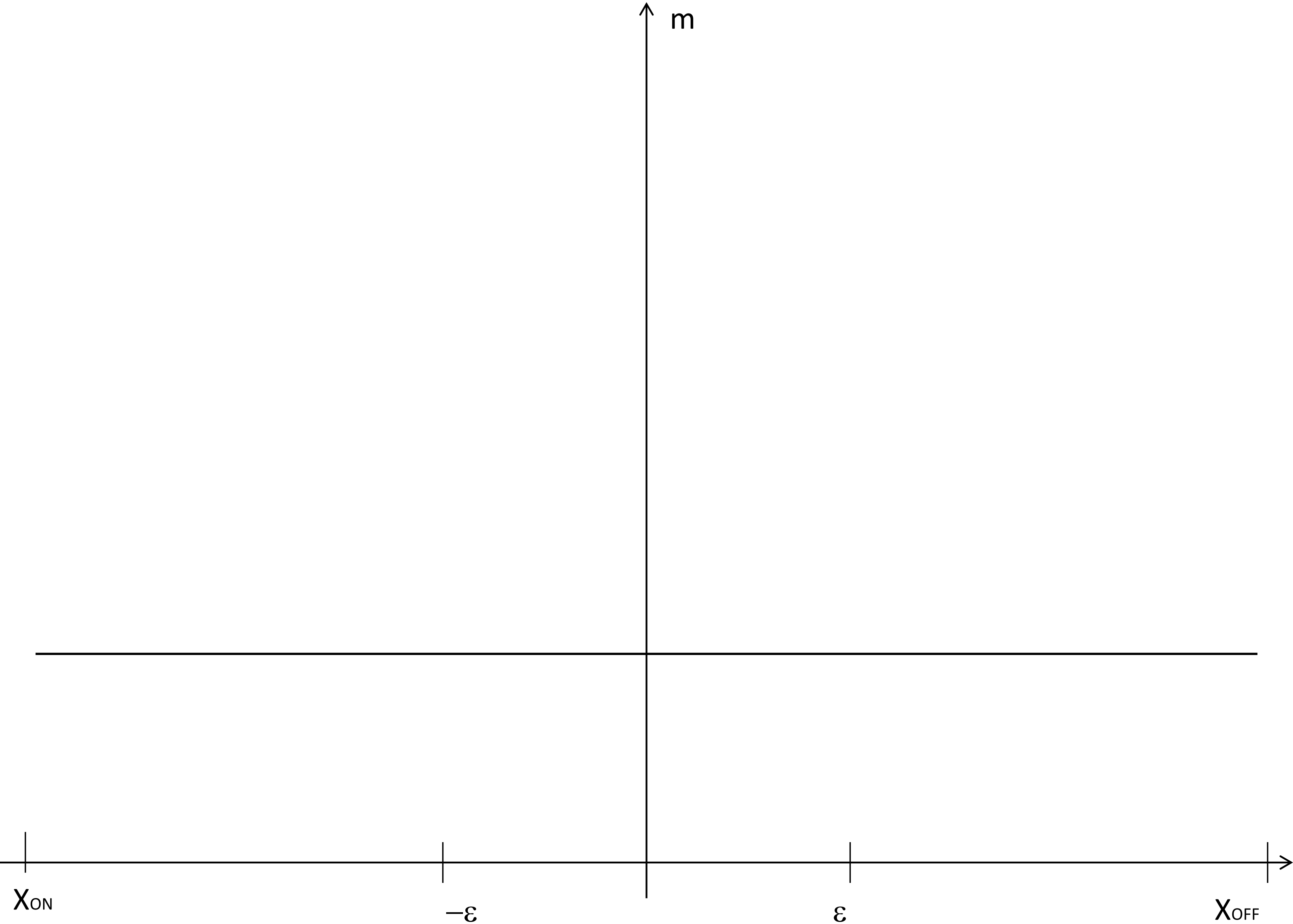
The feedback law $u(x,z,\eta_{01},\eta_{10},t)=z$ is optimal.



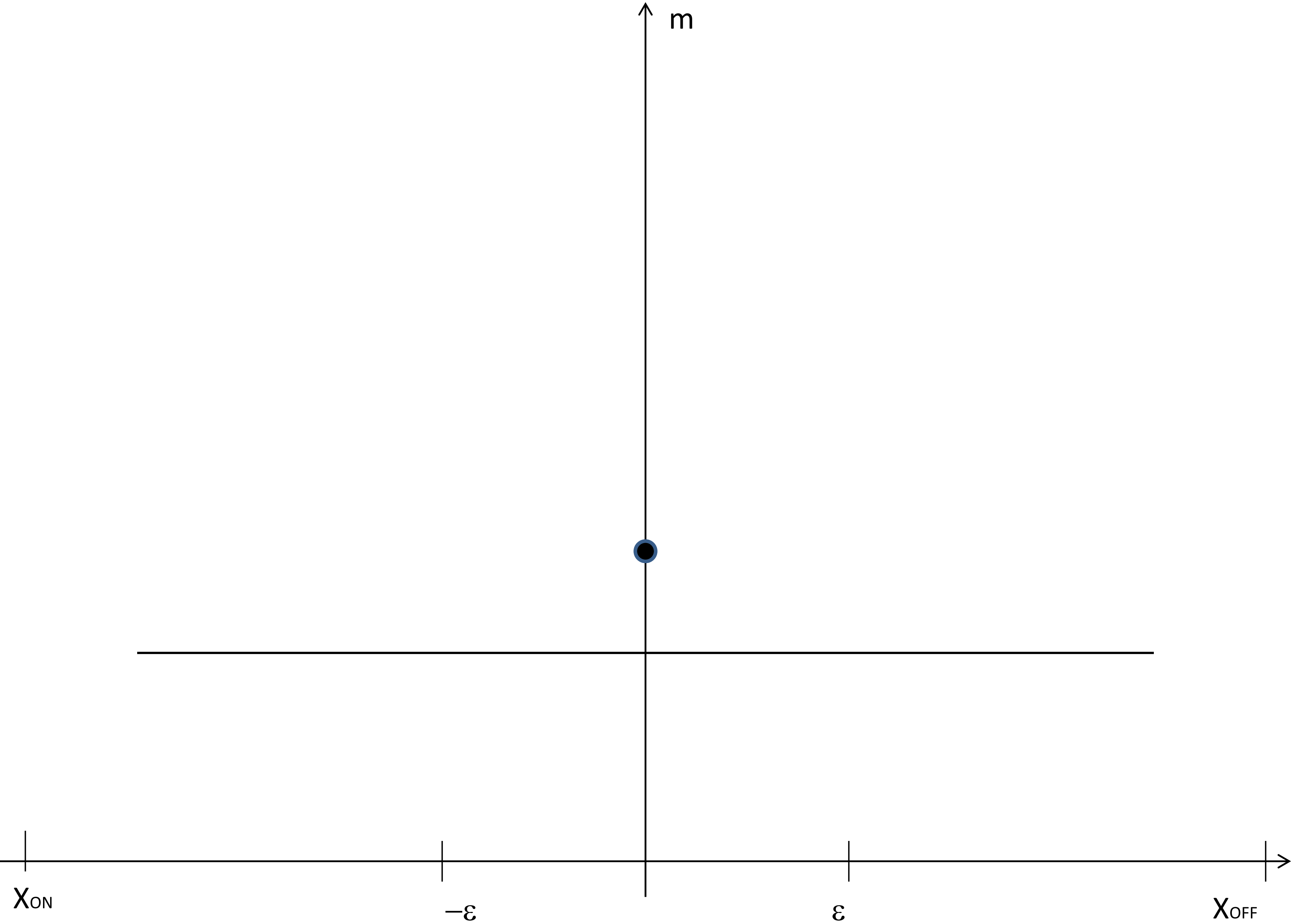


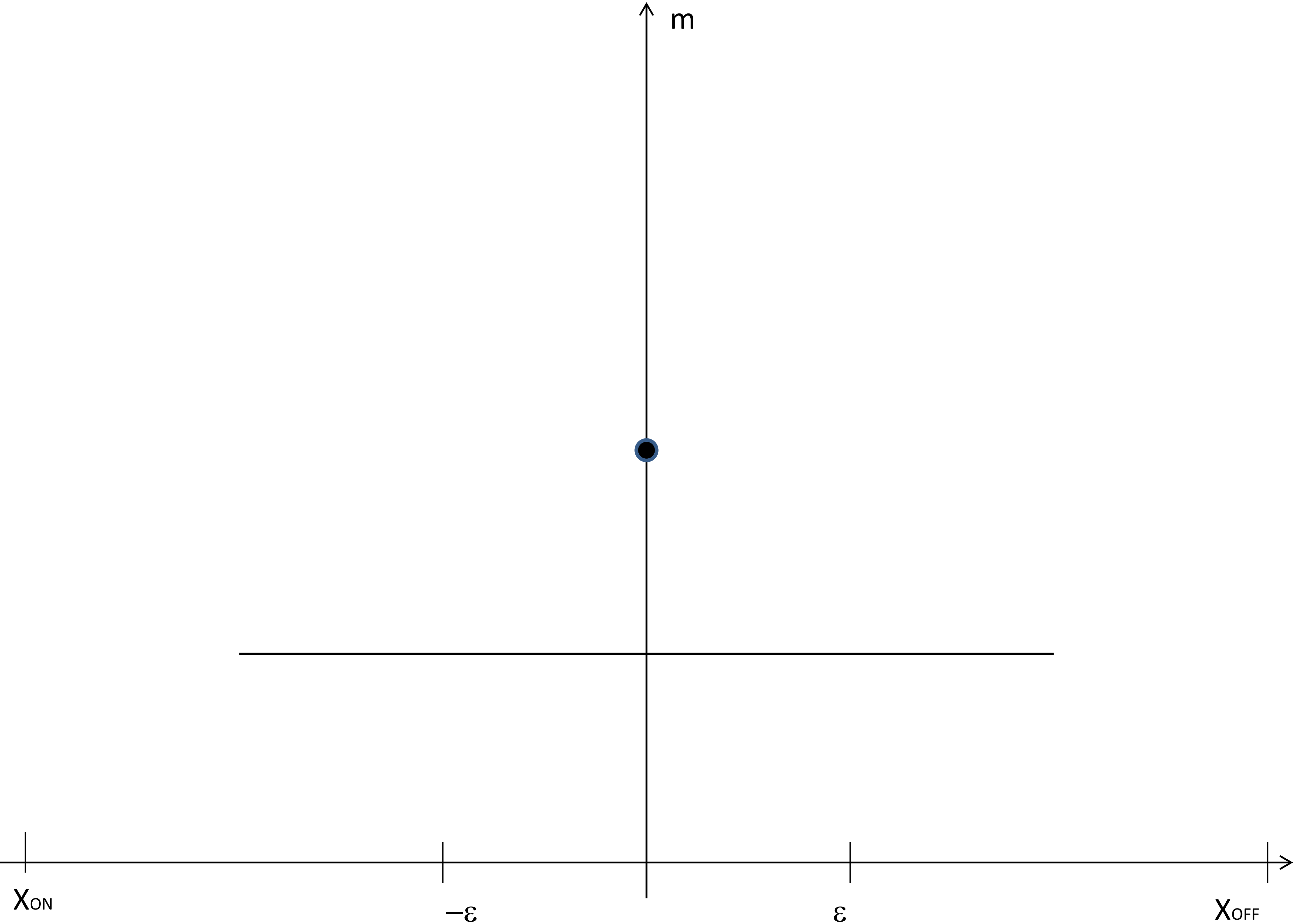
$$\widetilde{\Psi}(x(T),z(T),\eta_{01}(T),\eta_{10}(T))=\Psi^{z(T)}(x(T))-\xi_{01}\eta_{01}(T)-\xi_{10}\eta_{10}(T)$$

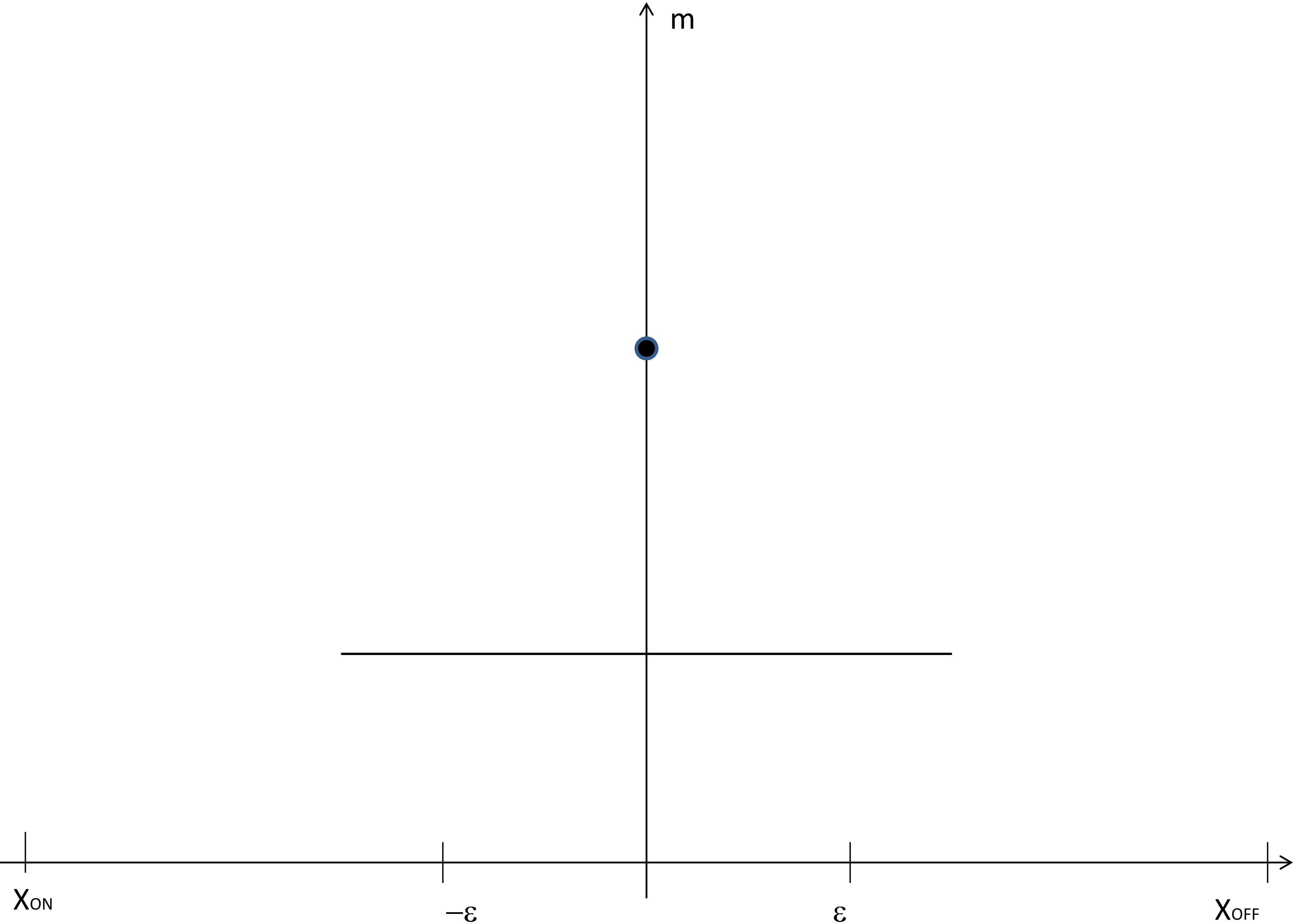
Thermostatic control problem, B. et al., some previous works



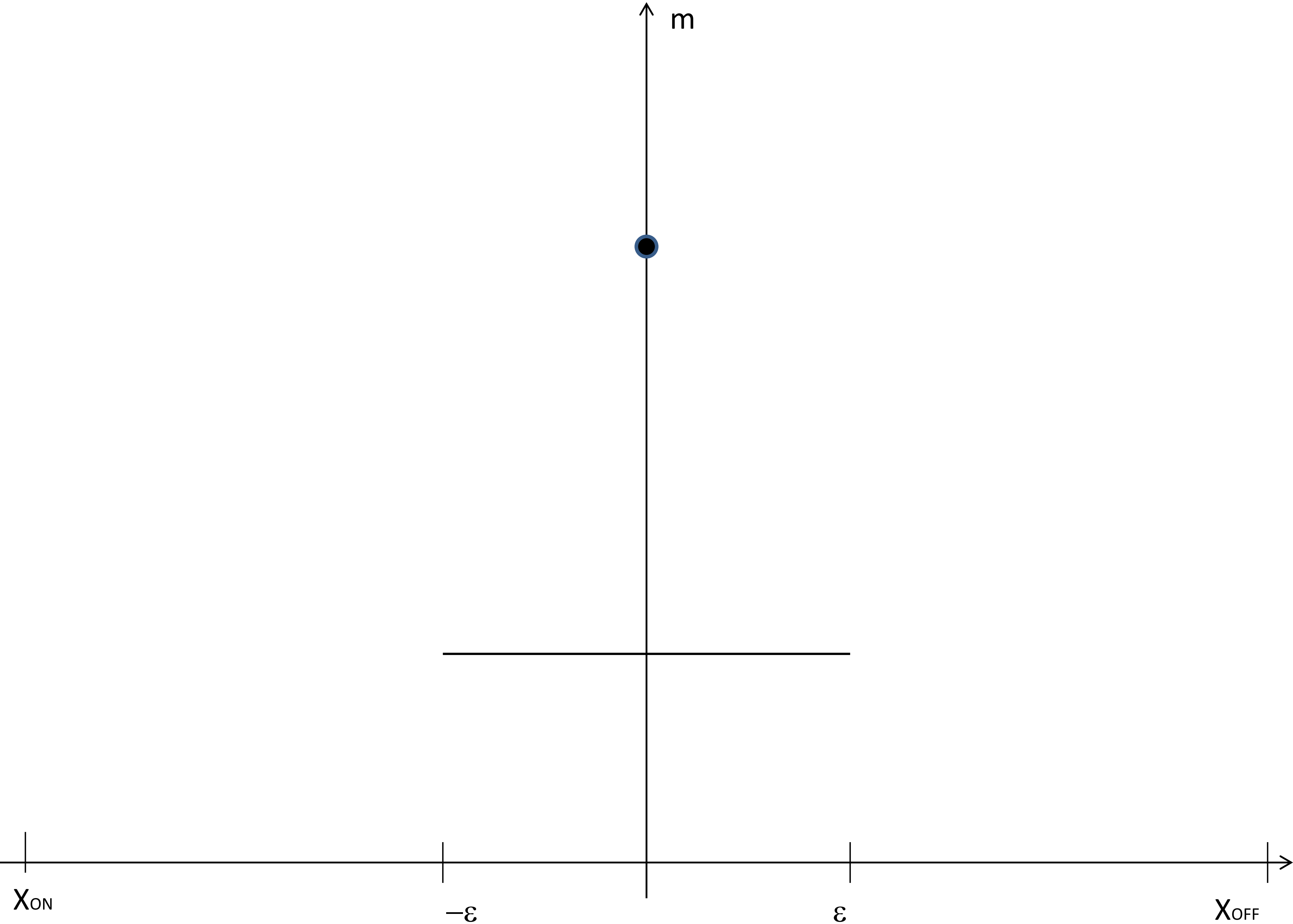
m
X_{ON}
$-\varepsilon$
ε
X_{OFF}


m
XON
−ε
ε
XOFF

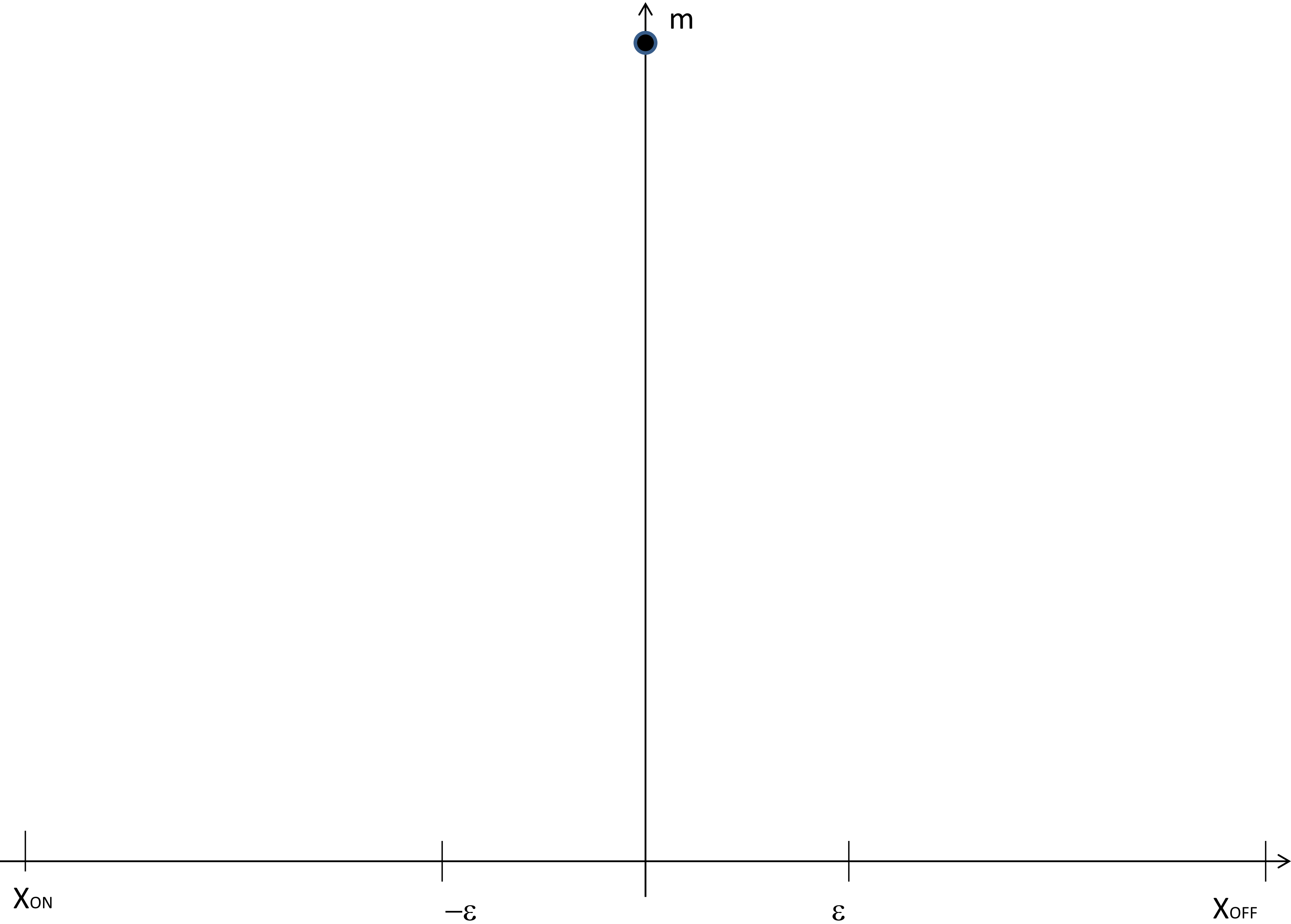

m
X_{ON}
$-\varepsilon$
ε
X_{OFF}



m
XON
−ε
ε
XOFF


m
X_{ON}
$-\varepsilon$
ε
X_{OFF}


m
XON
−ε
ε
XOFF



m
XON
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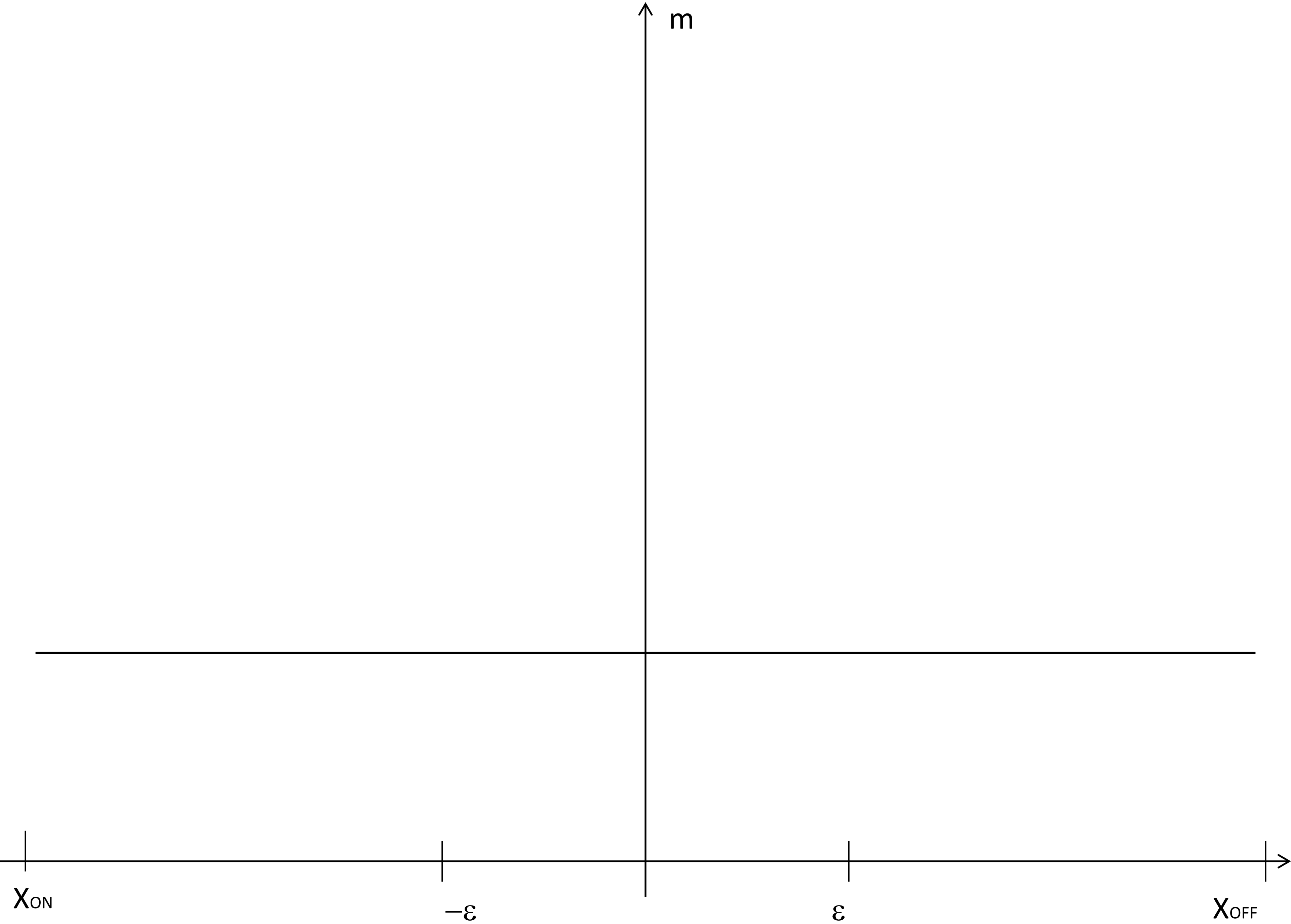
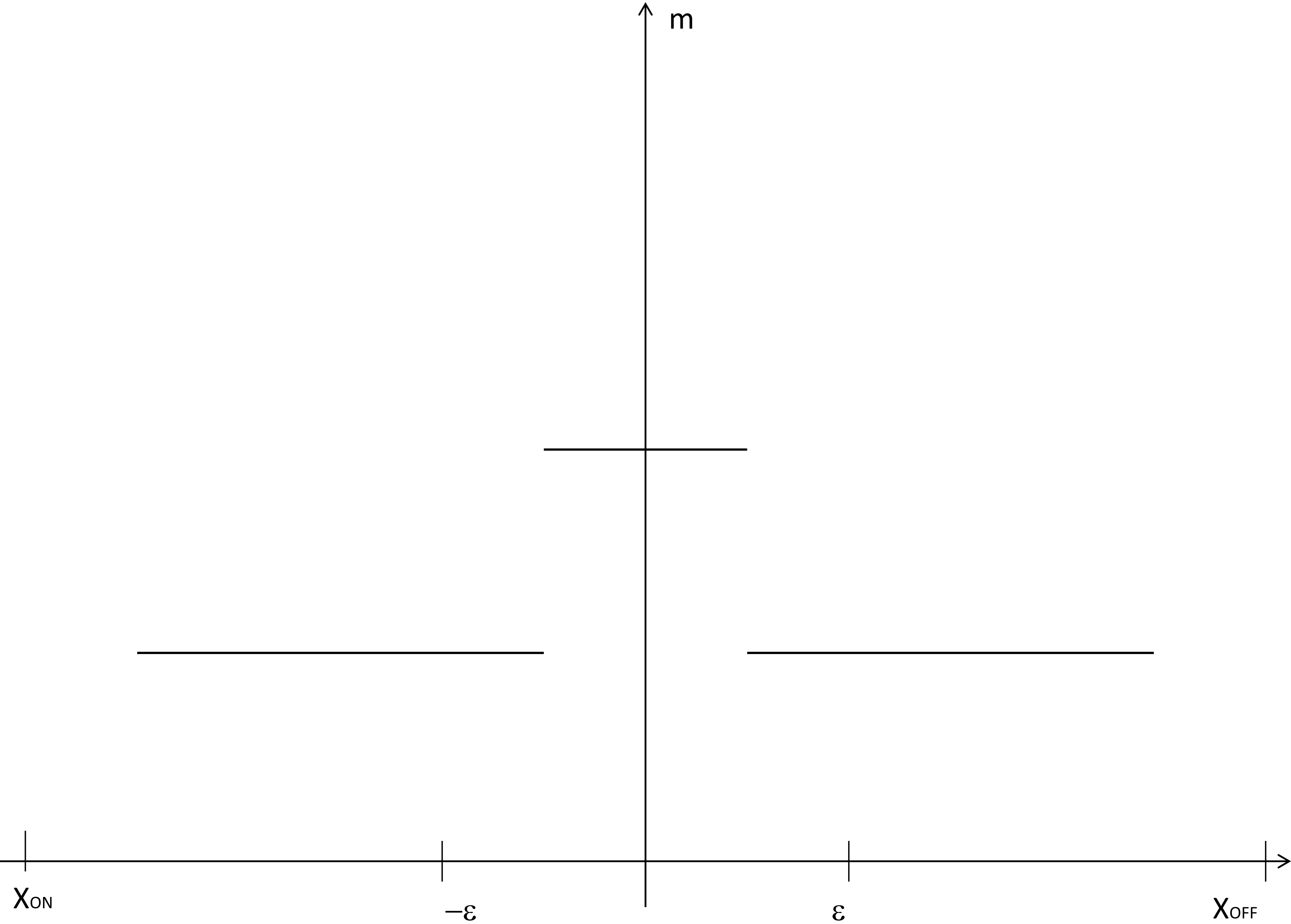
ε
XOFF


m
X_{ON}
$-\varepsilon$
ε
X_{OFF}


m
X_{ON}
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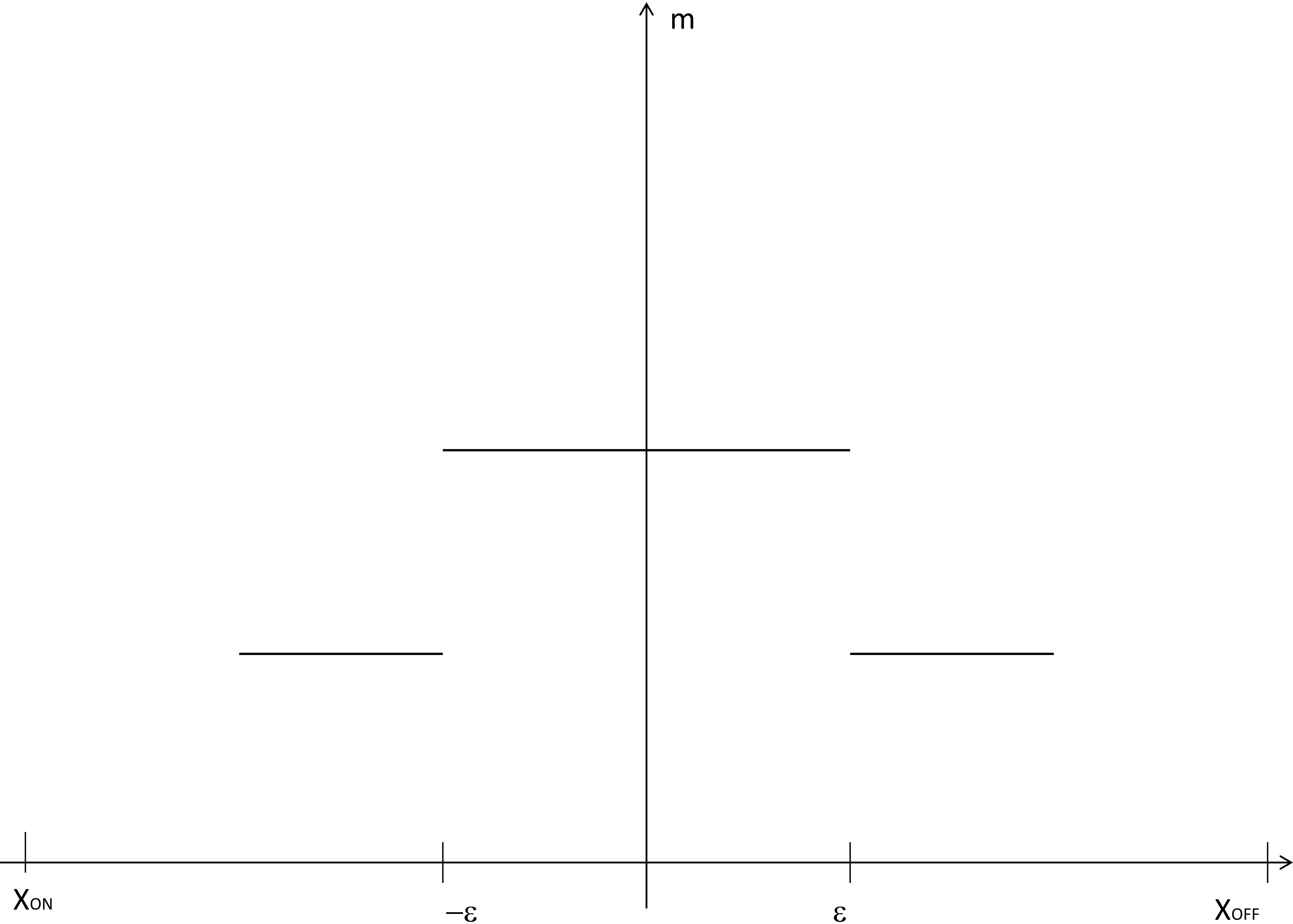
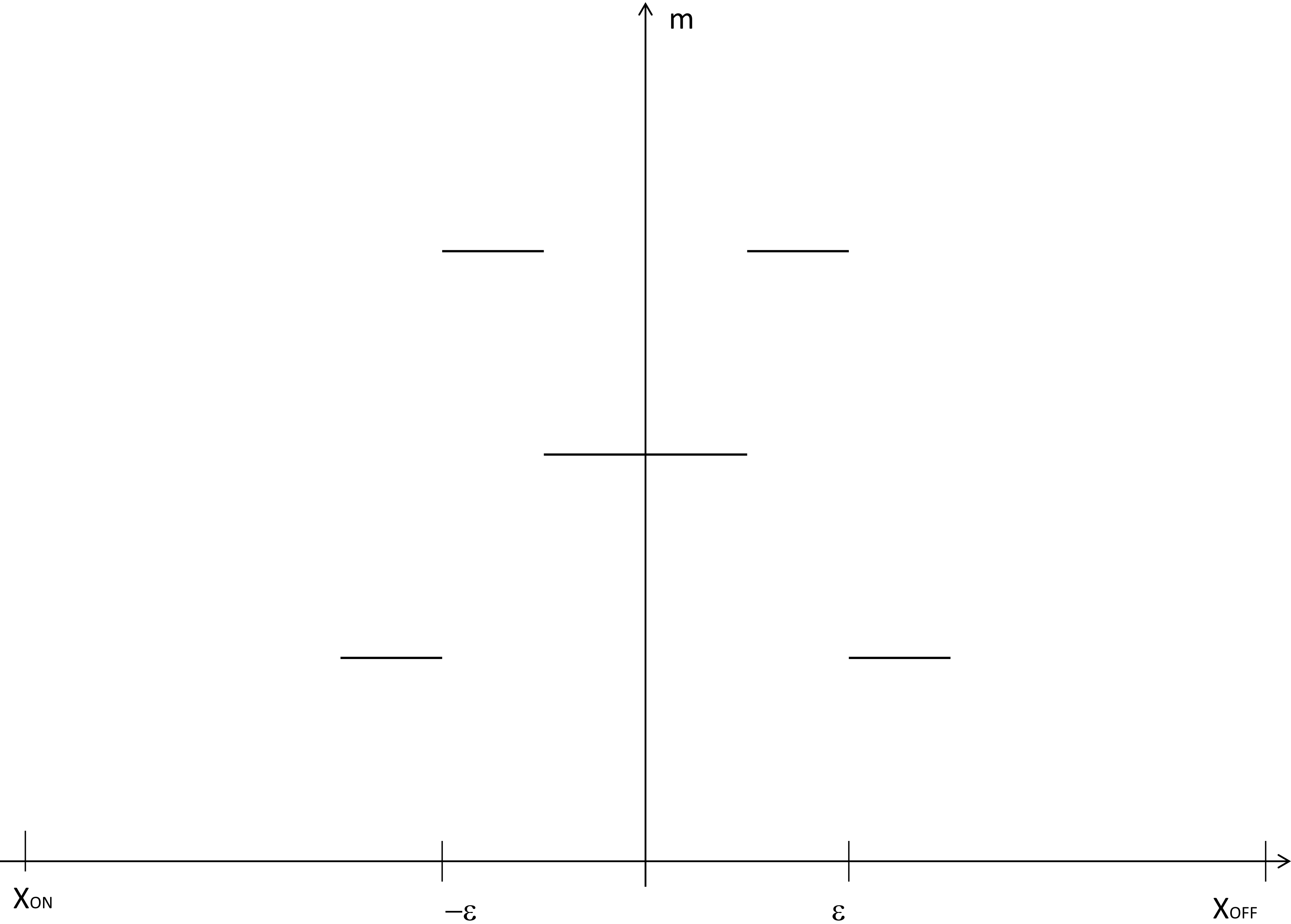
X_{OFF}


m
XON
−ε
ε
XOFF



m
X_{ON}
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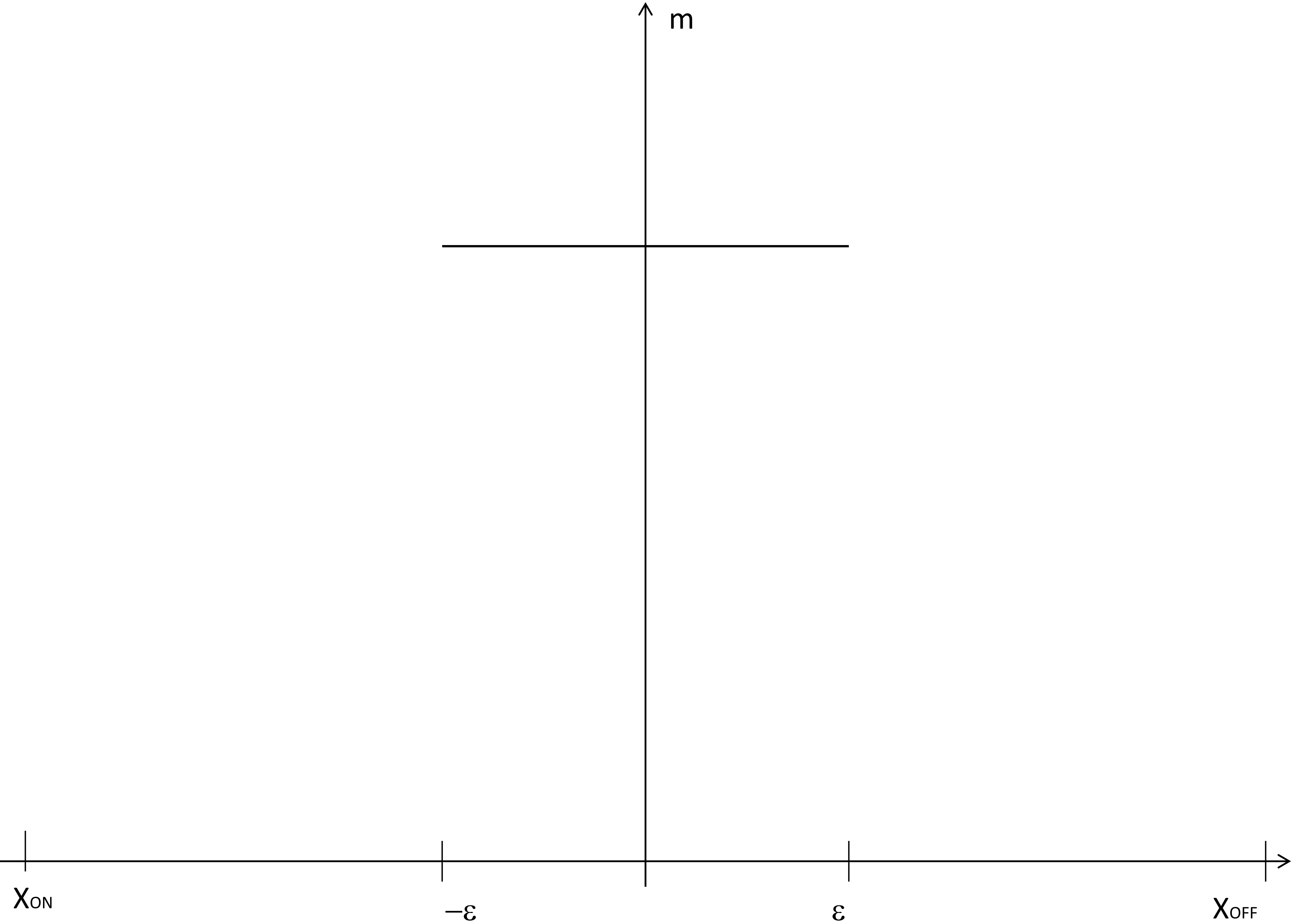
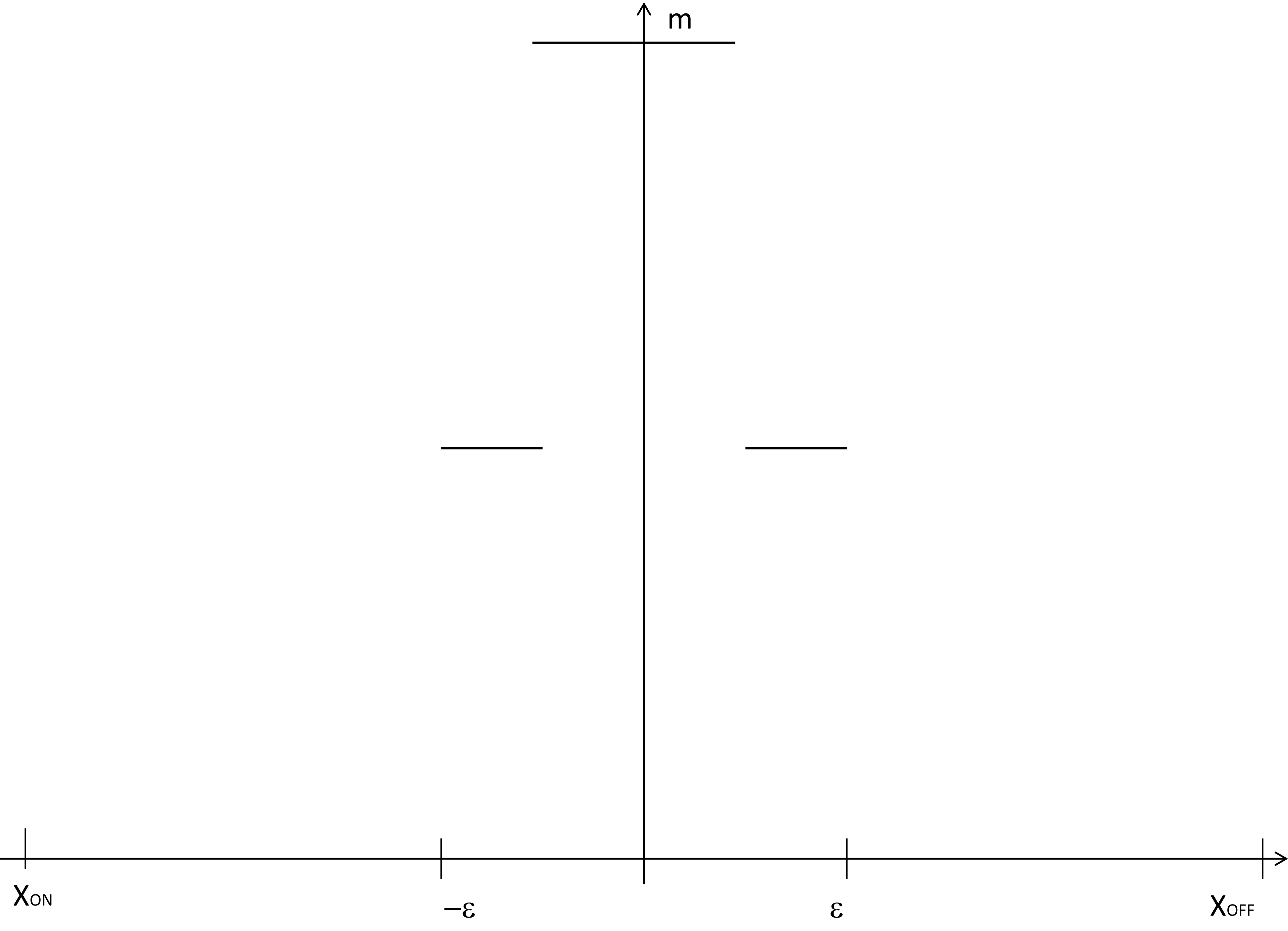

m
X_ON
−ε
ε
X_OFF


m
X_{ON}
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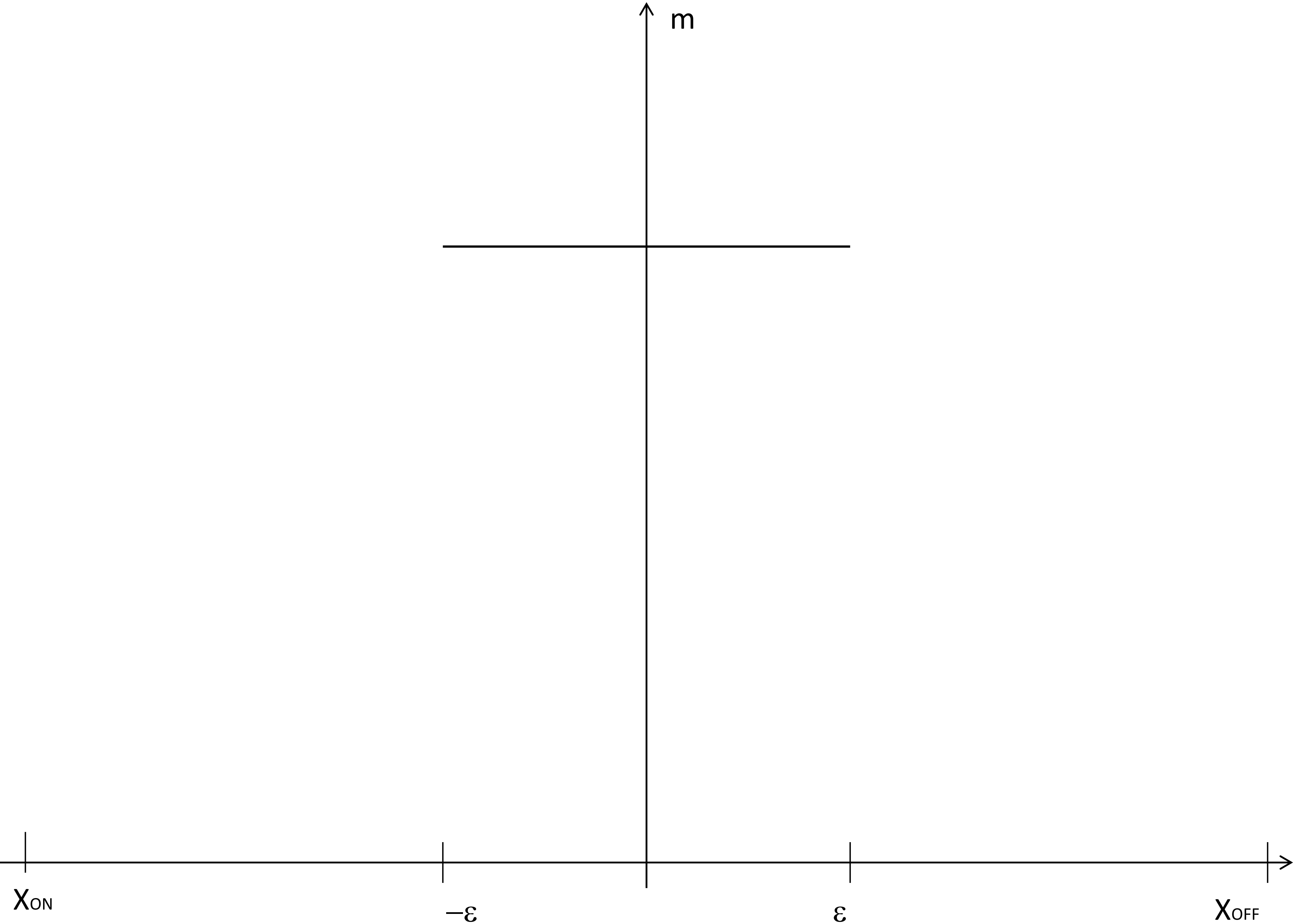

m
XON
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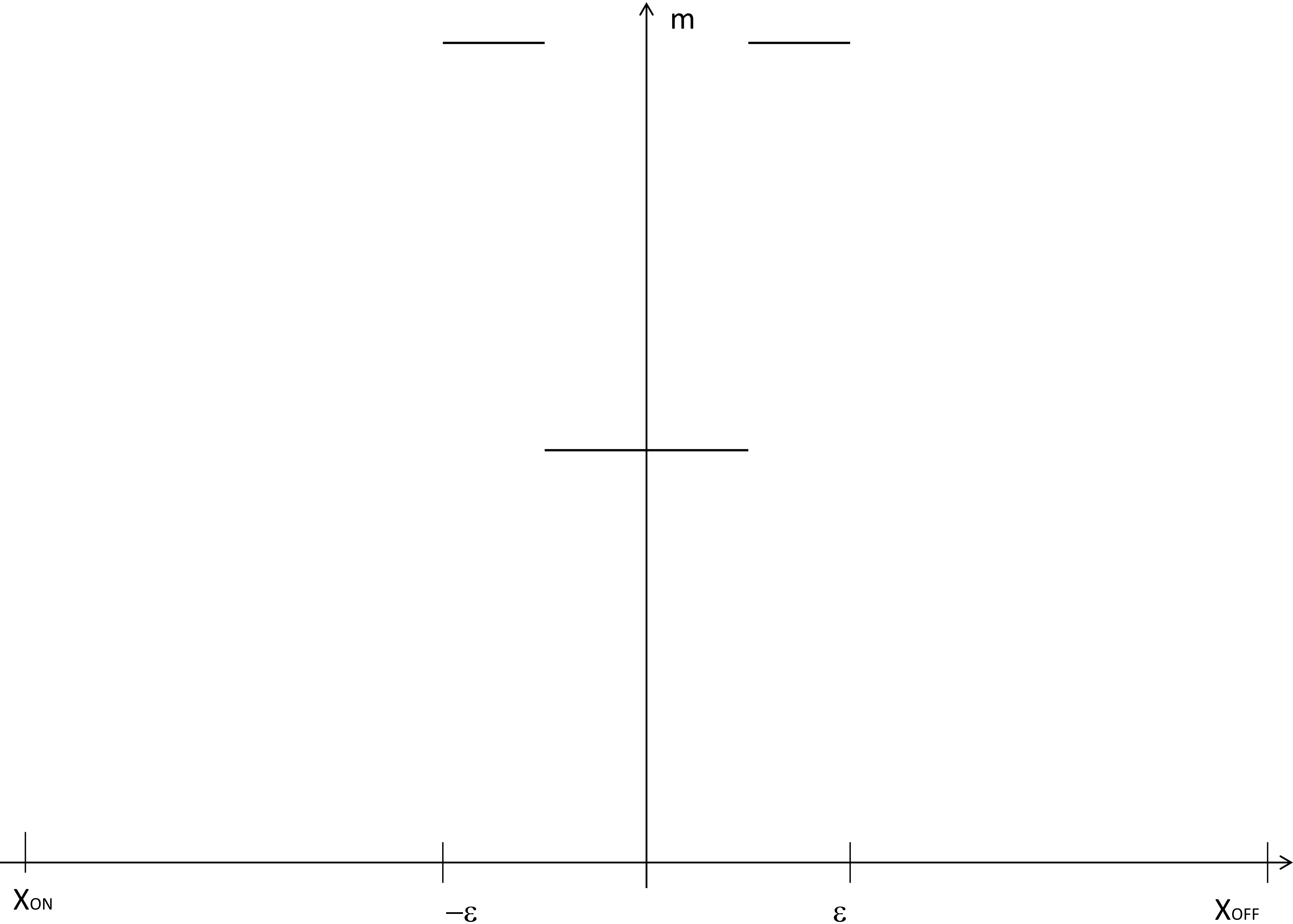
ε
XOFF


m
XON
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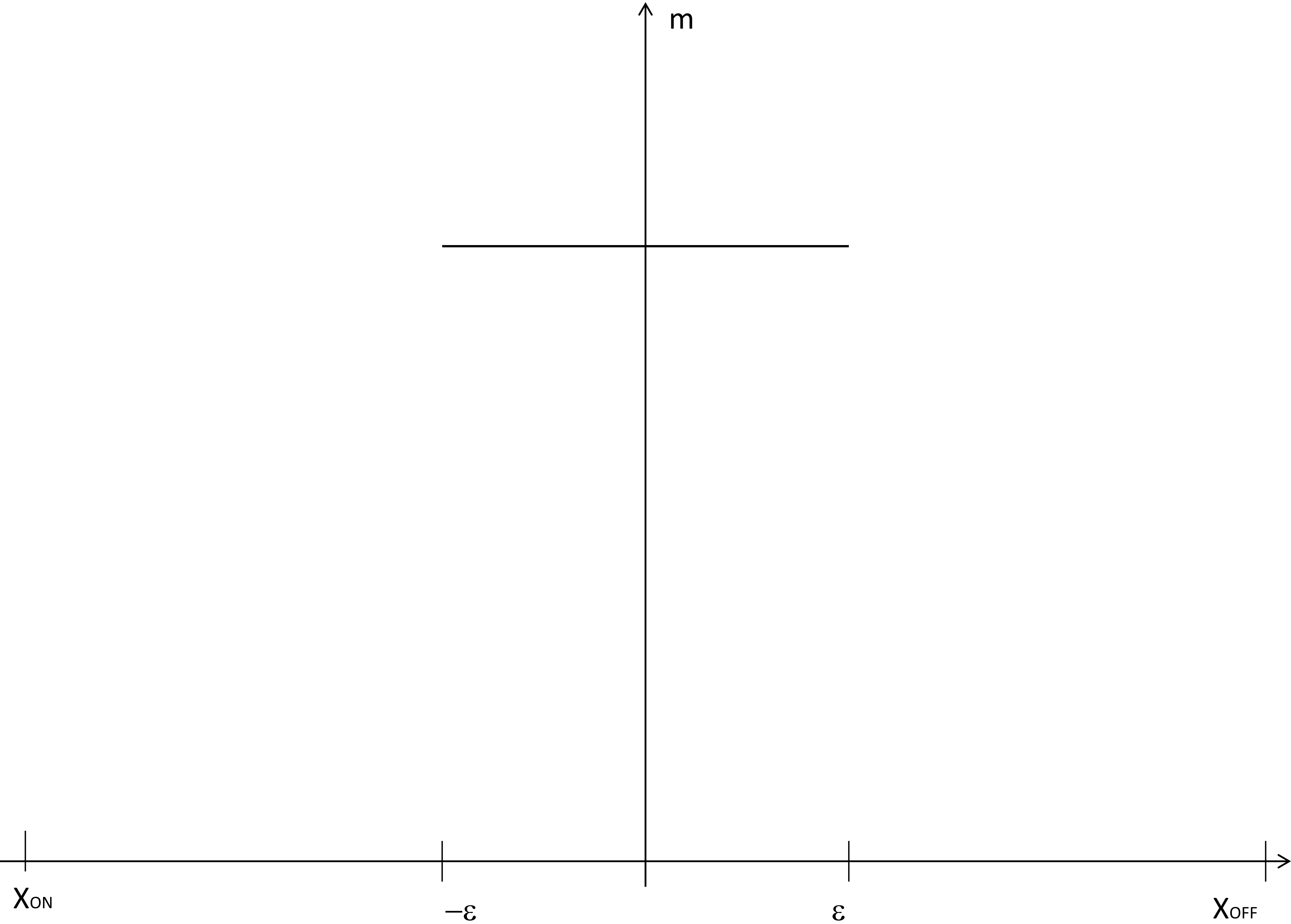
m
XON
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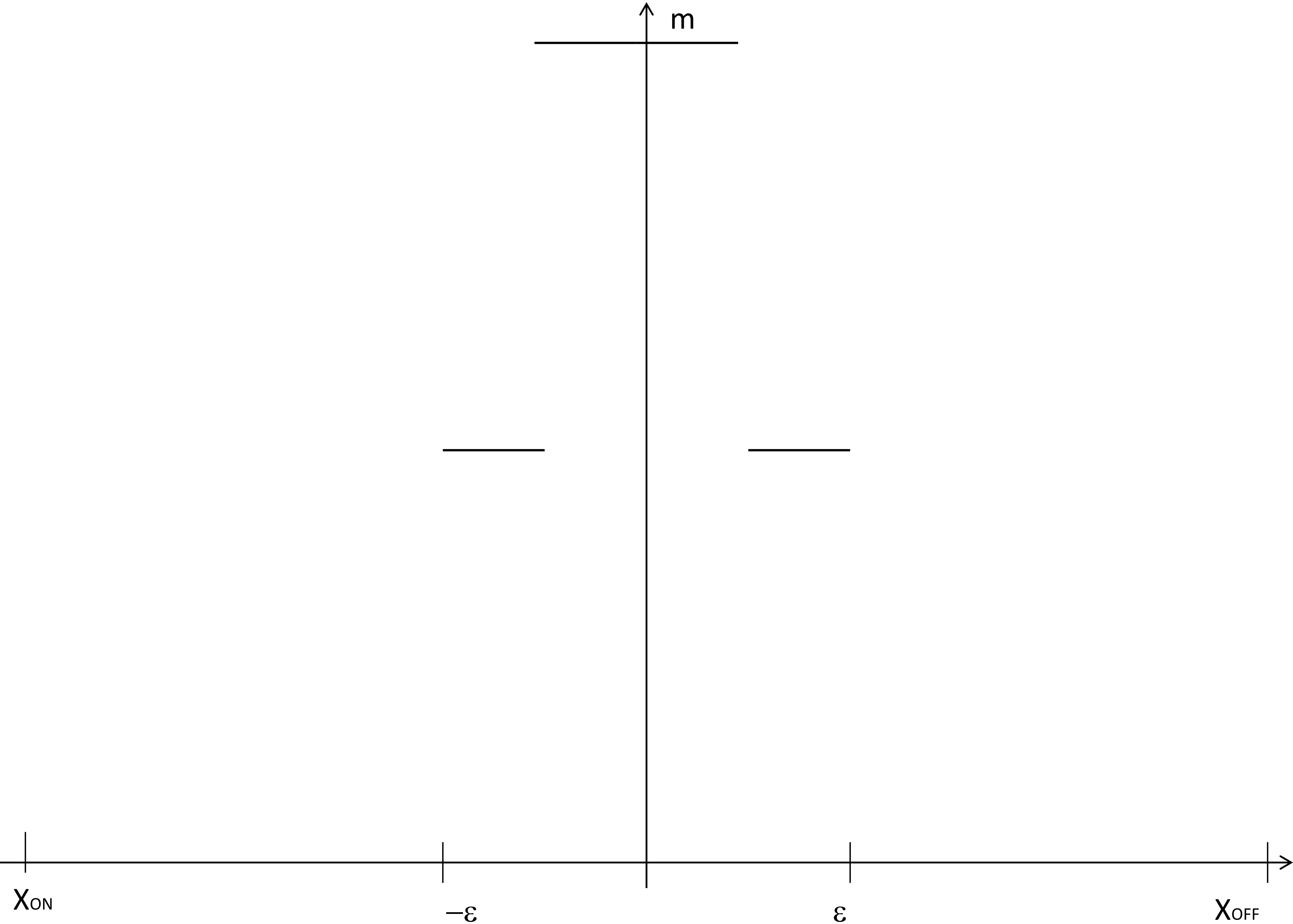
XOFF


m
XON
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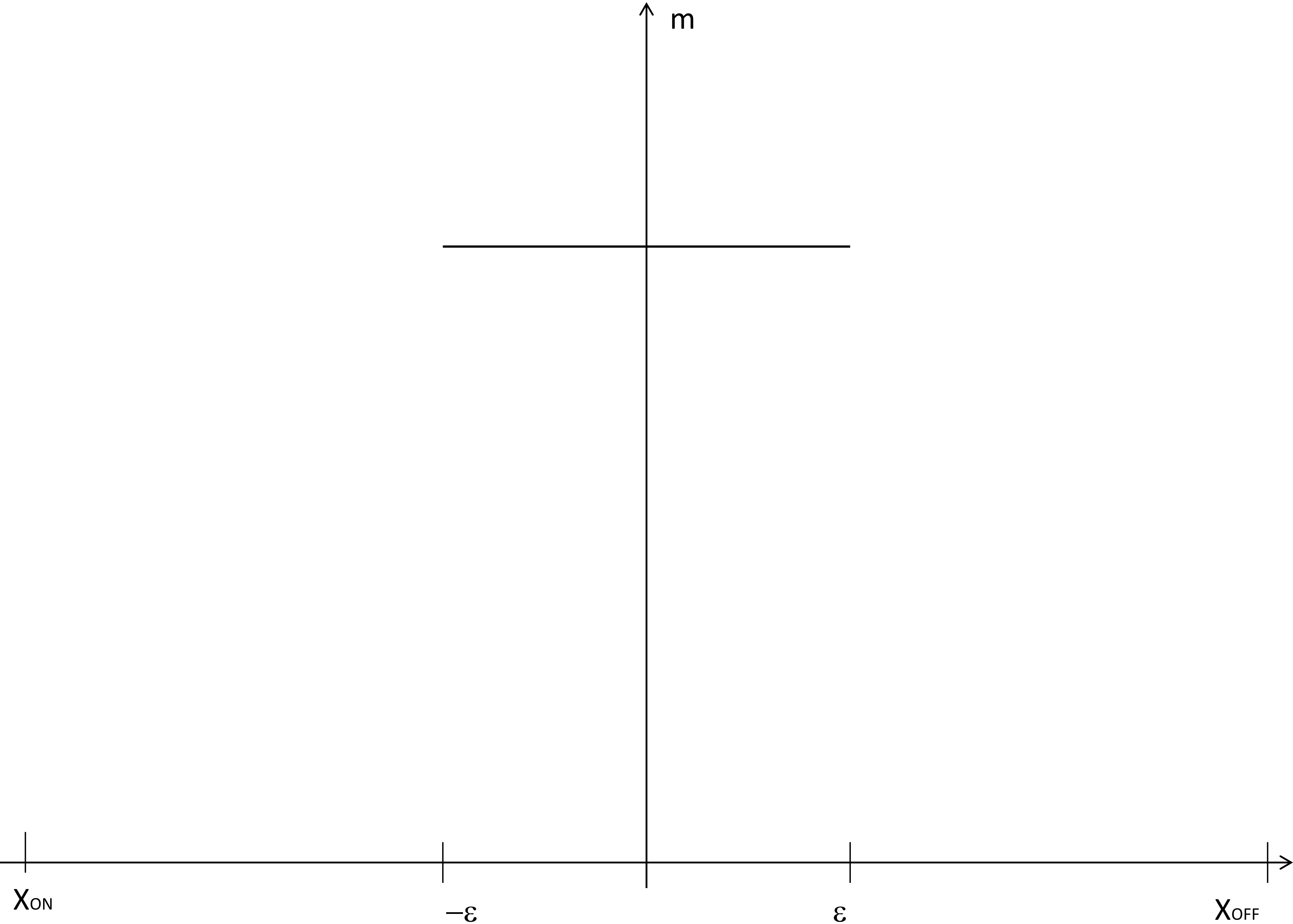
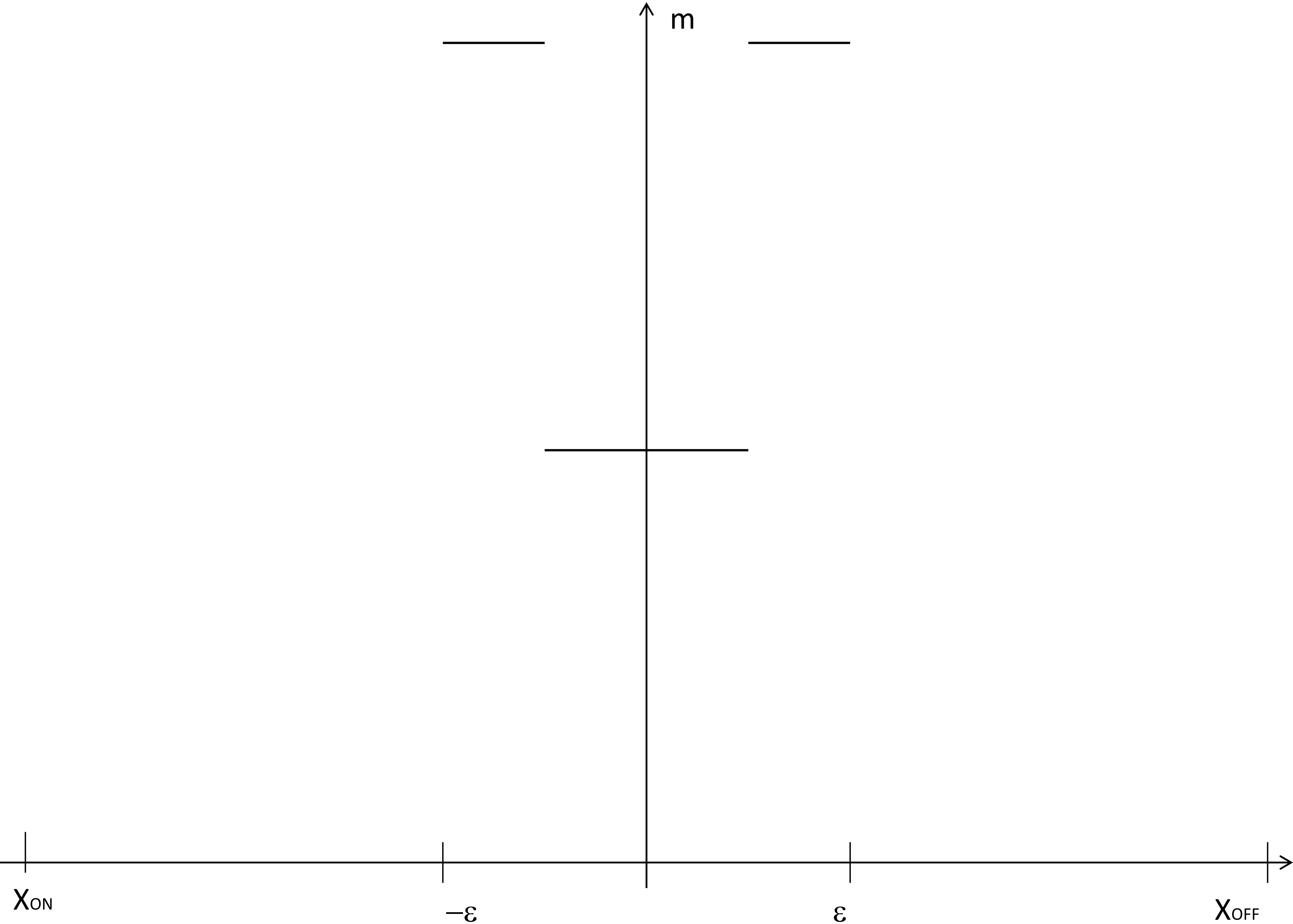


m
X_{ON}
$-\varepsilon$
ε
X_{OFF}


m
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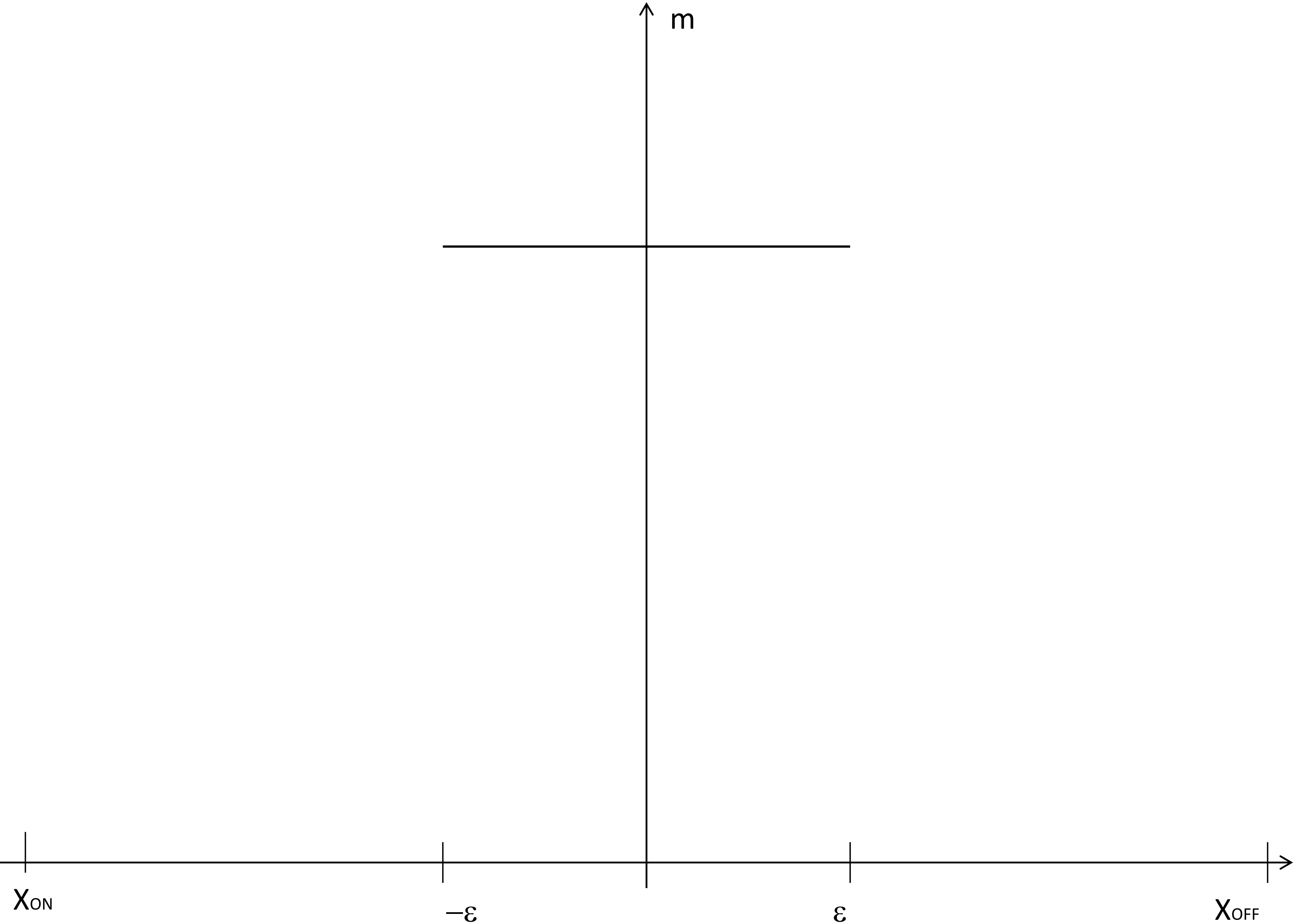
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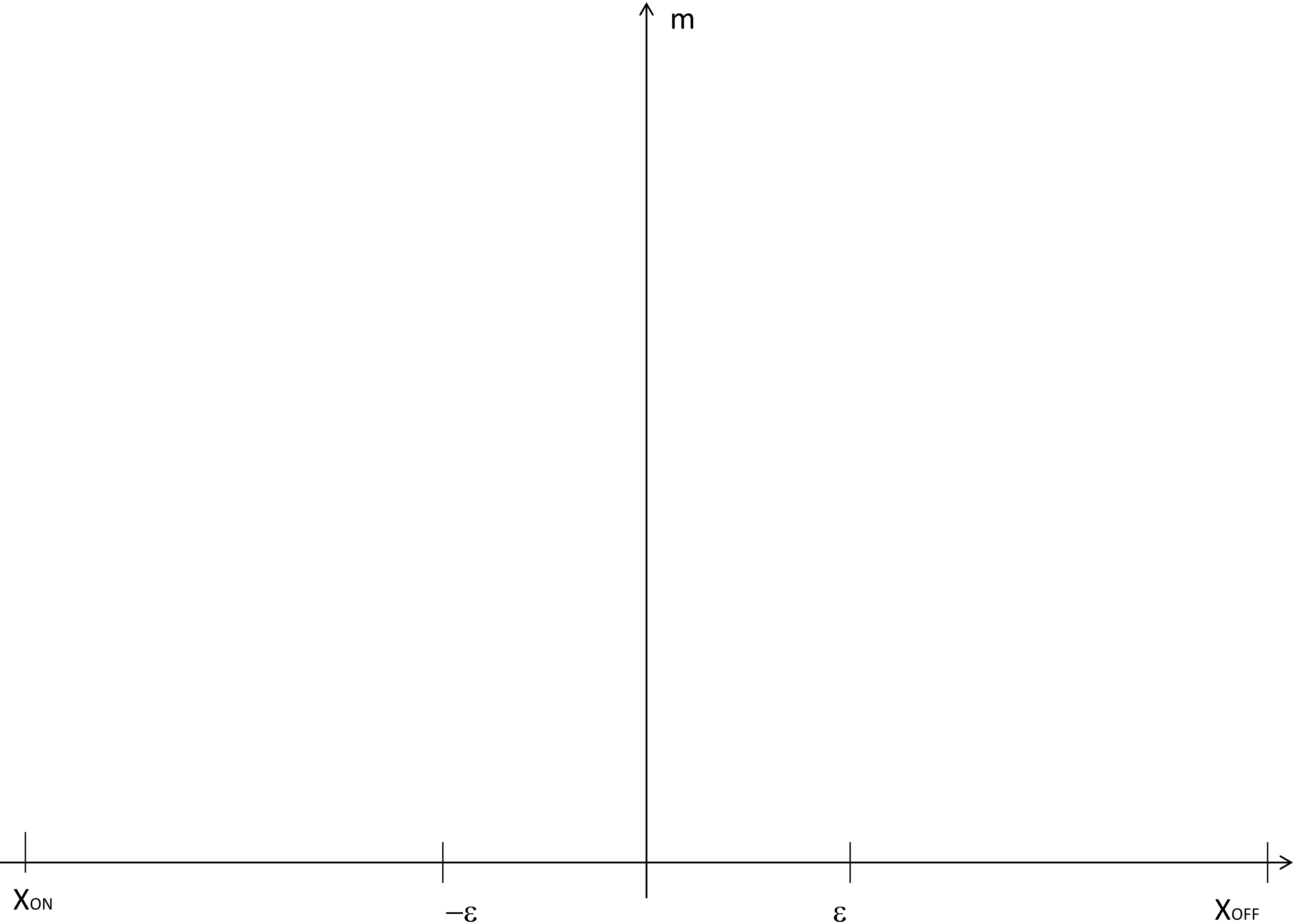
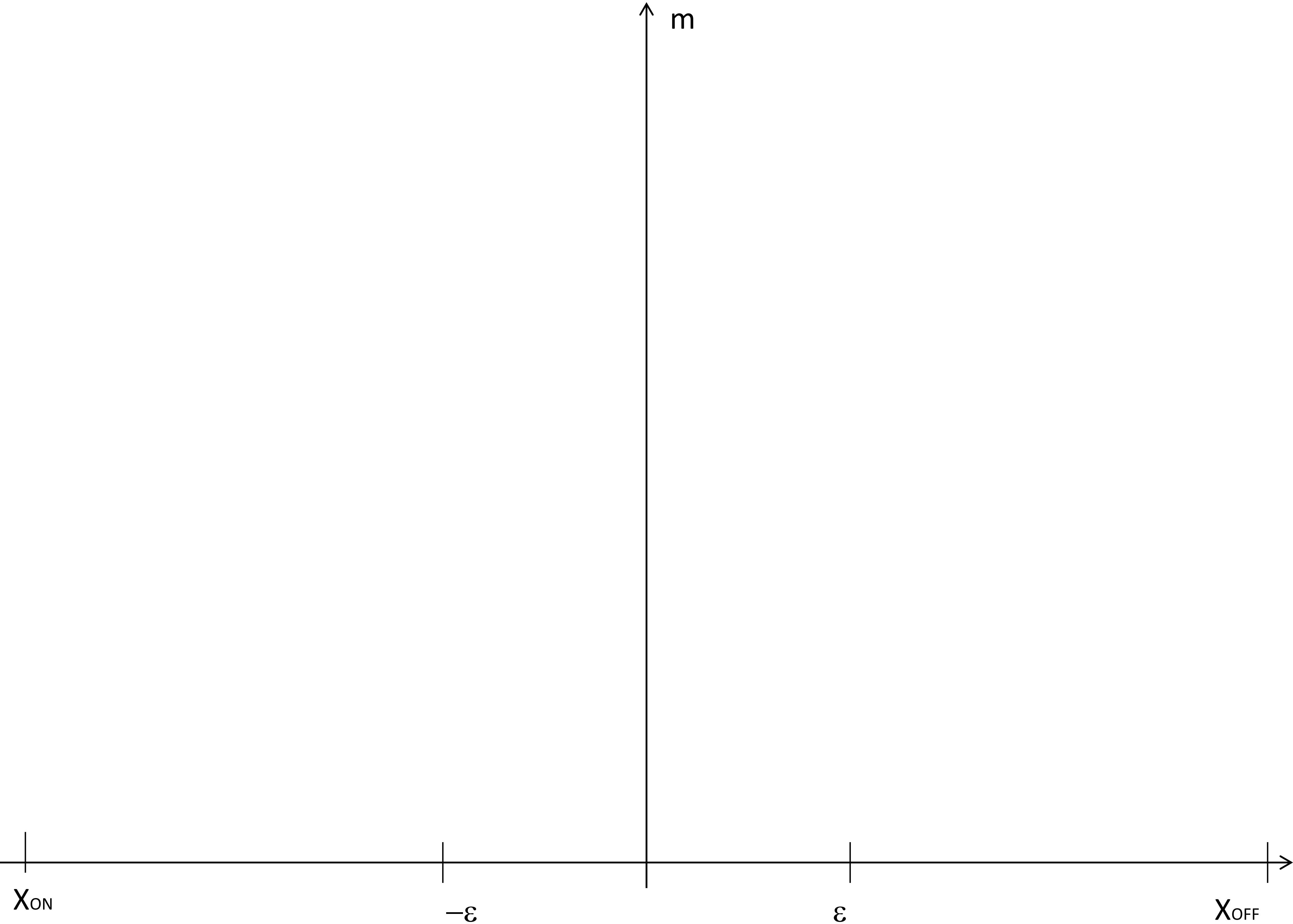
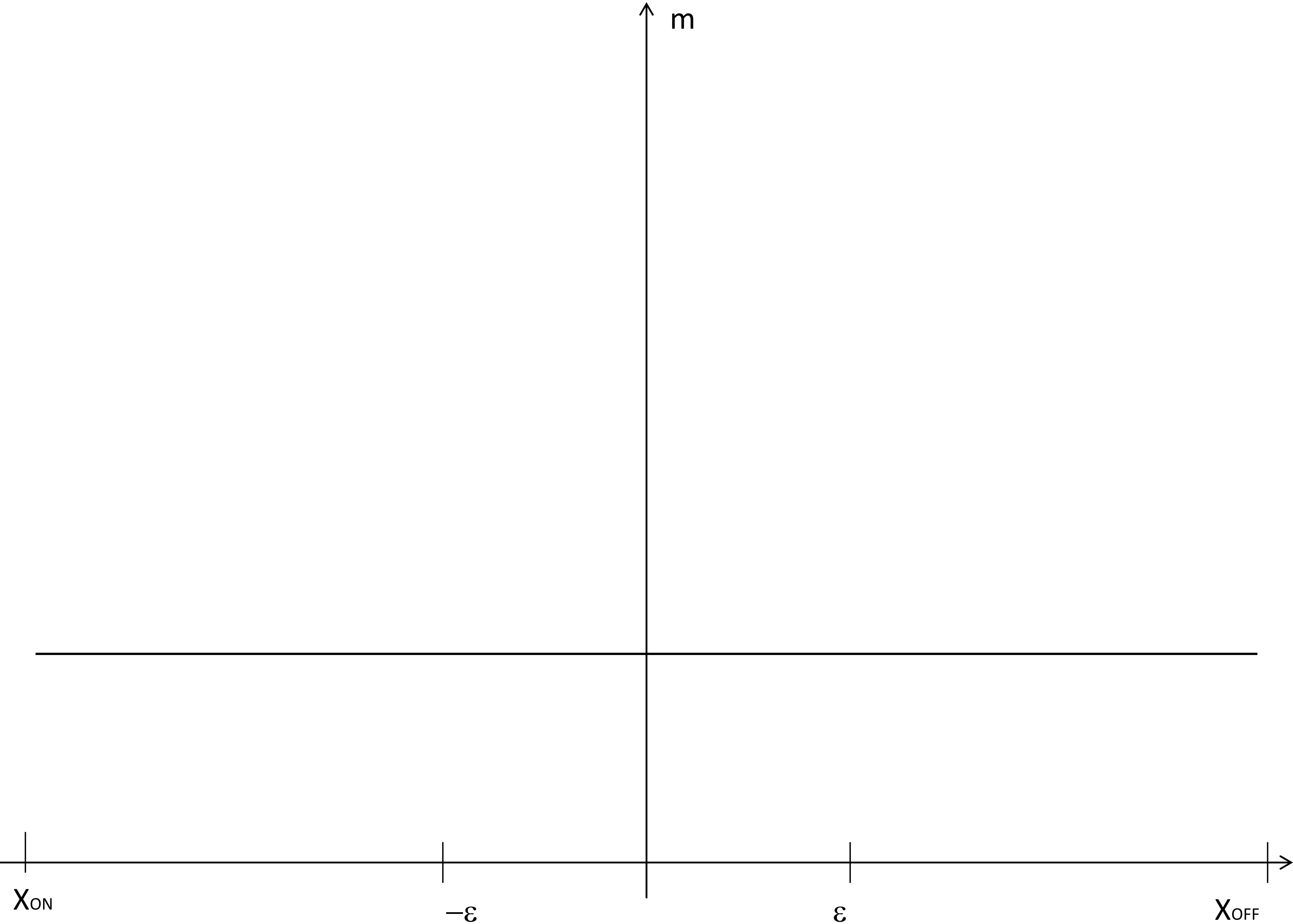
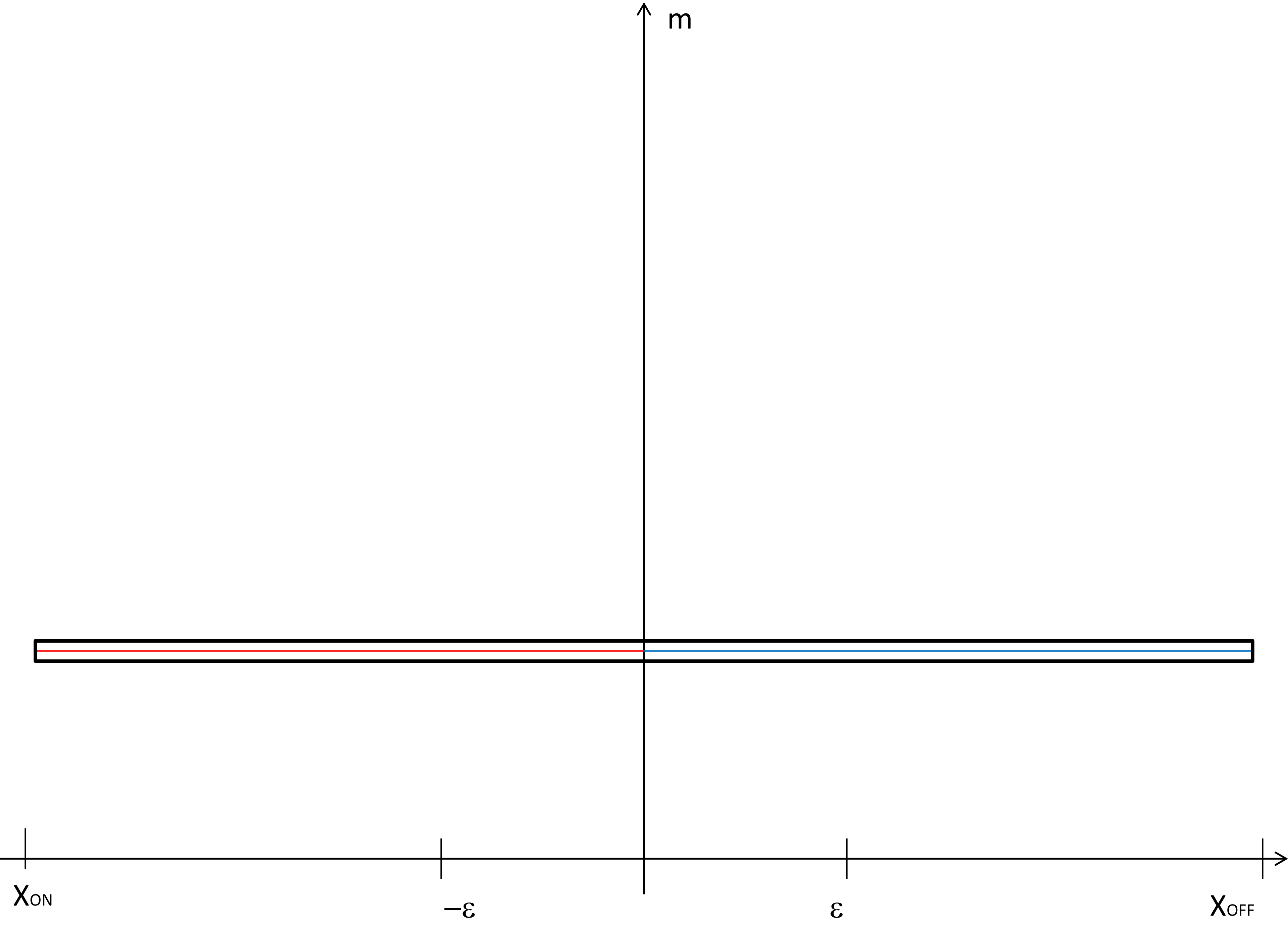
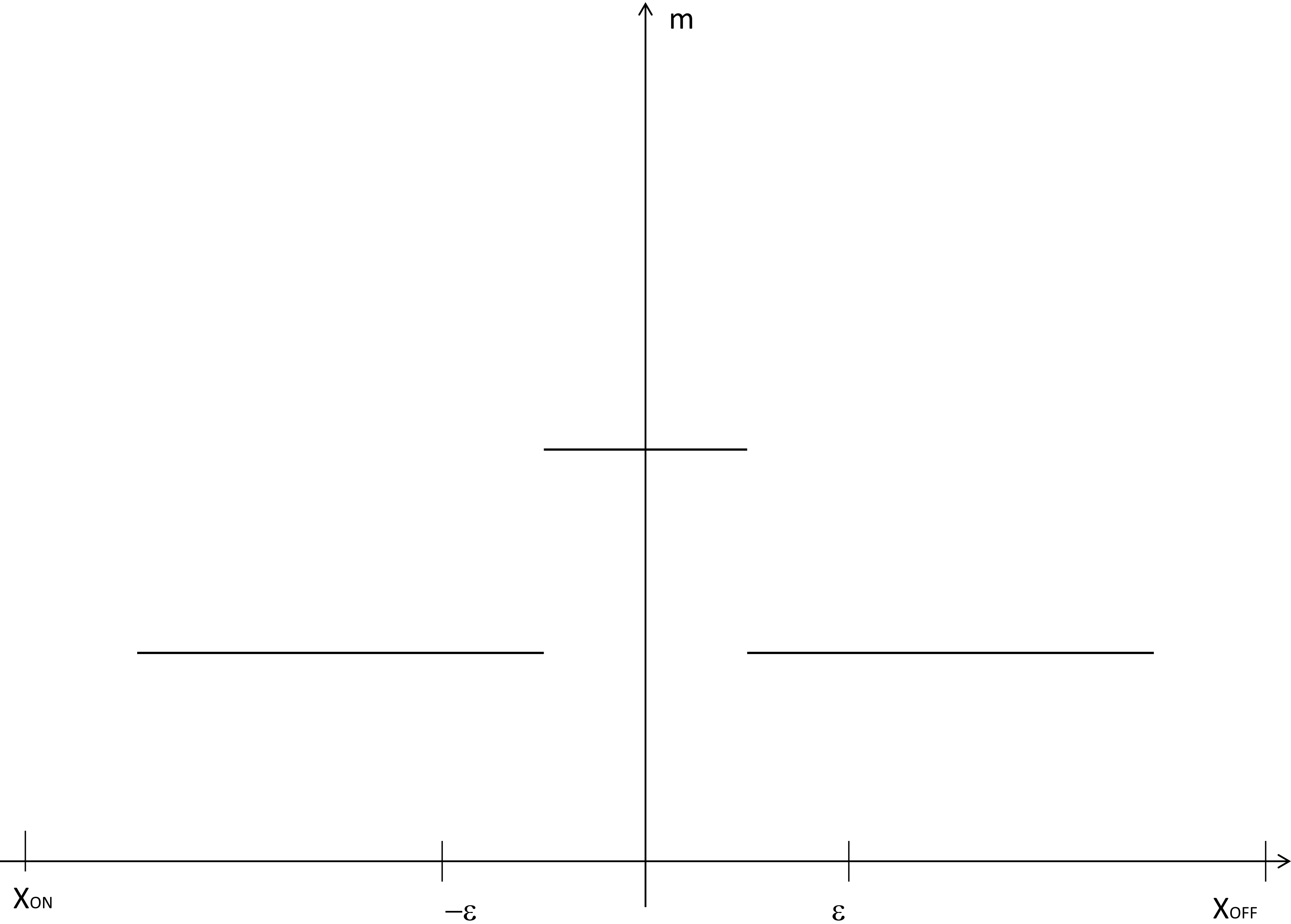
XOFF

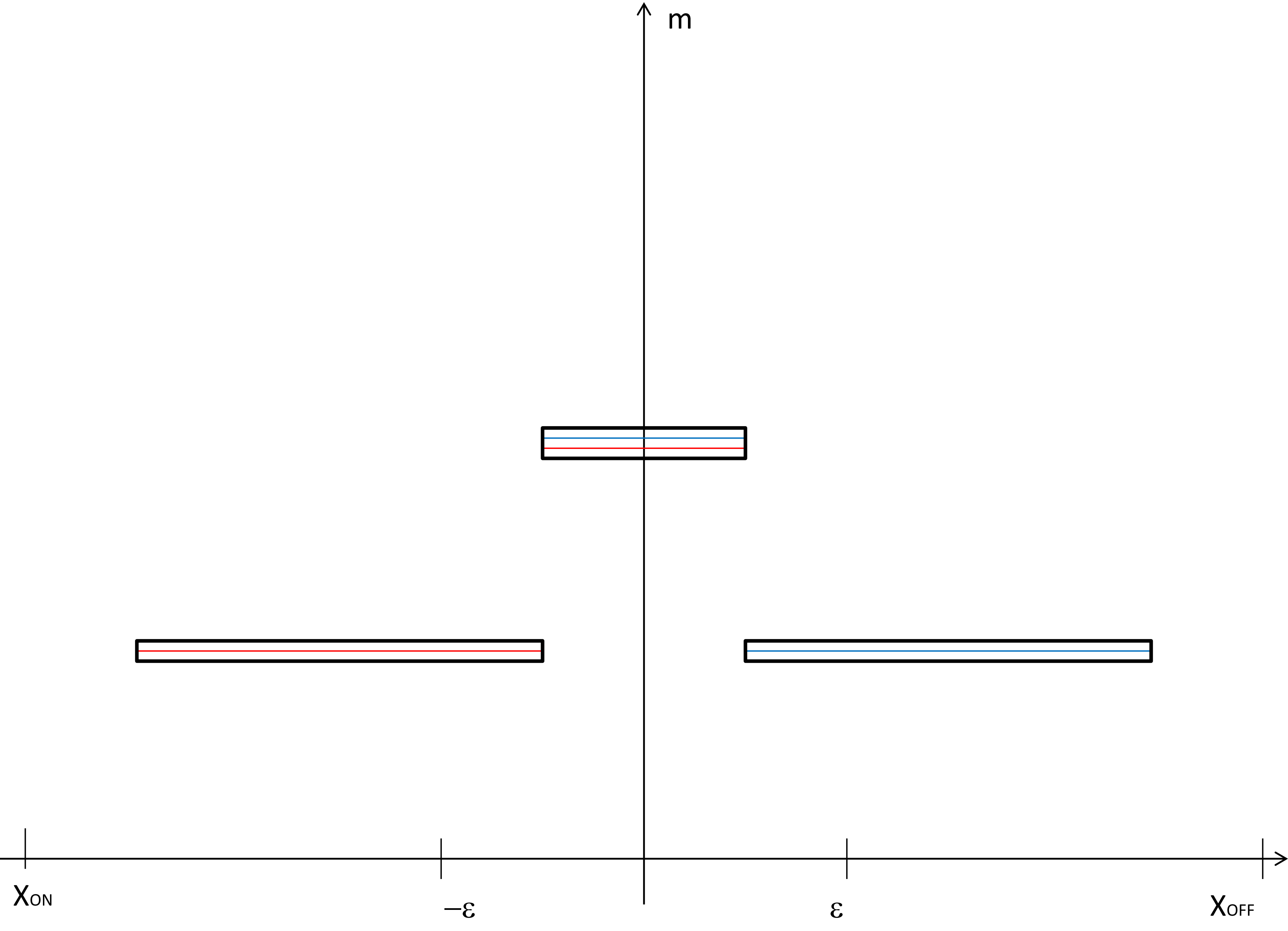

m
X_ON
−ε
ε
X_OFF


m
XON
−ε
ε
XOFF

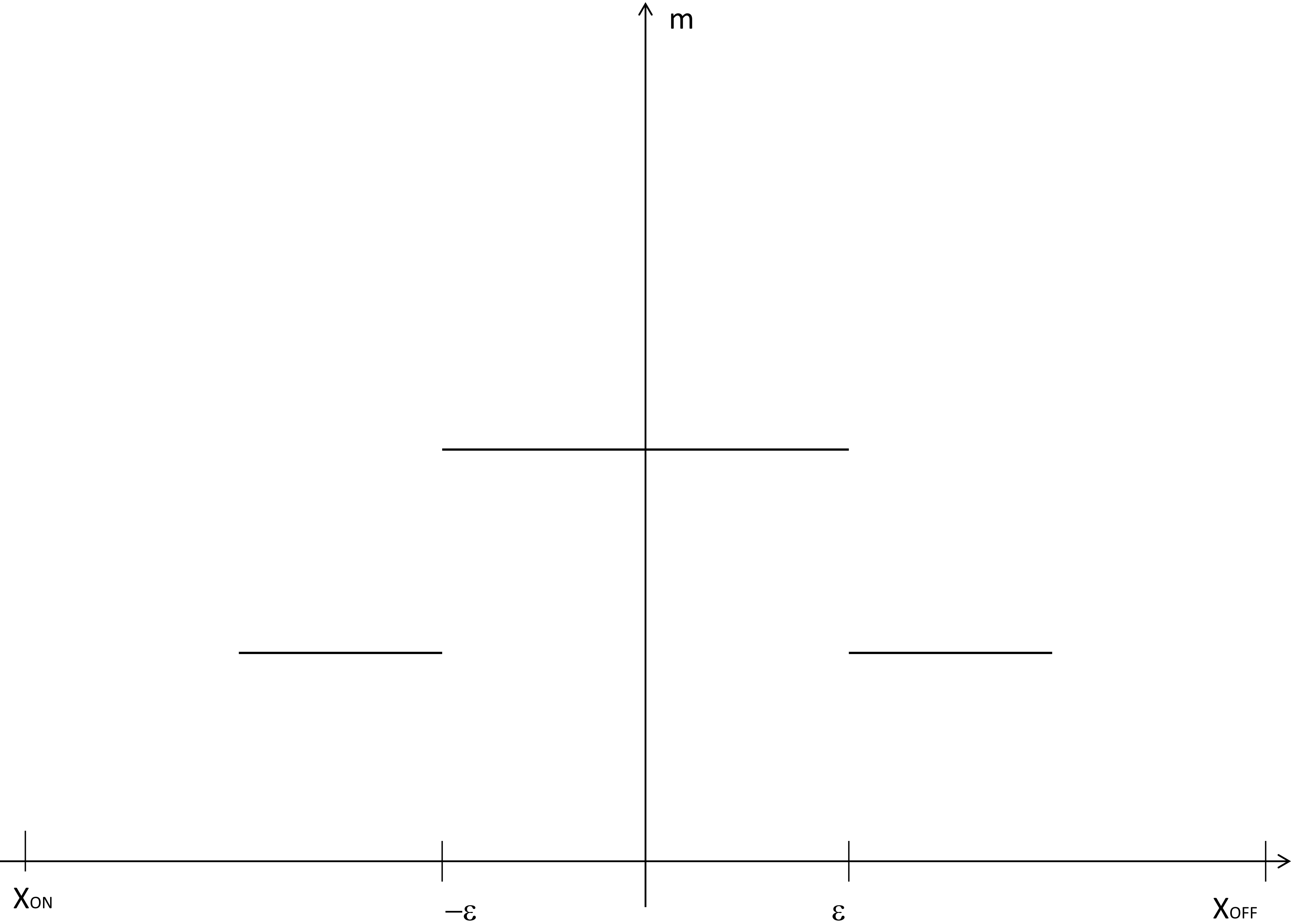

m
X_{ON}
$-\varepsilon$
ε
X_{OFF}


m
XON
−ε
ε
XOFF

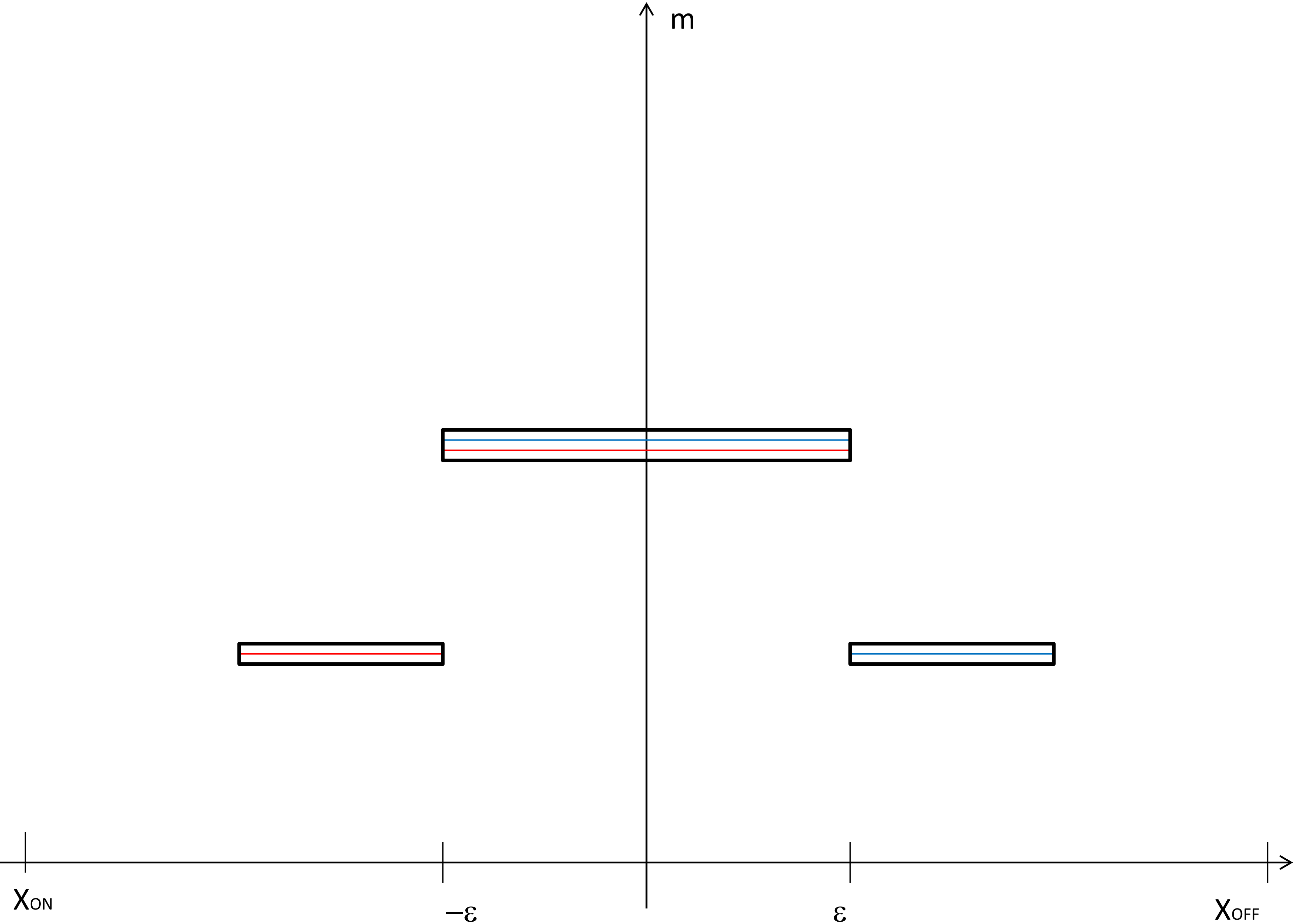

m
XON
−ε
ε
XOFF

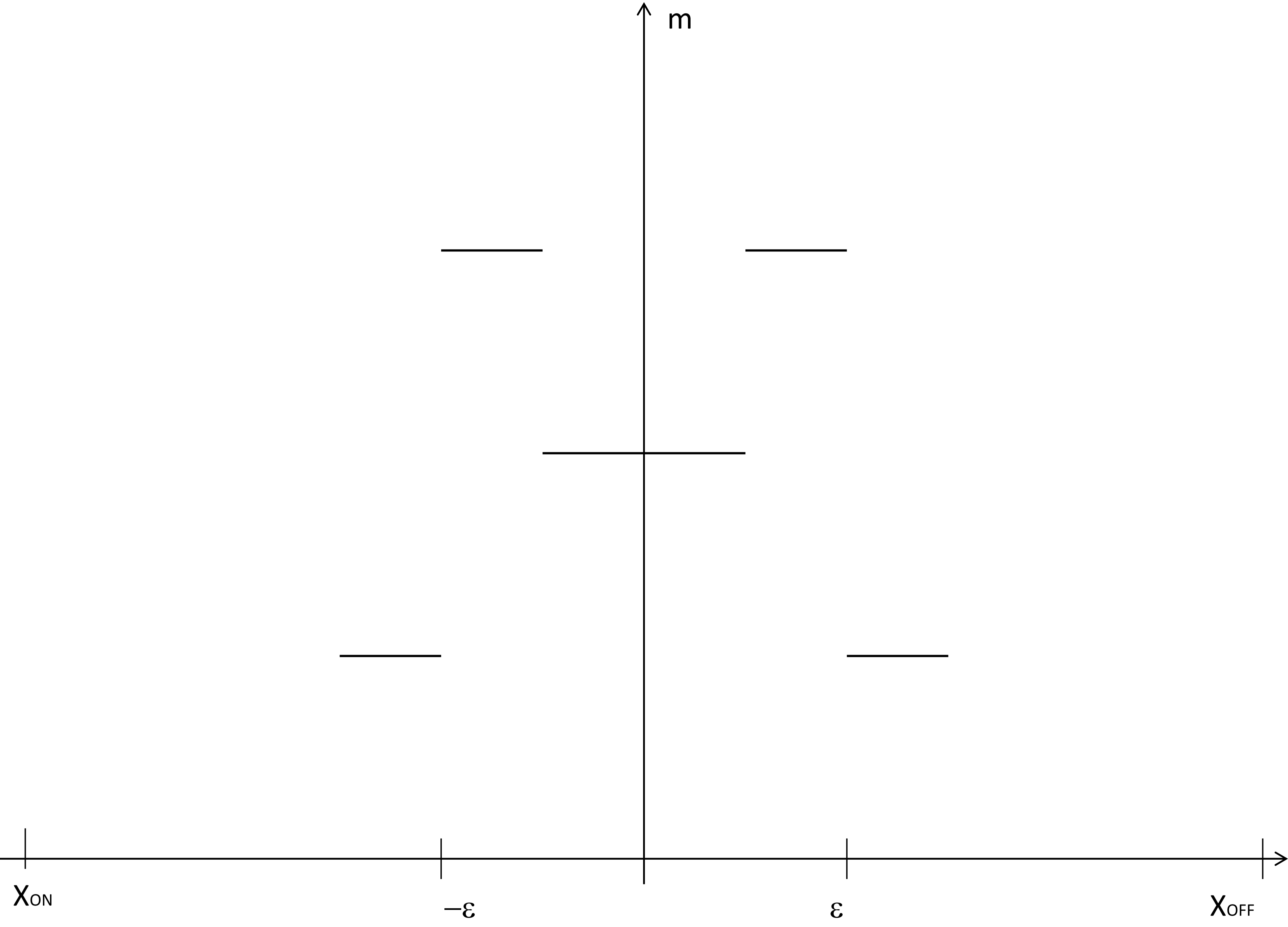


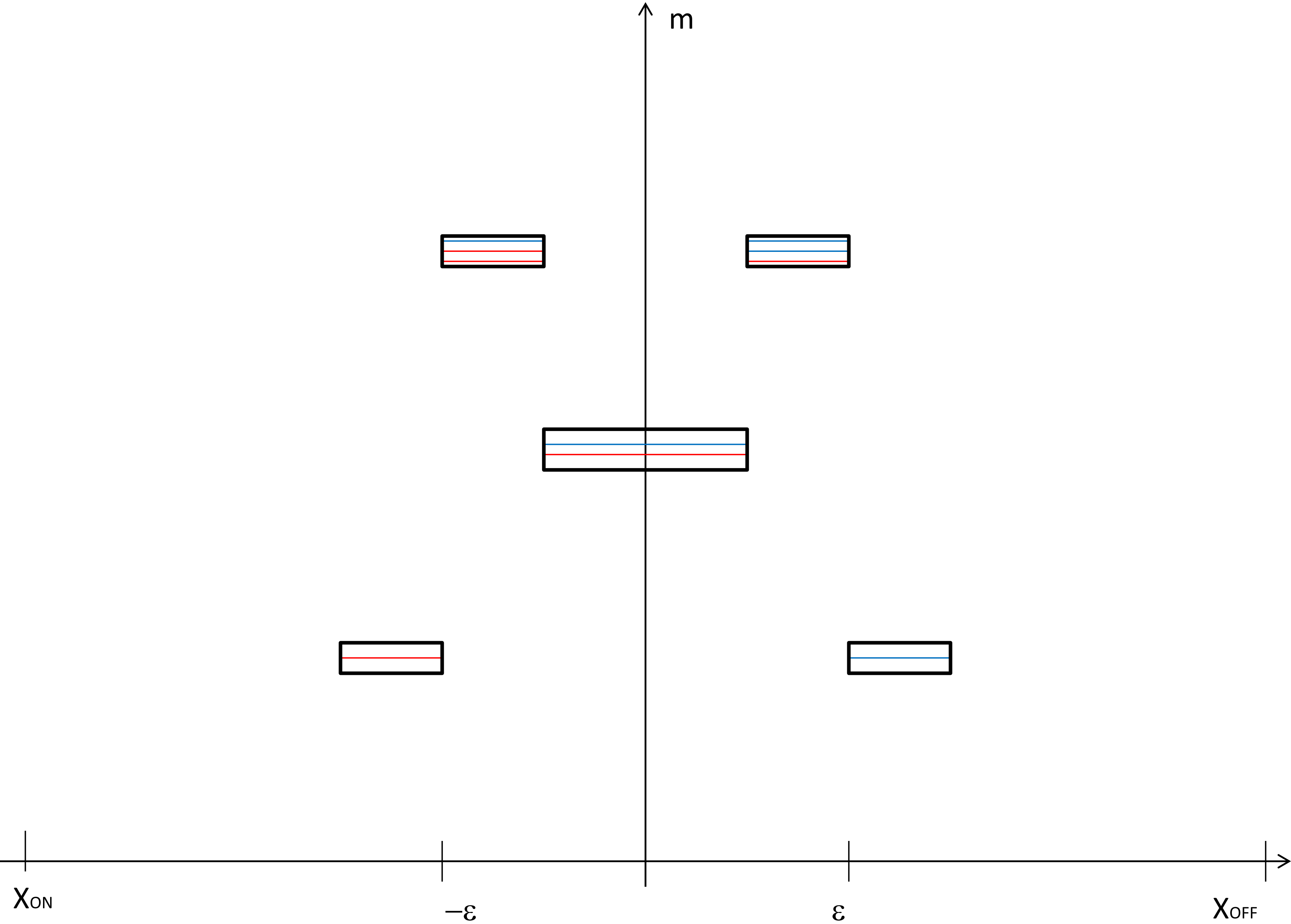
m
X_{ON}
$-\varepsilon$
ε
X_{OFF}

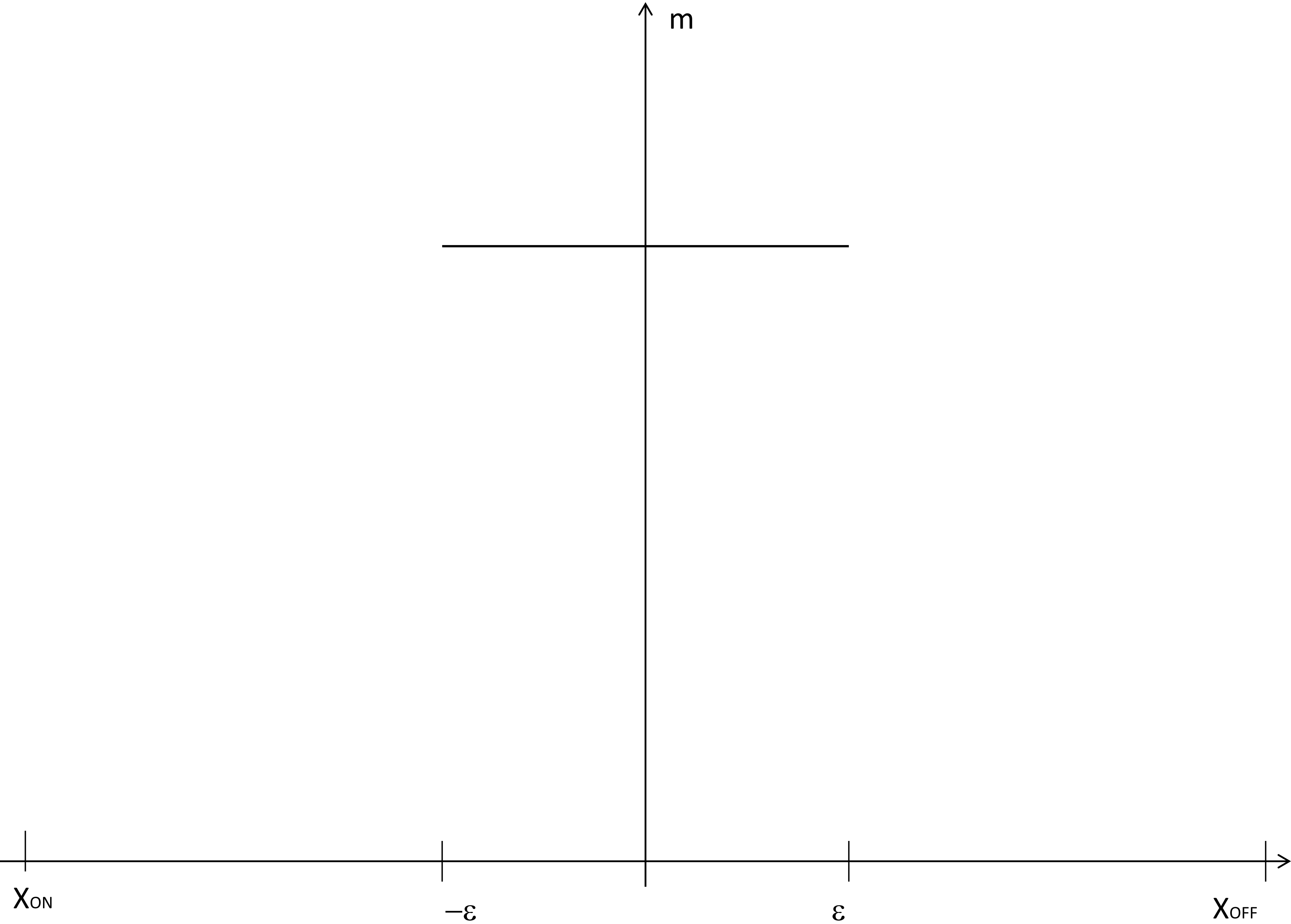


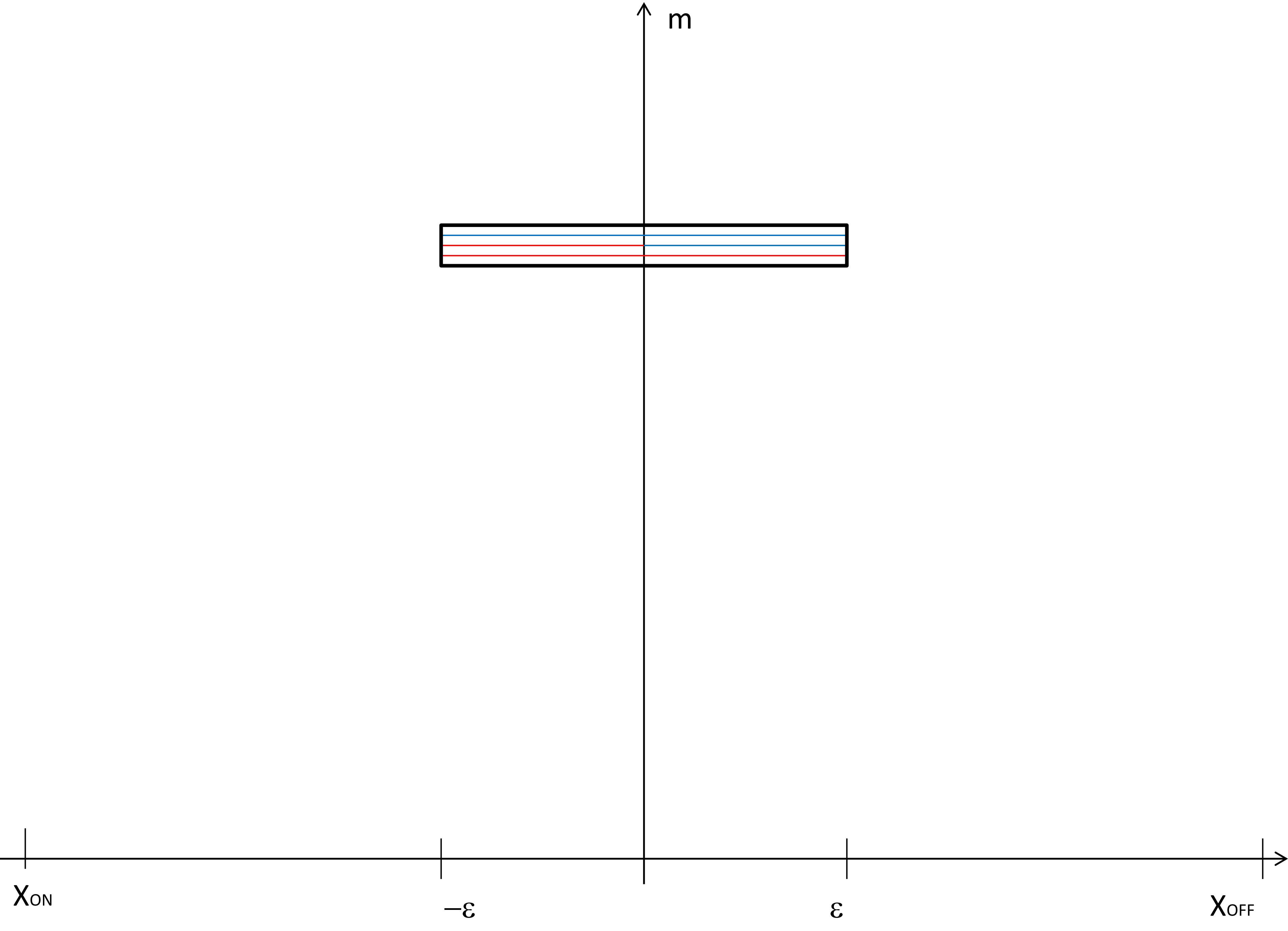
m
X_ON
−ε
ε
X_OFF


m
X_{ON}
−ε
ε
X_{OFF}

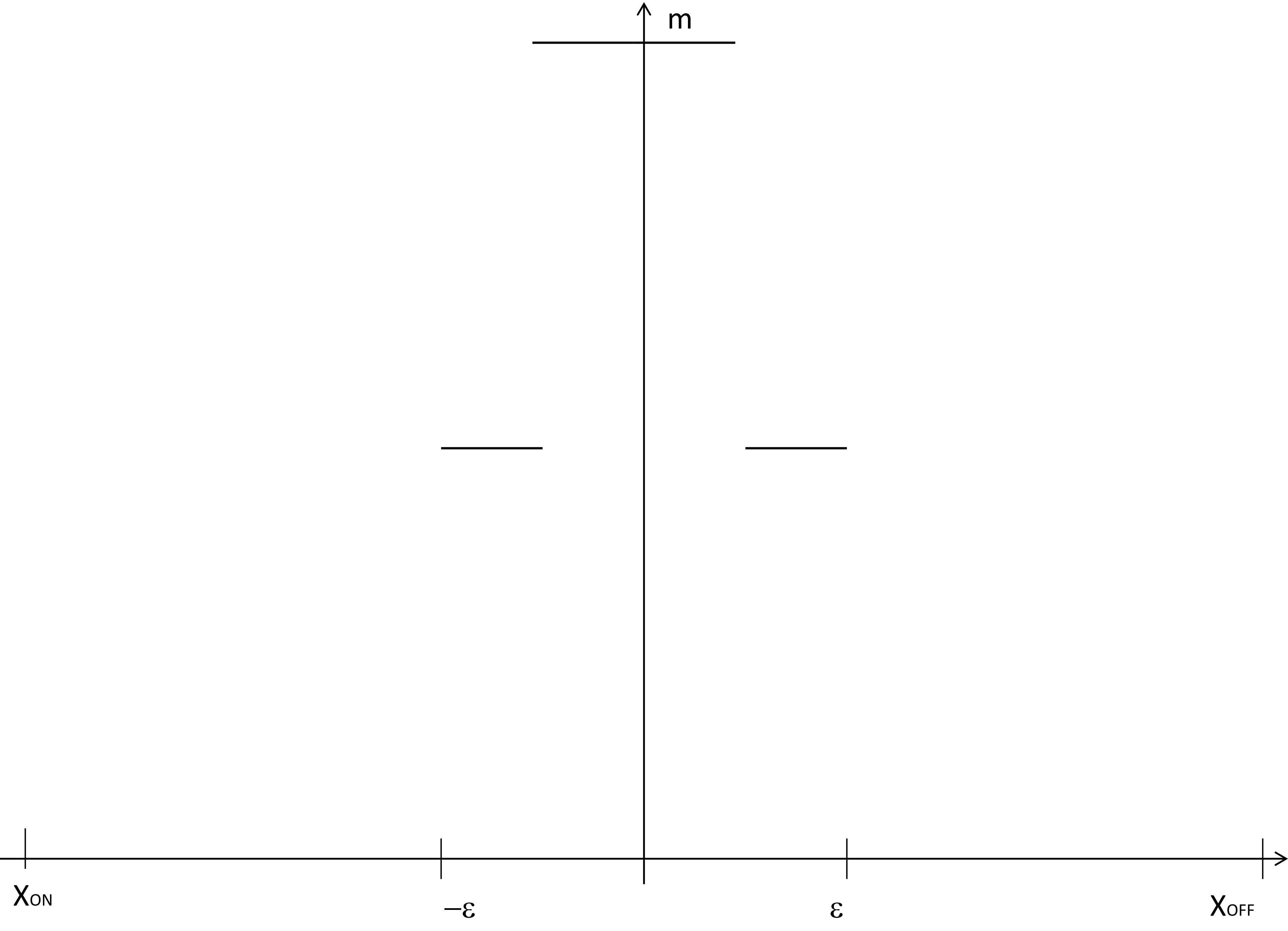

m
XON
−ε
ε
XOFF

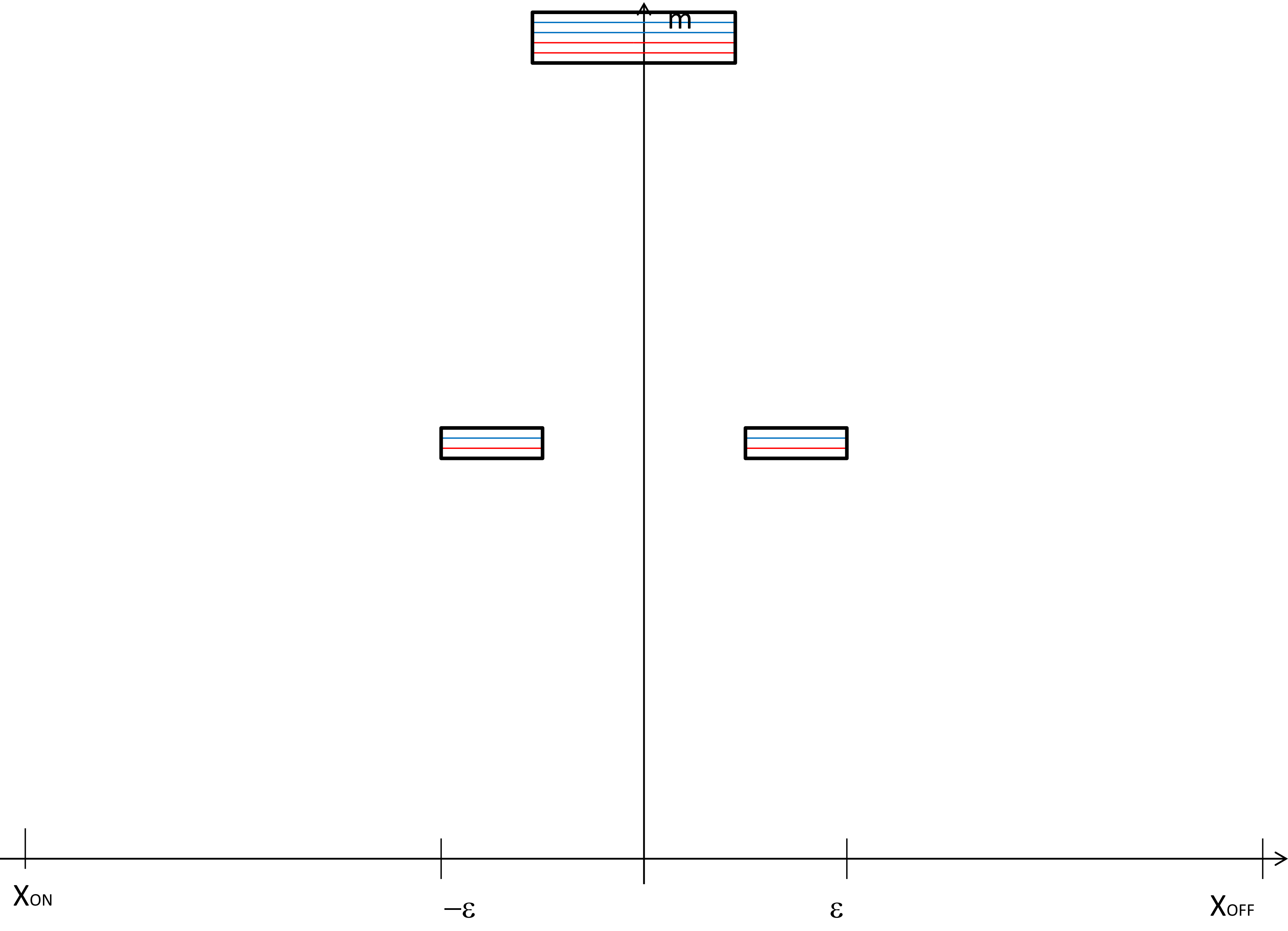

m
X_{ON}
$-\varepsilon$
ε
X_{OFF}

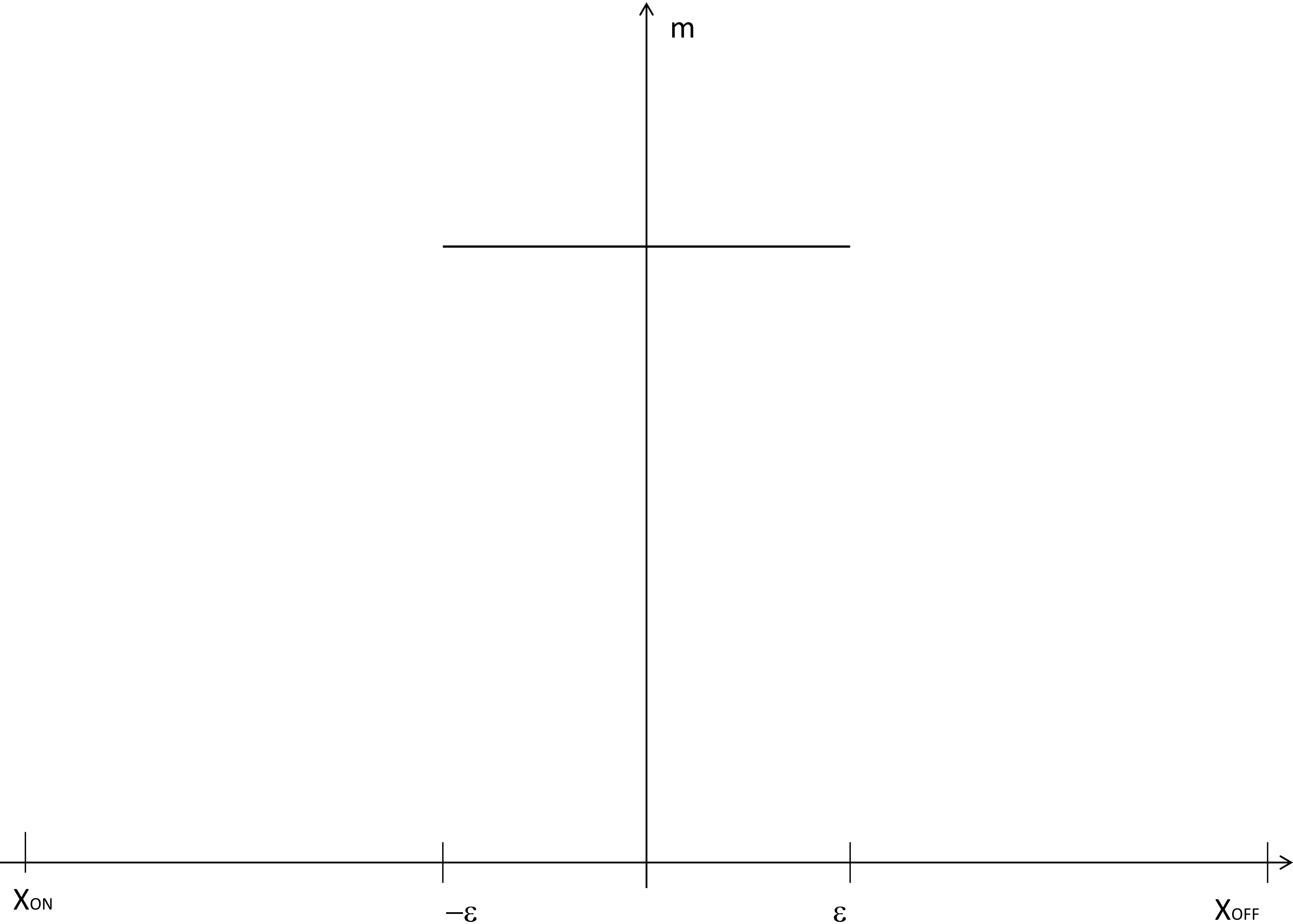

m
X_{ON}
$-\varepsilon$
ε
X_{OFF}



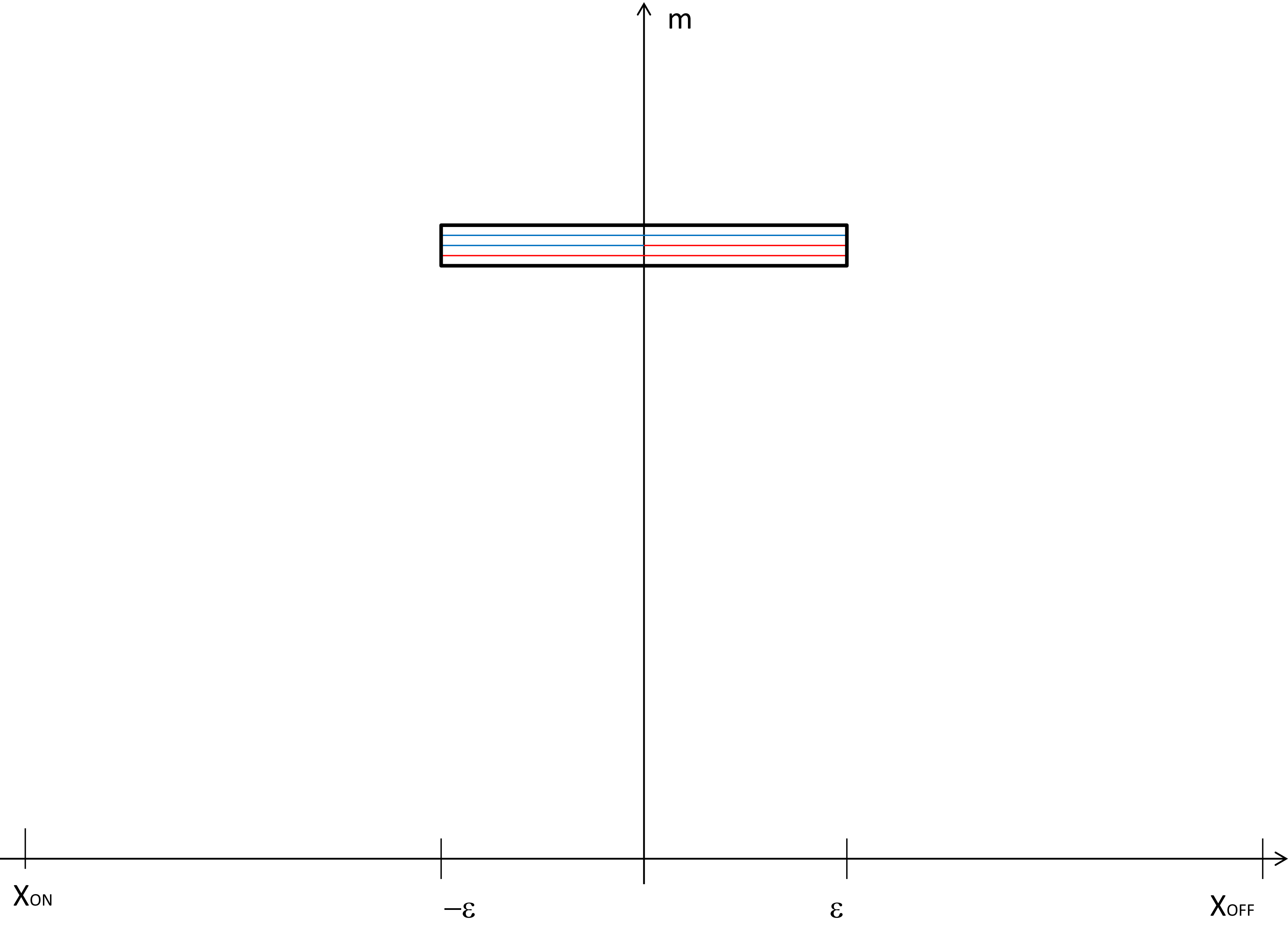
m
X_{ON}
$-\varepsilon$
ε
X_{OFF}


m
XON
−ε
ε
XOFF

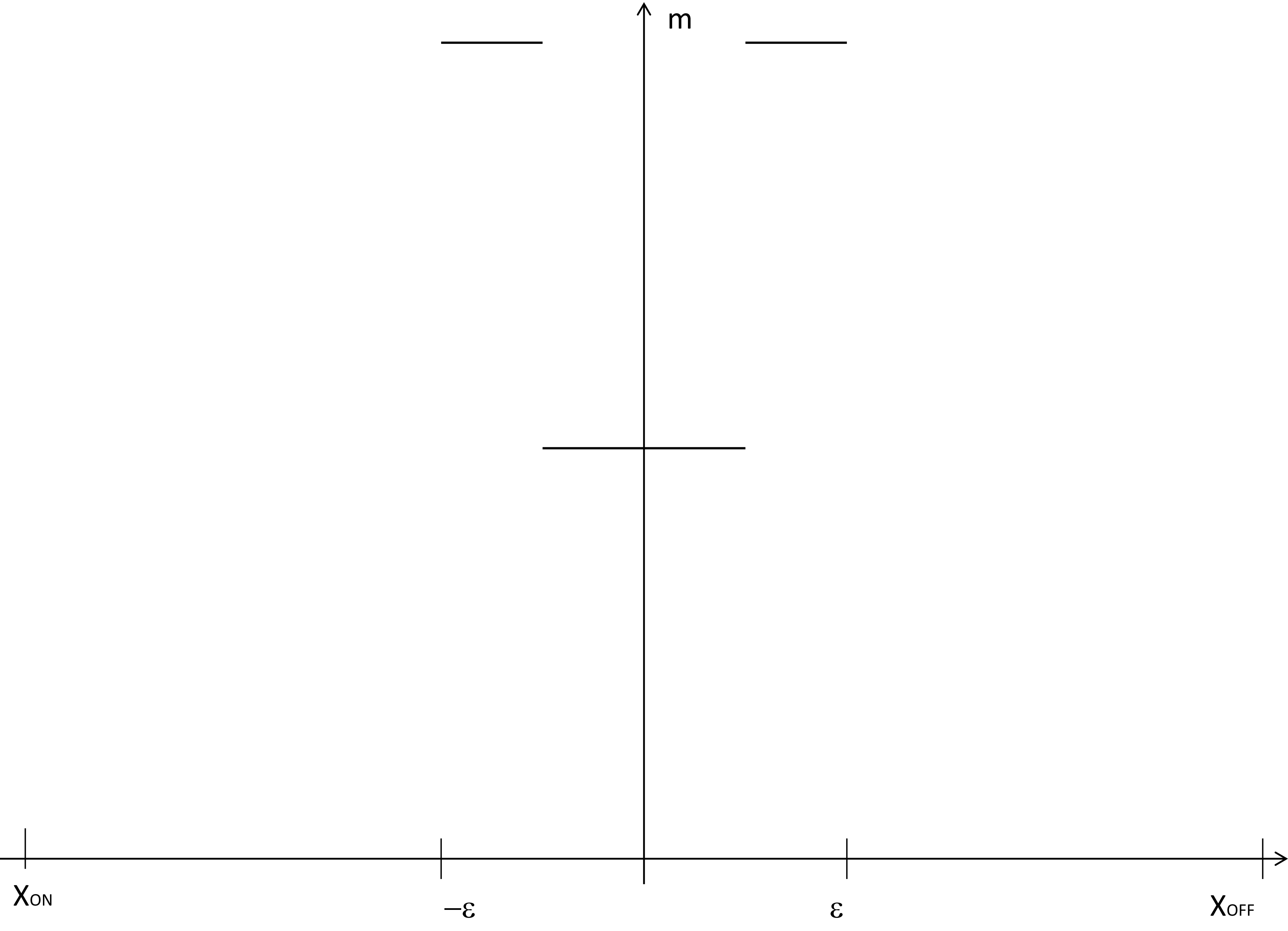

m
X_{ON}
$-\varepsilon$
ε
X_{OFF}

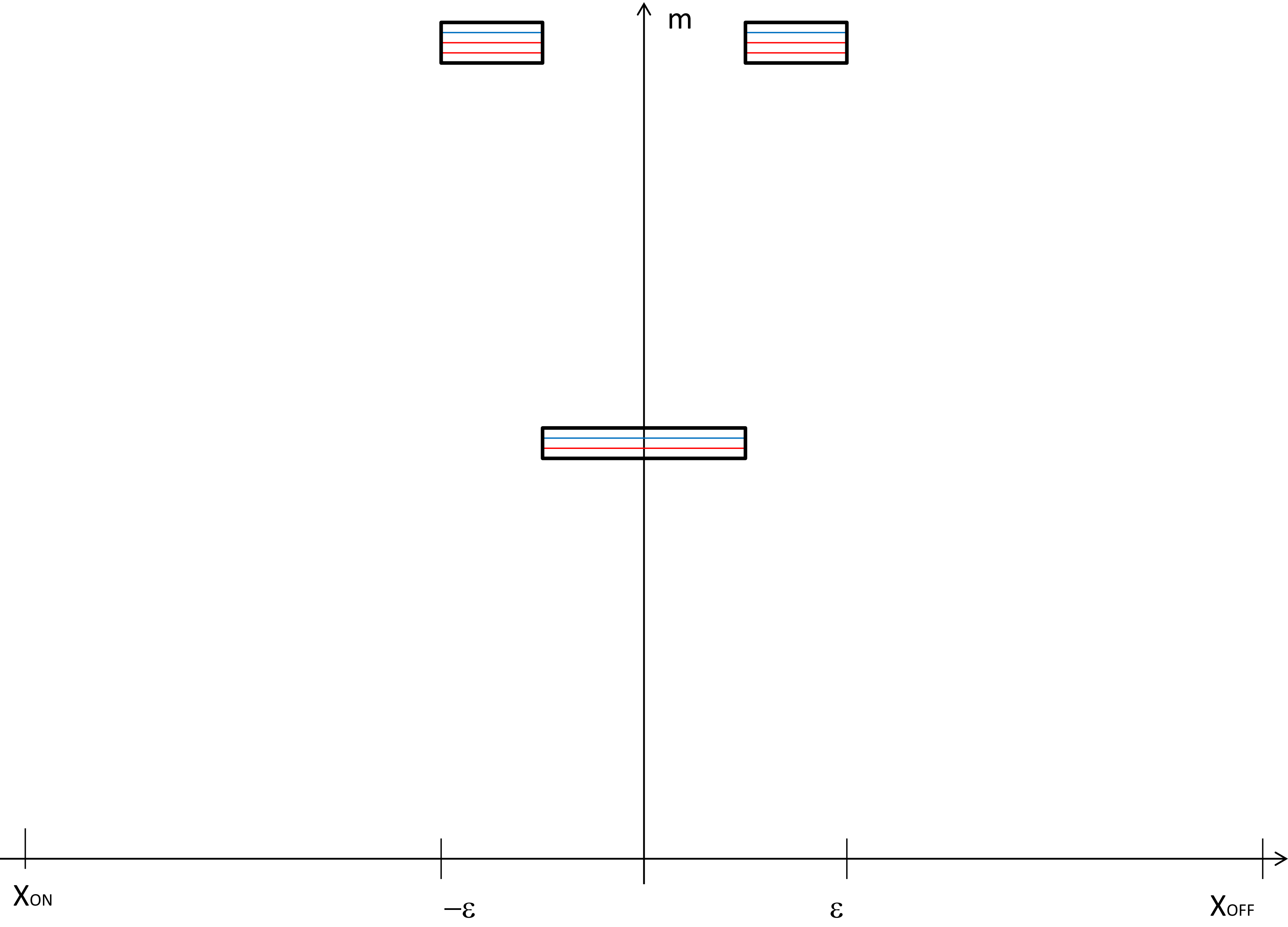


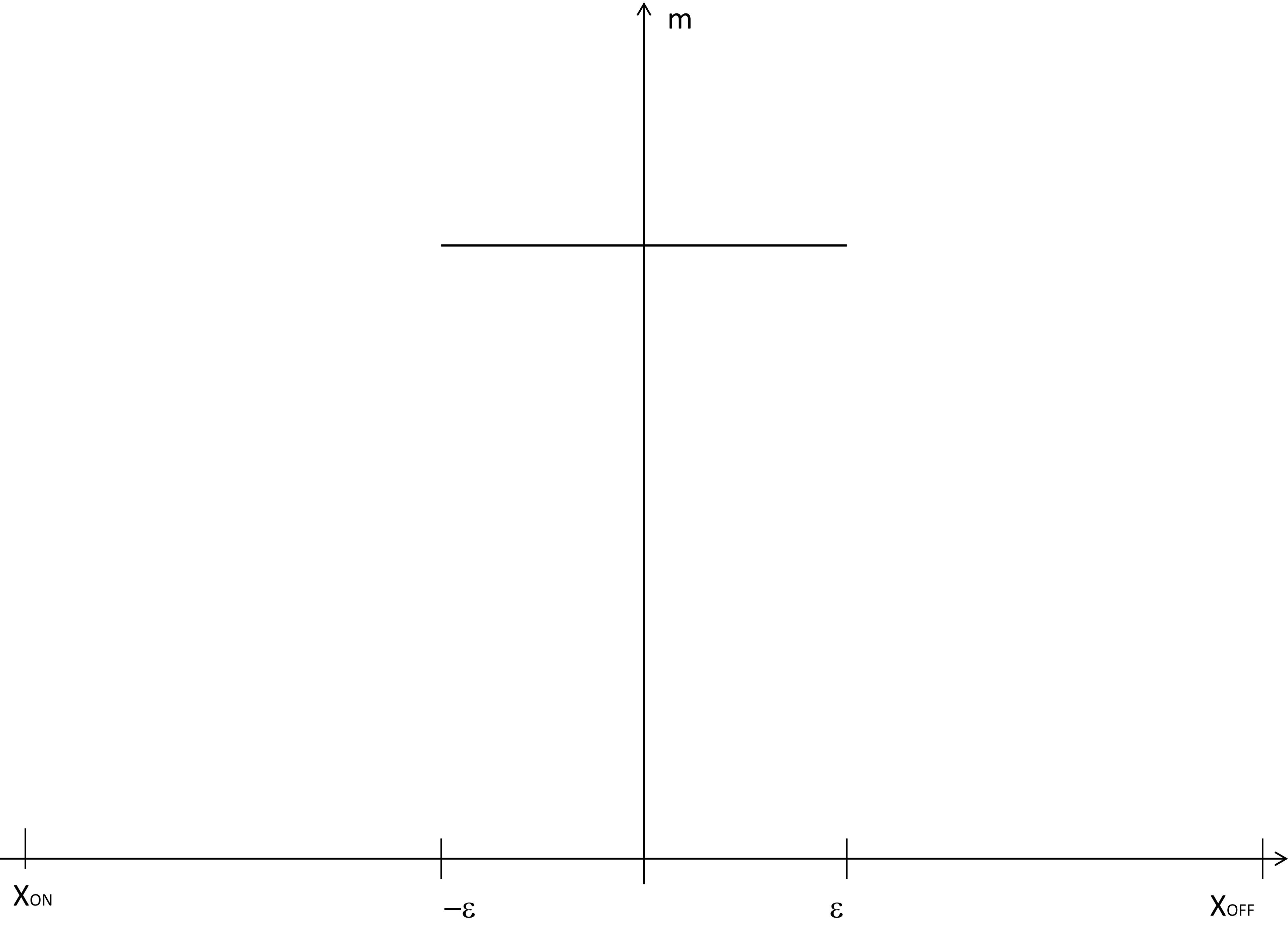
m
XON
−ε
ε
XOFF

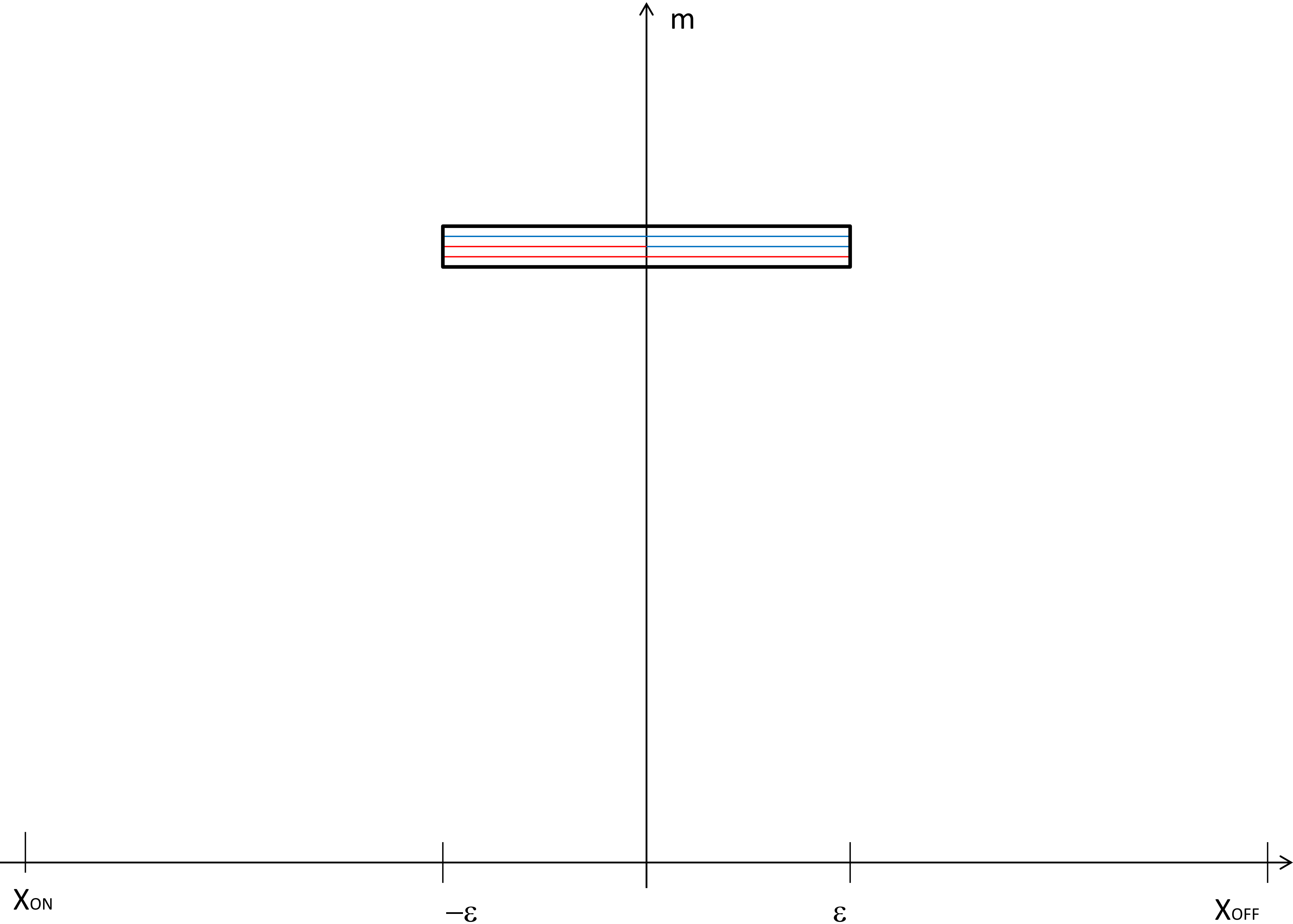


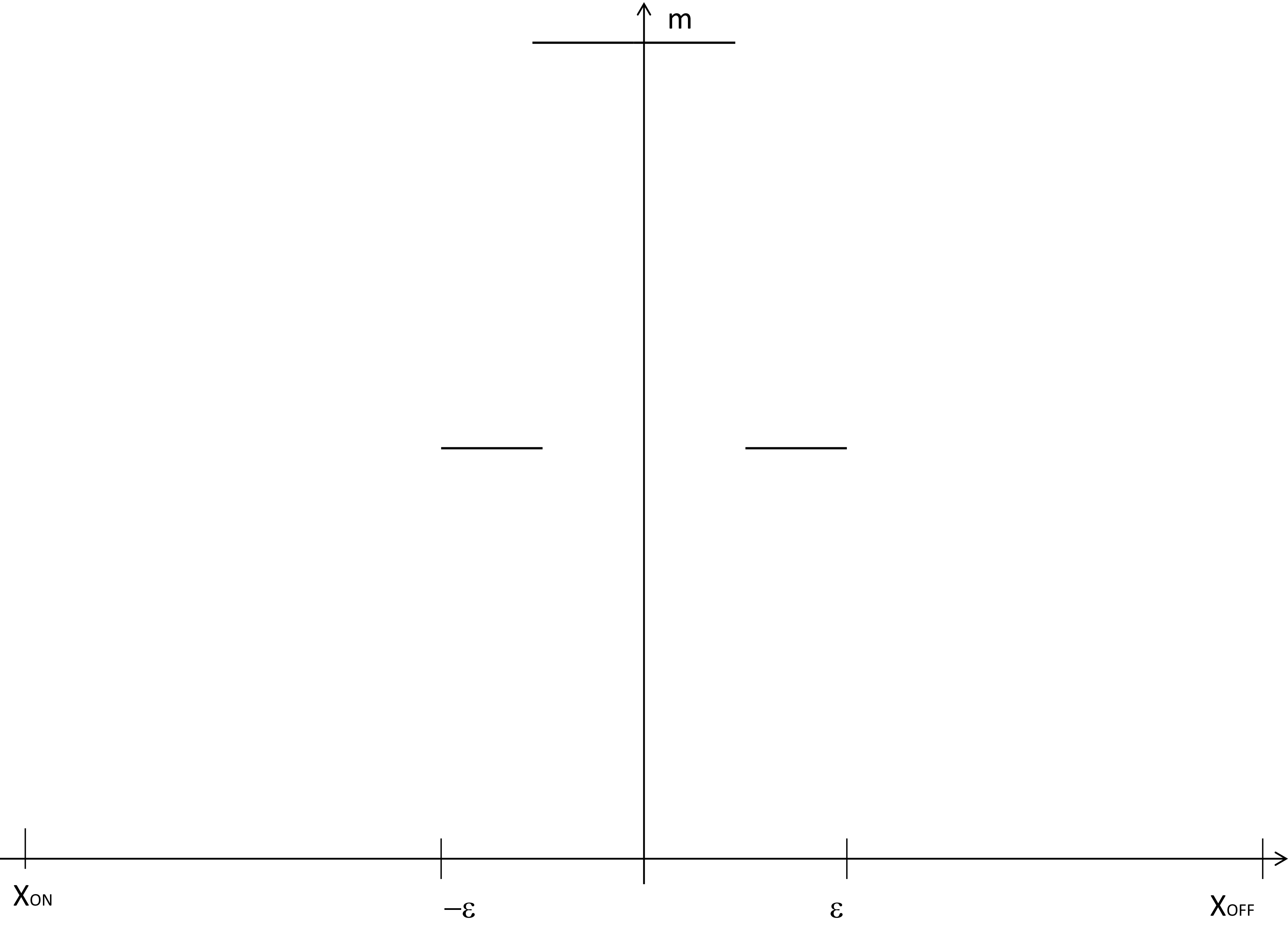
m
X_{ON}
$-\varepsilon$
ε
X_{OFF}


m
XON
−ε
ε
XOFF

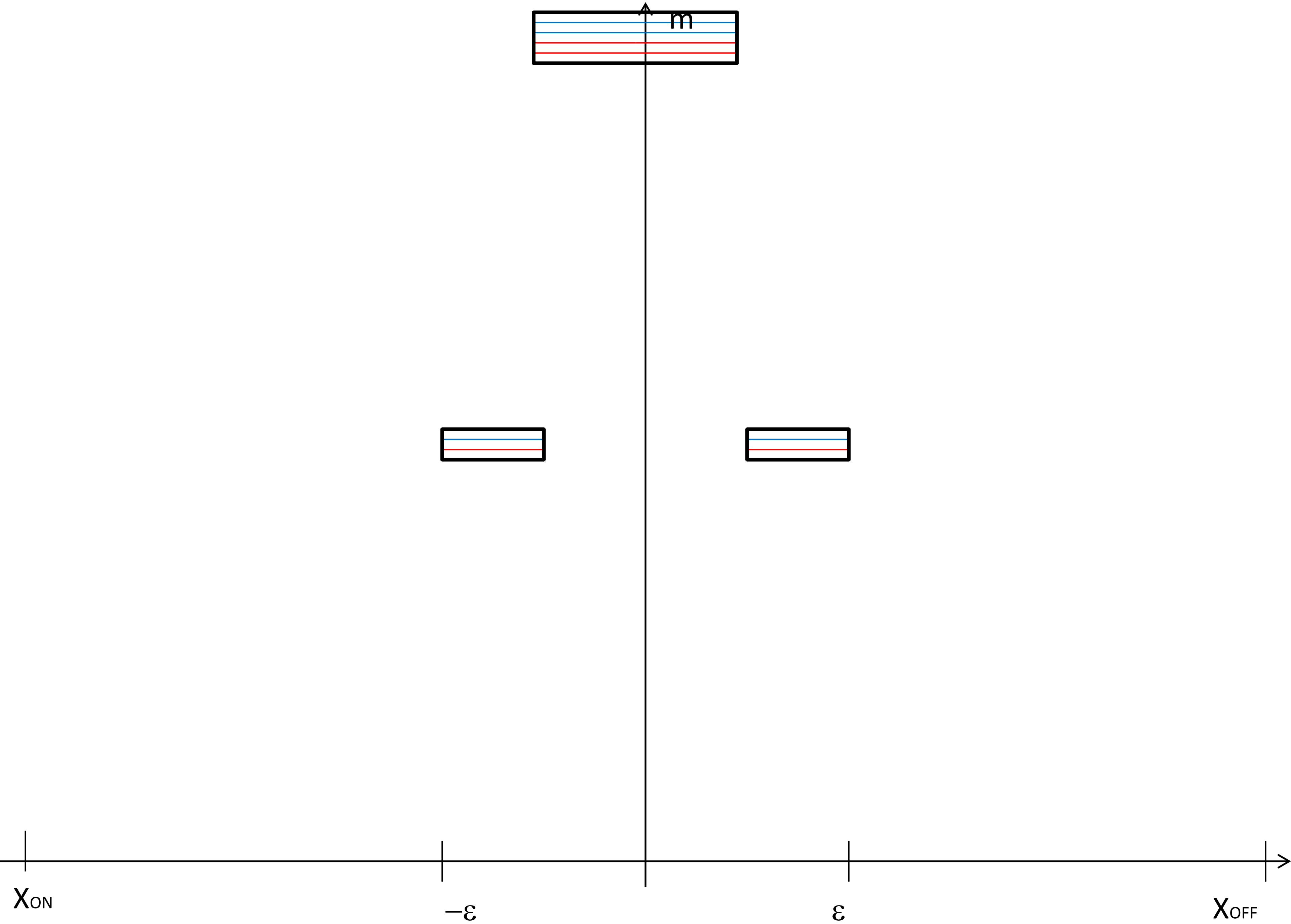

m
XON
−ε
ε
XOFF

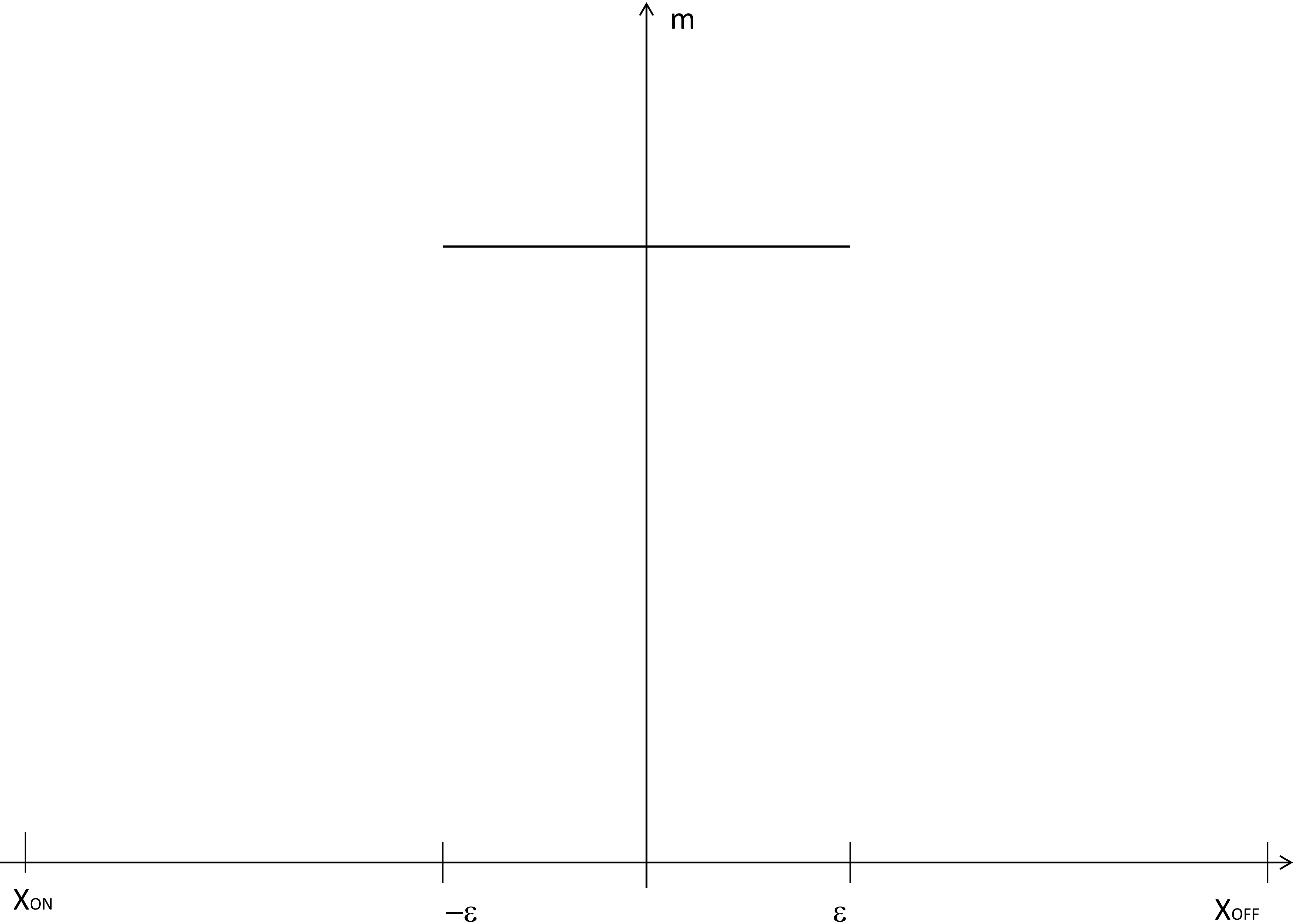

m
X ON
−ε
ε
X OFF


m
X_{ON}
$-\varepsilon$
ε
X_{OFF}

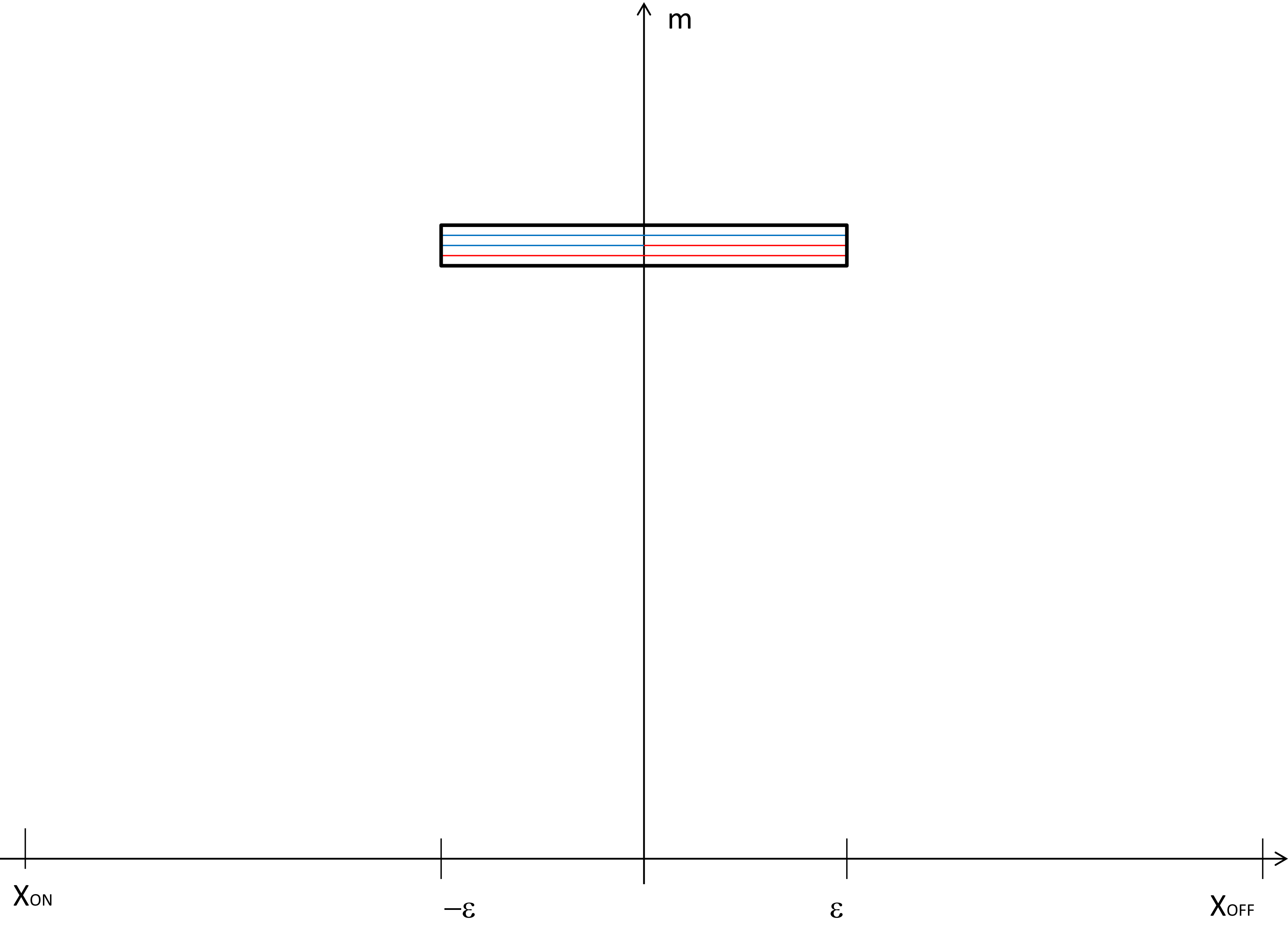


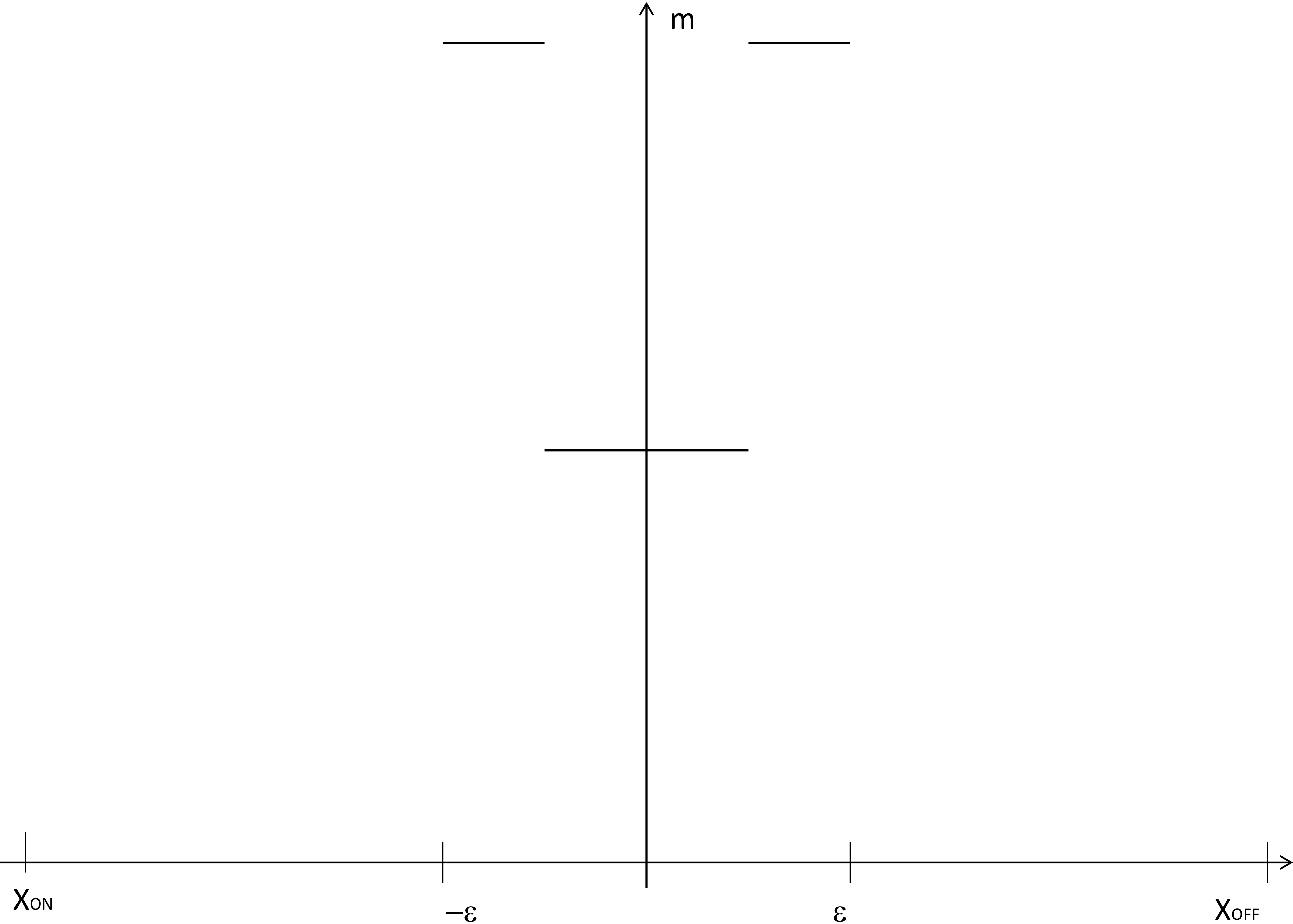
m
XON
−ε
ε
XOFF


m
XON
−ε
ε
XOFF

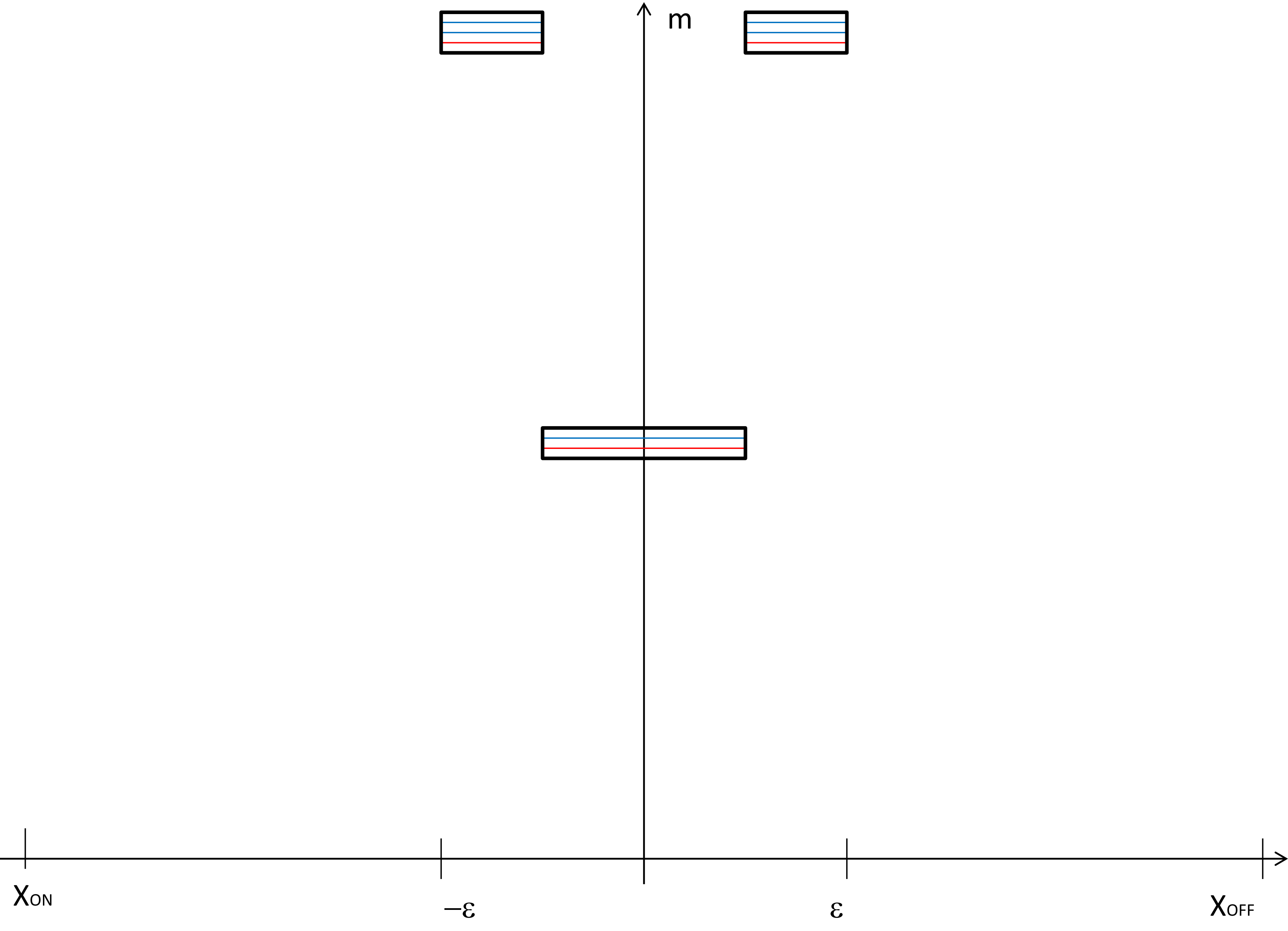


m
X_{ON}
$-\varepsilon$
ε
X_{OFF}

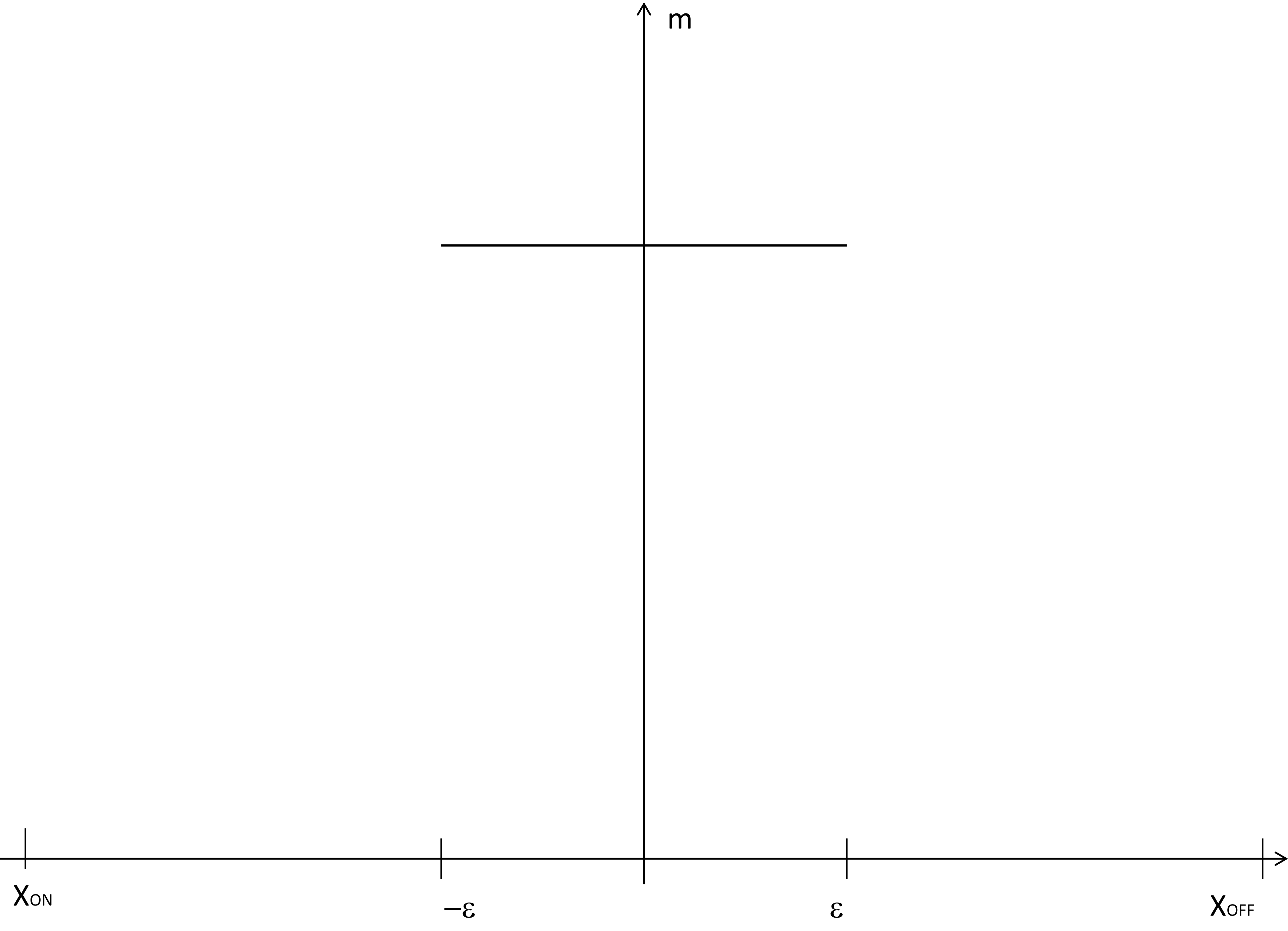

m
X_{ON}
$-\varepsilon$
ε
X_{OFF}



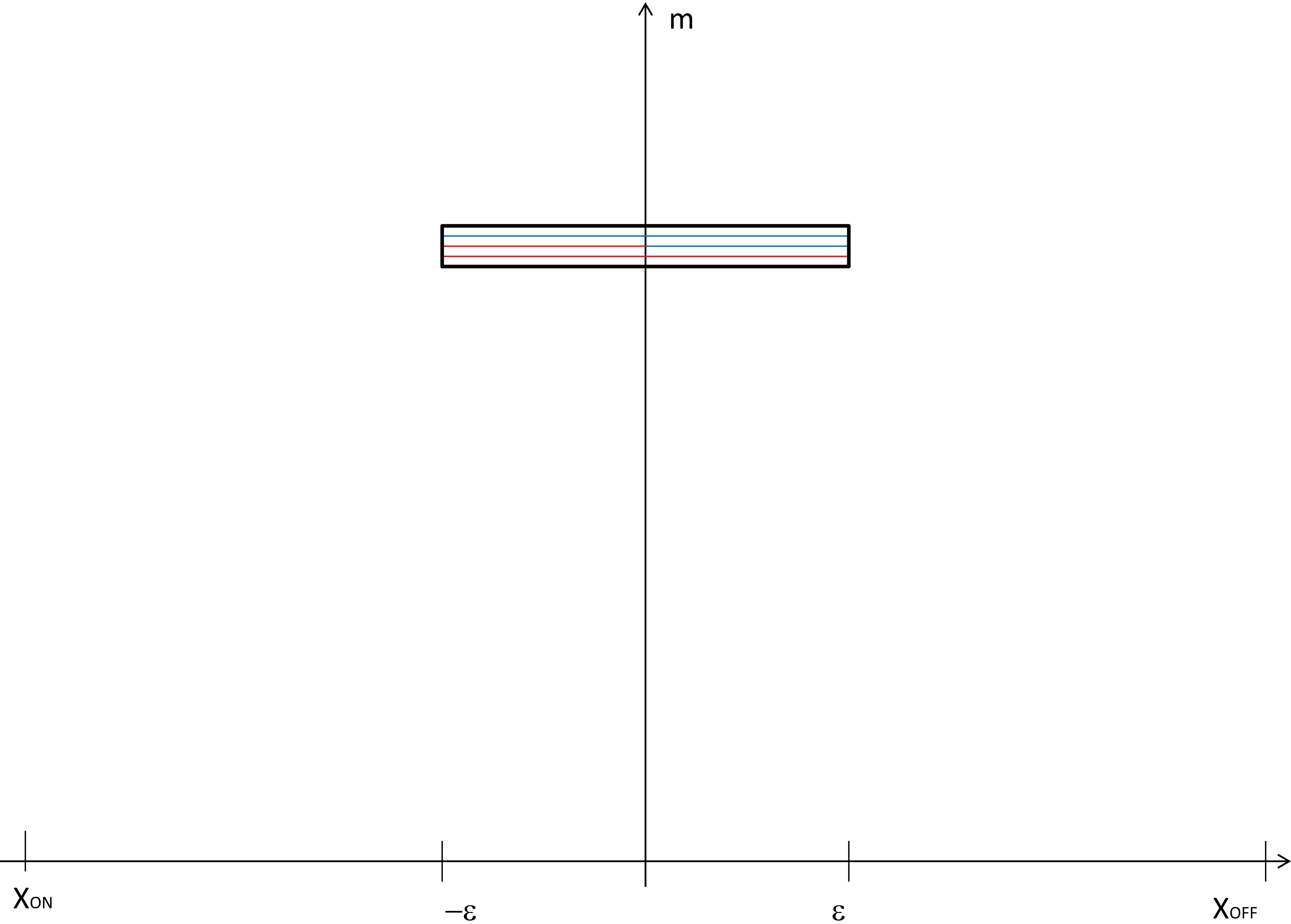
m
X ON
−ε
ε
X OFF



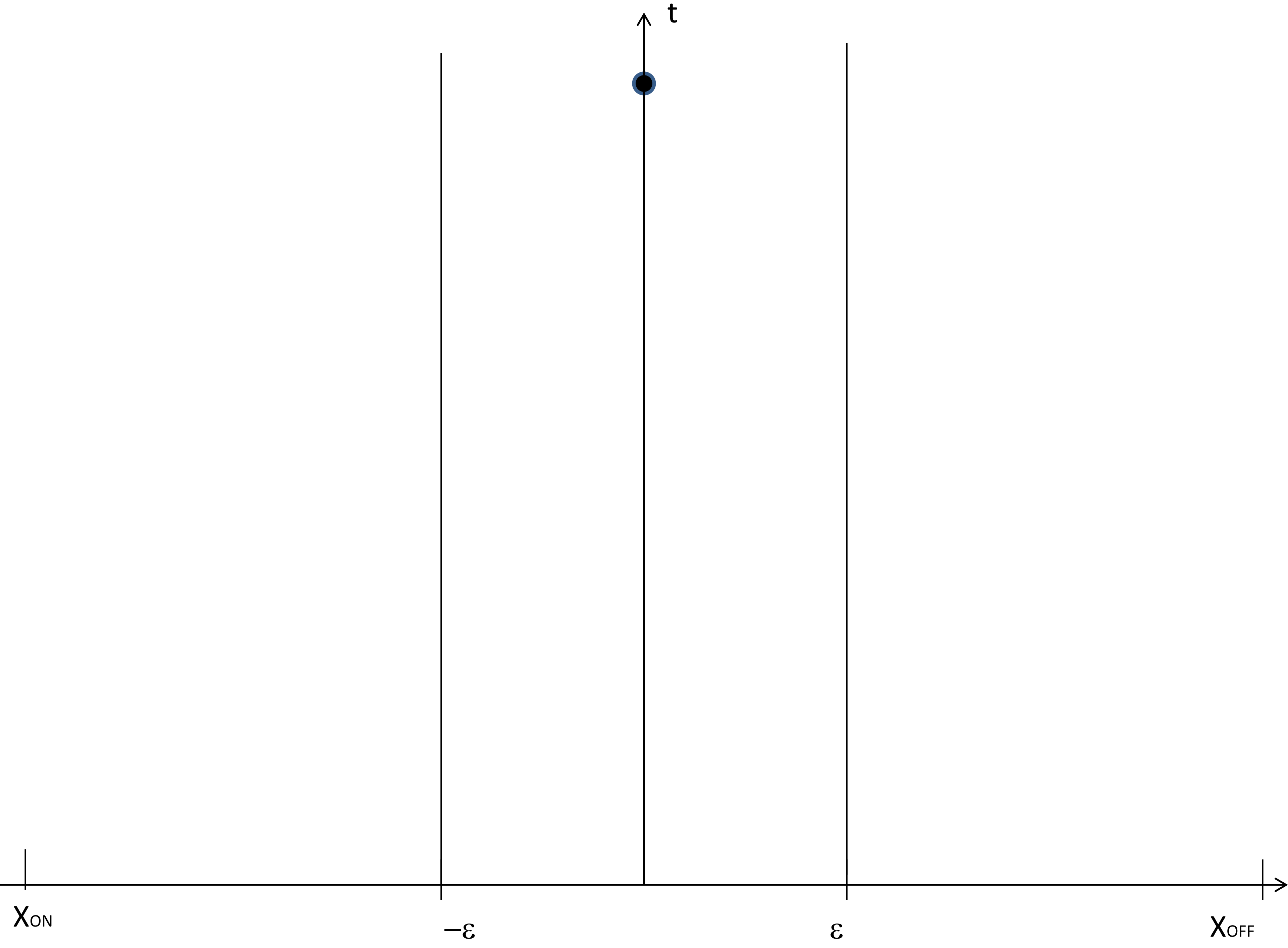
m
XON
−ε
ε
XOFF

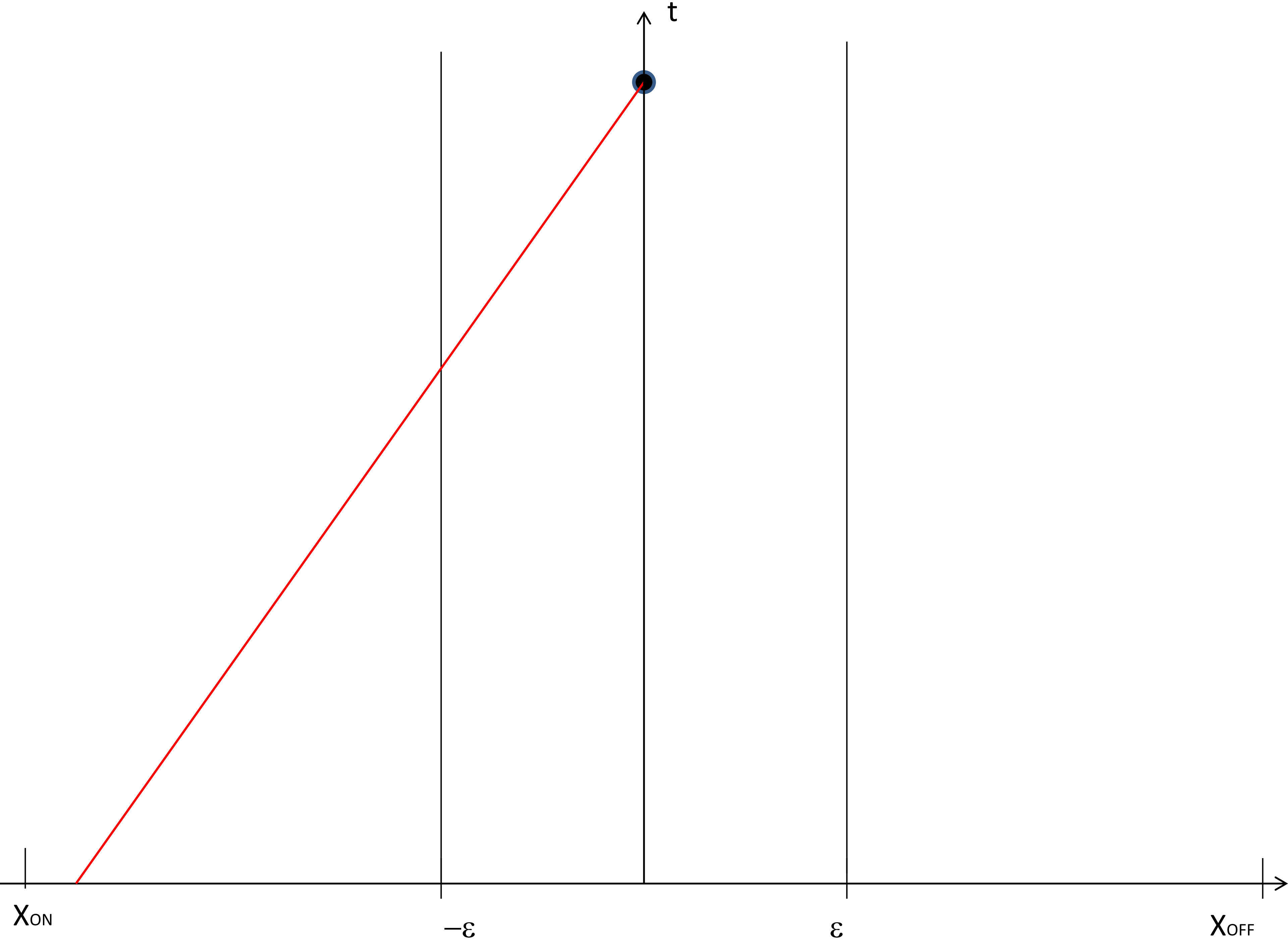


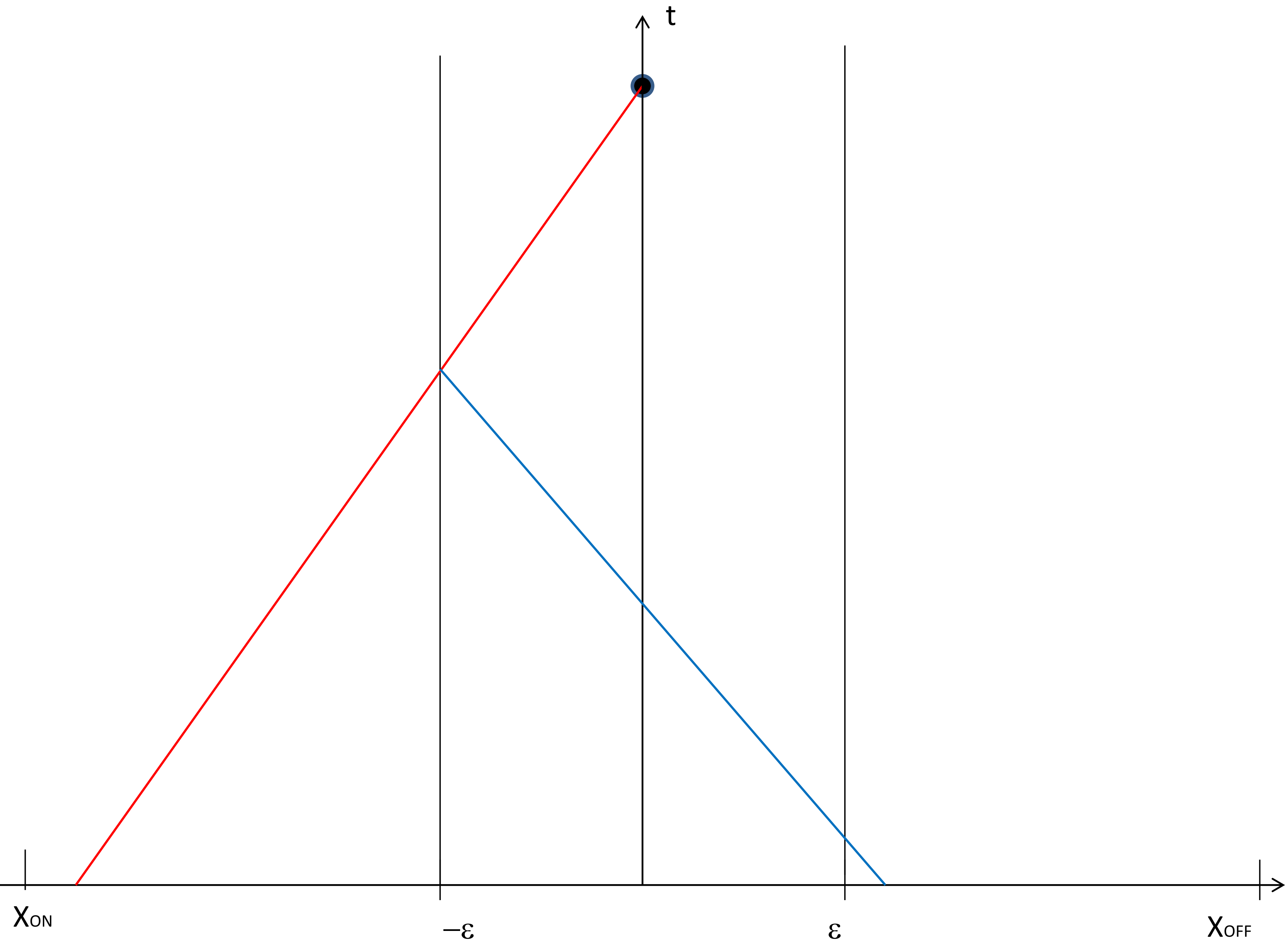
m
X_{ON}
$-\varepsilon$
ε
X_{OFF}

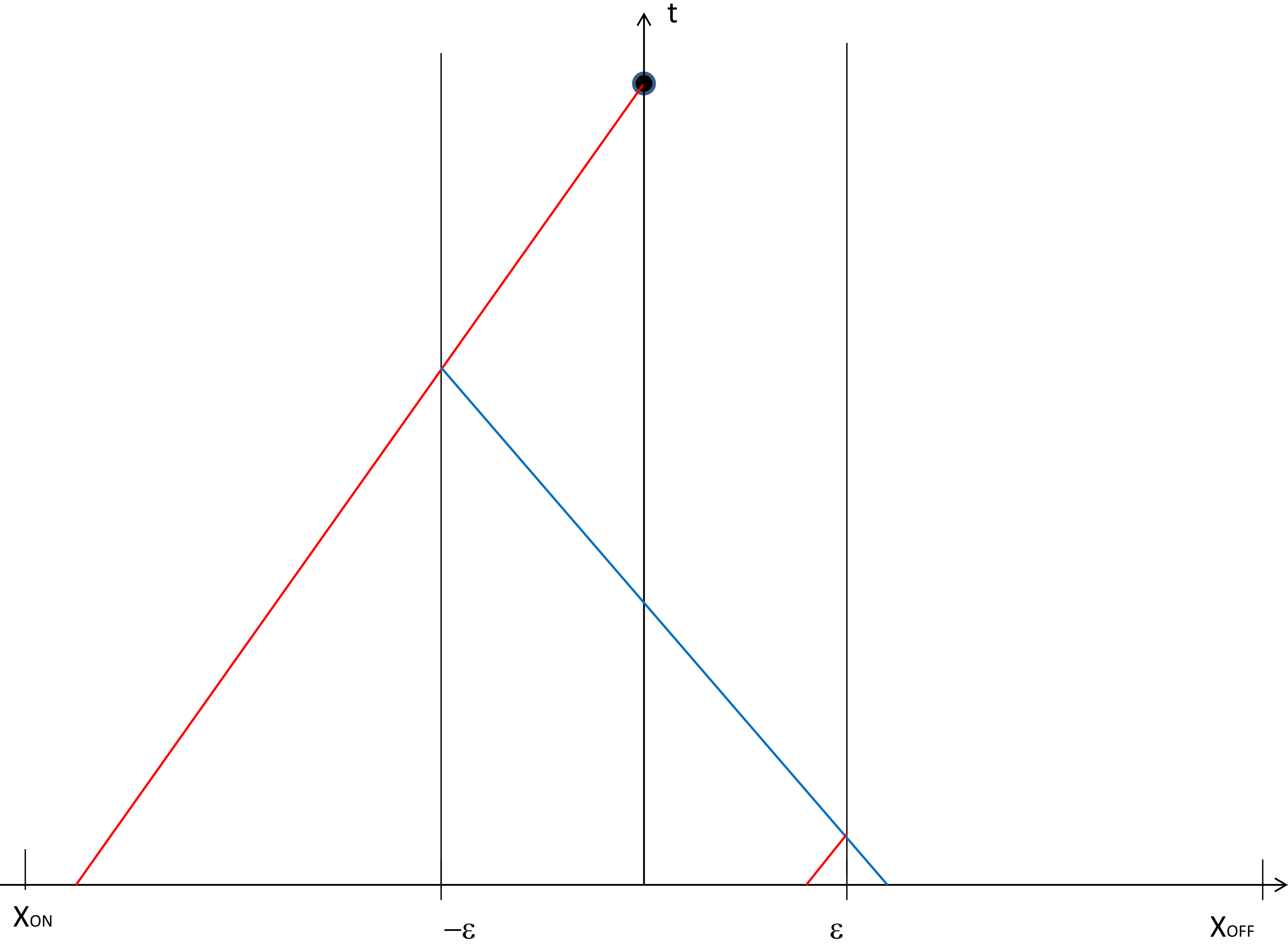

m
XON
−ε
ε
XOFF

- Does this kind of evolution of the distribution satisfy a suitable (Kolmogorov, transport) differential equation?

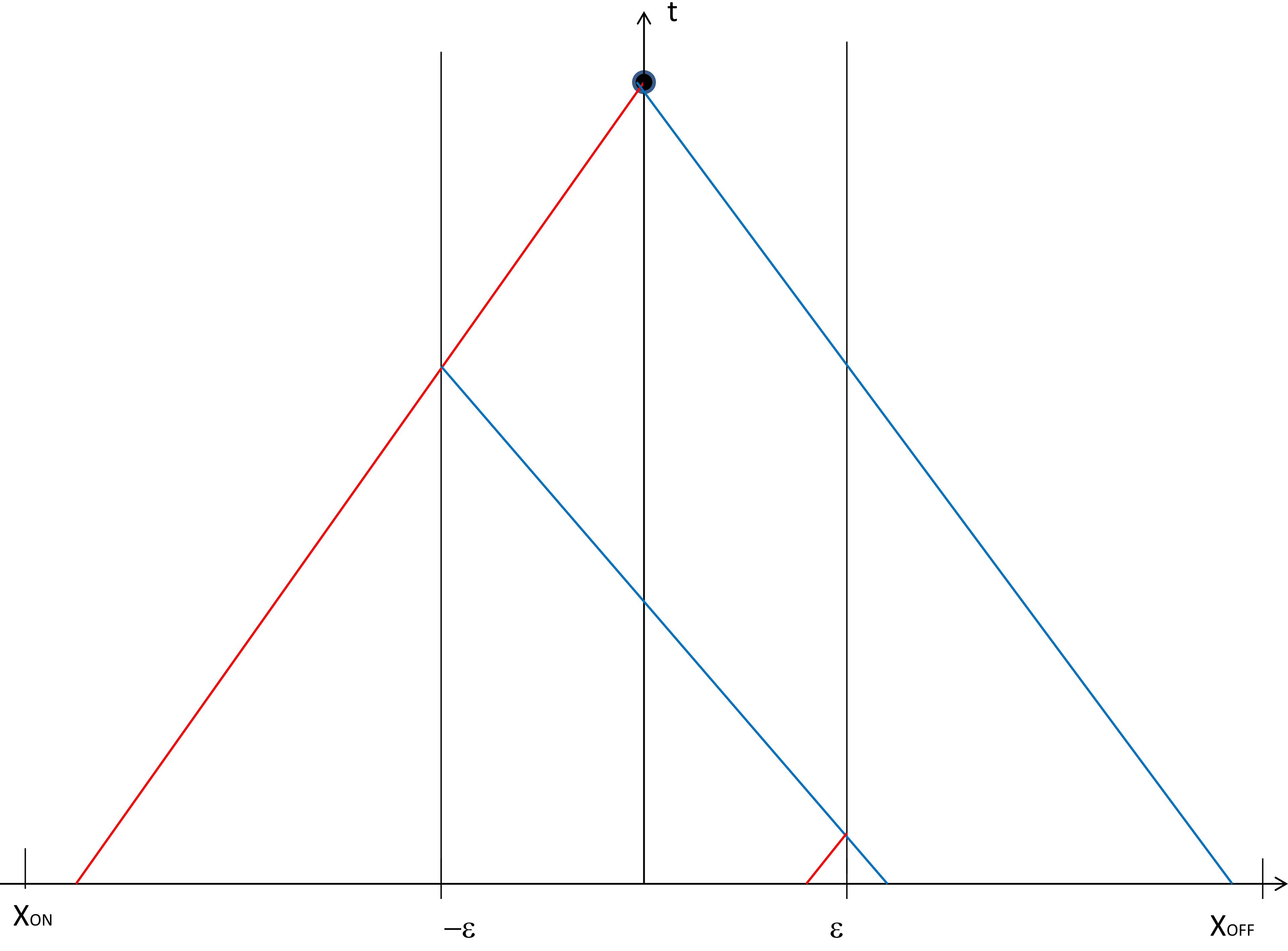

t
X_{ON}
$-\varepsilon$
ε
X_{OFF}


t
XON
−ε
ε
XOFF

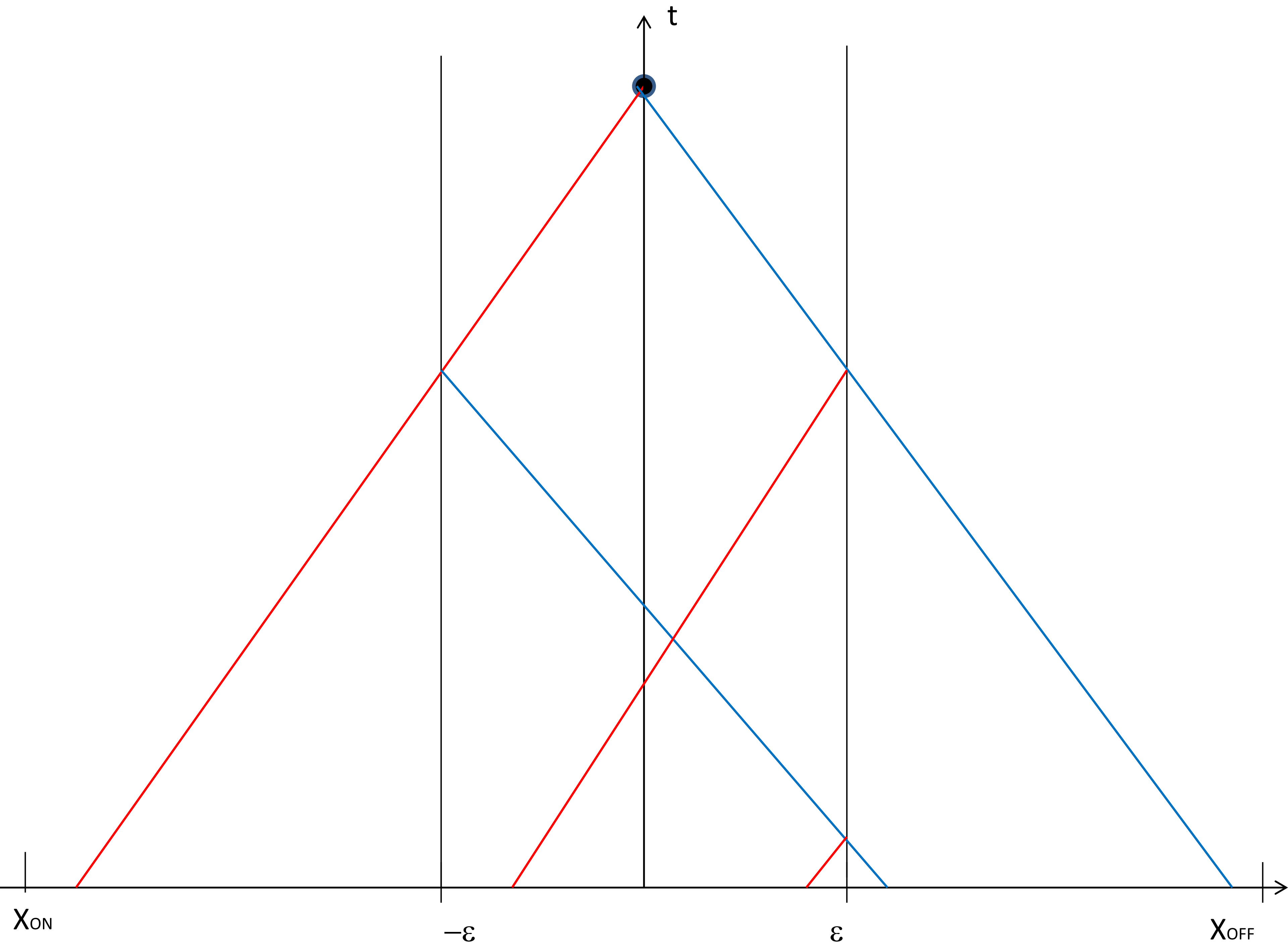

t
X_{ON}
$-\varepsilon$
ε
X_{OFF}



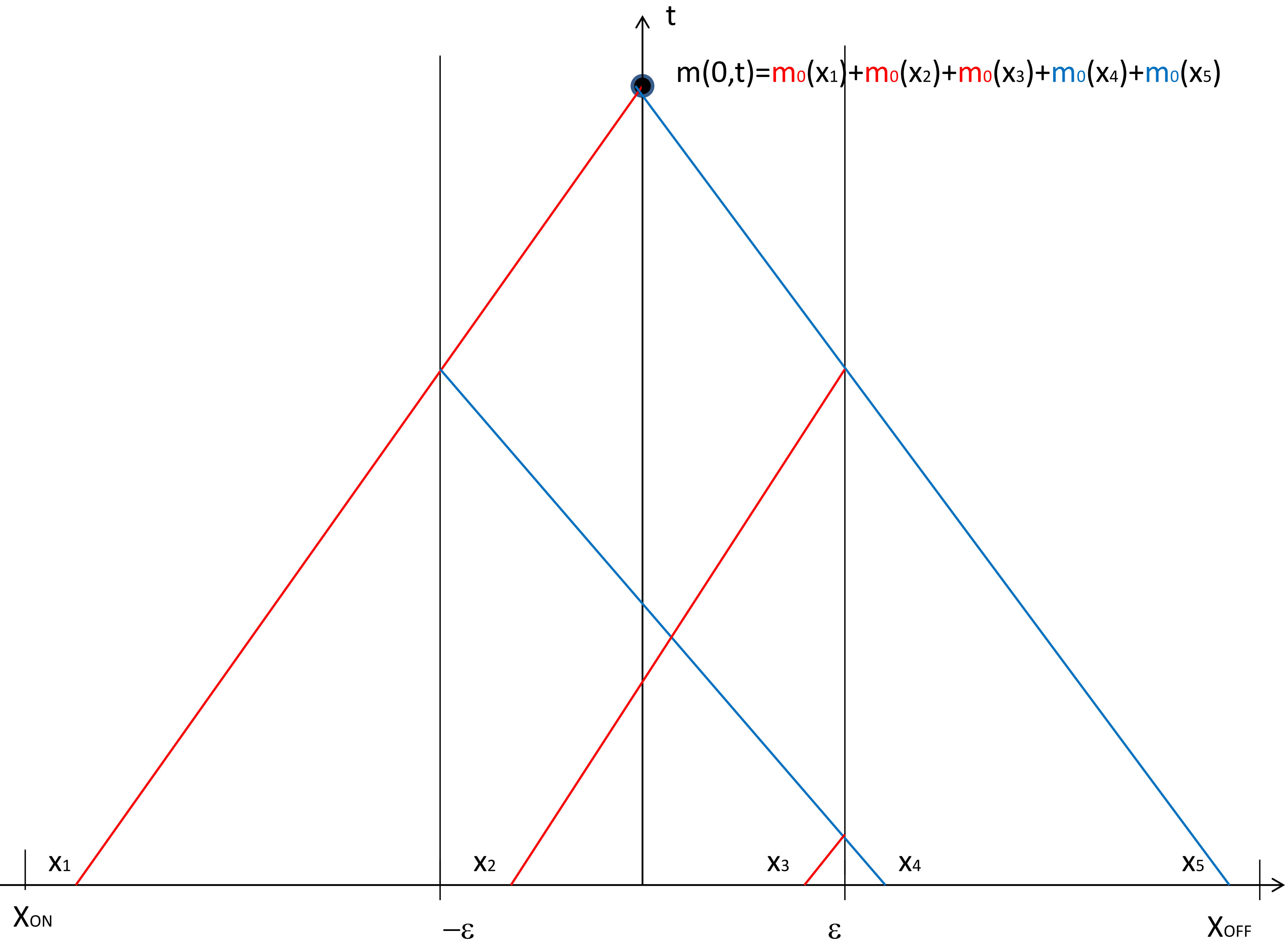
t
X_{ON}
$-\varepsilon$
ε
X_{OFF}

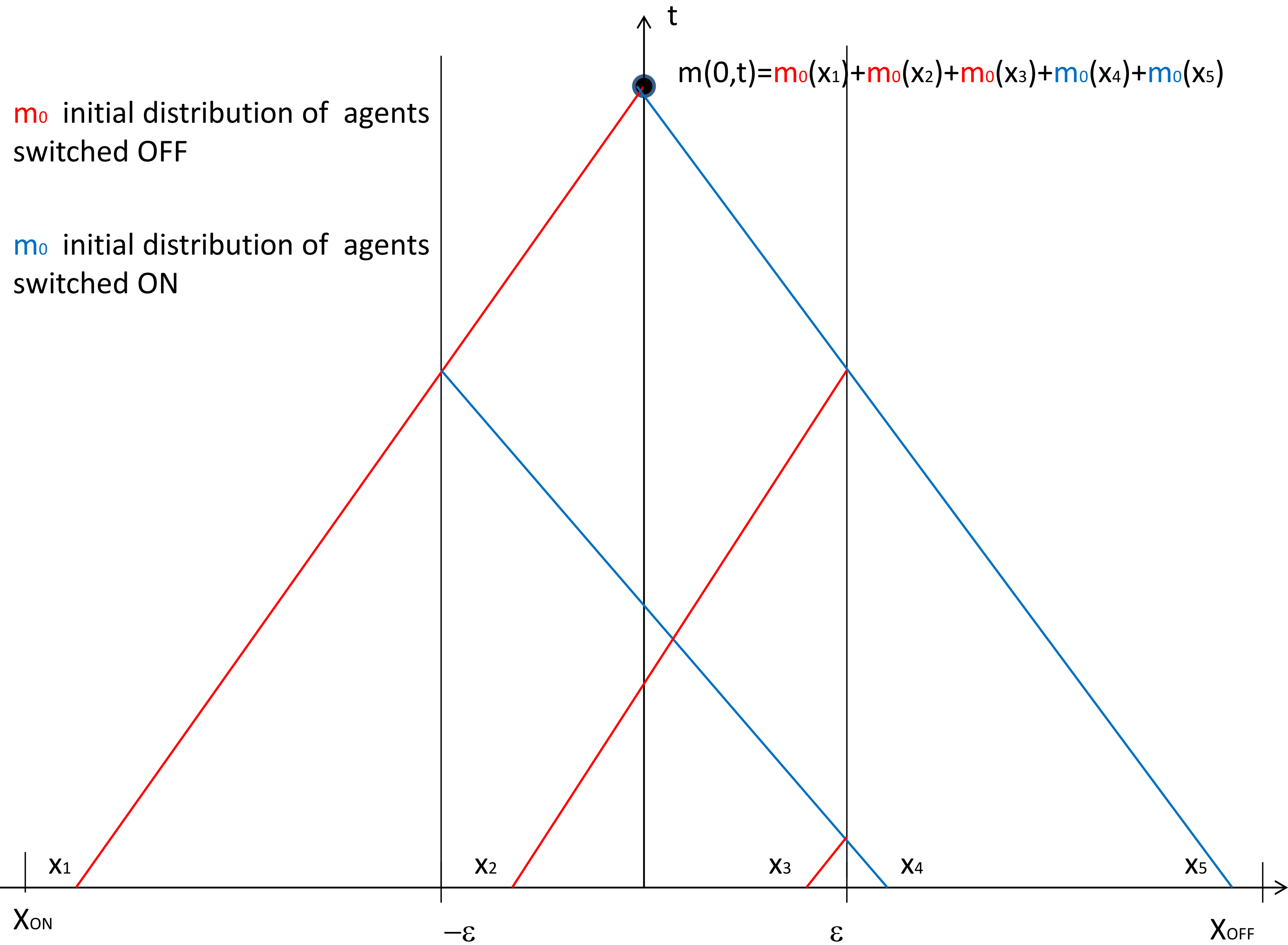


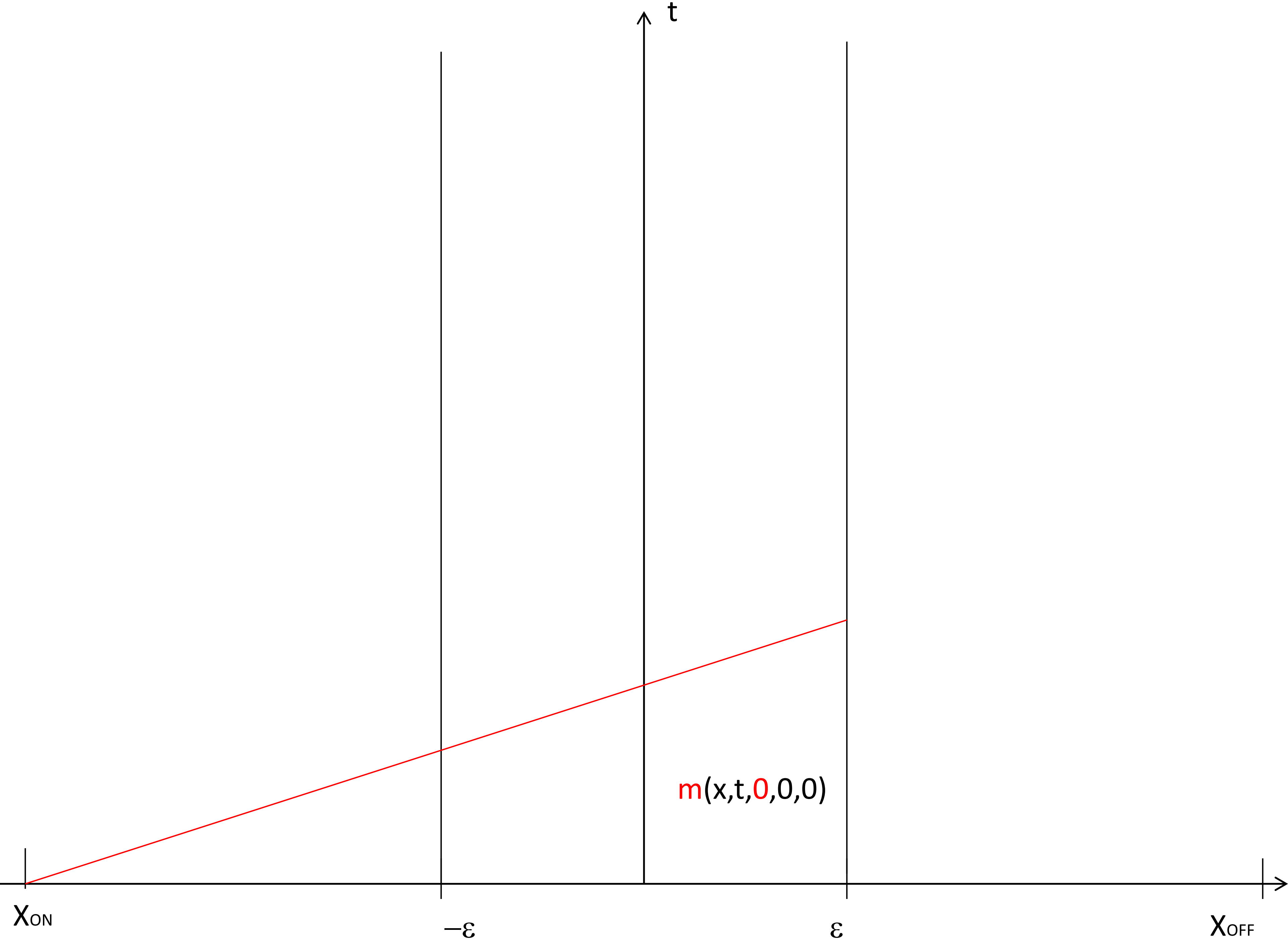
t
X ON
−ε
ε
X OFF

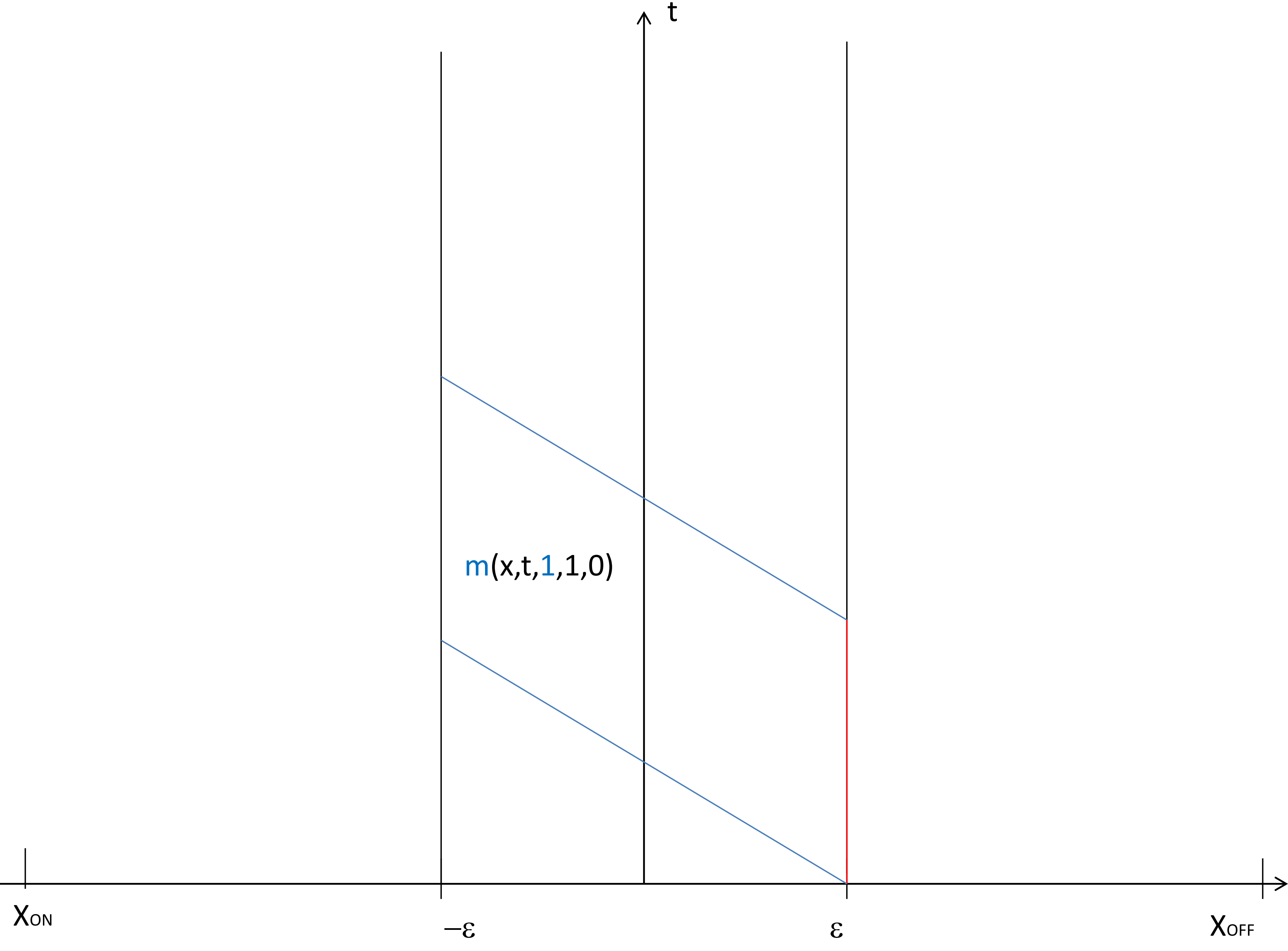


t
XON
−ε
ε
XOFF

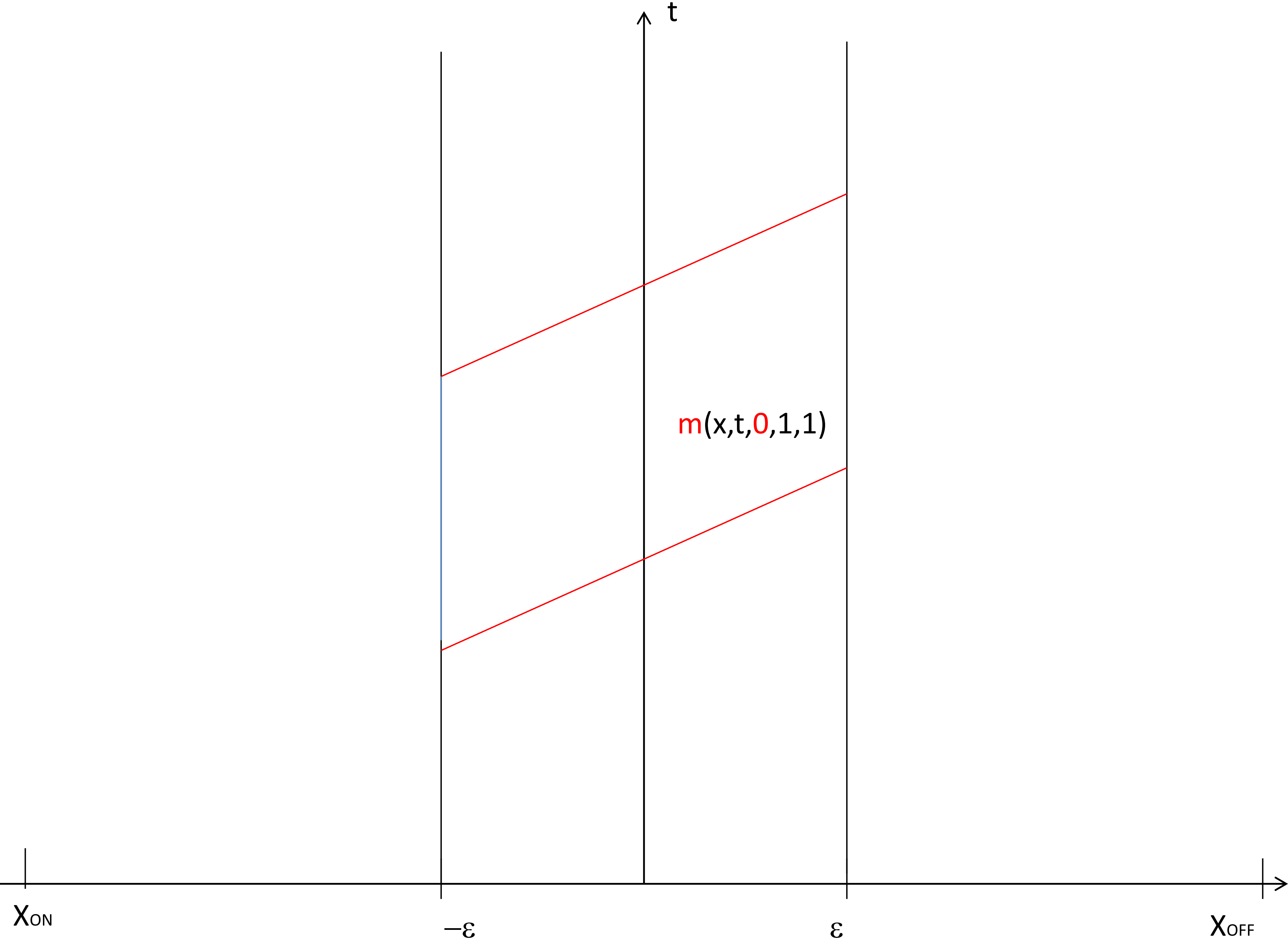

t
m(0,t)=m0(x1)+m0(x2)+m0(x3)+m0(x4)+m0(x5)
x1
x2
x3
x4
x5
XON
−ε
ε
XOFF


t
m(0,t)=m0(x1)+m0(x2)+m0(x3)+m0(x4)+m0(x5)
m0 initial distribution of agents switched OFF
m0 initial distribution of agents switched ON
x1
x2
x3
x4
x5
XON
−ε
ε
XOFF

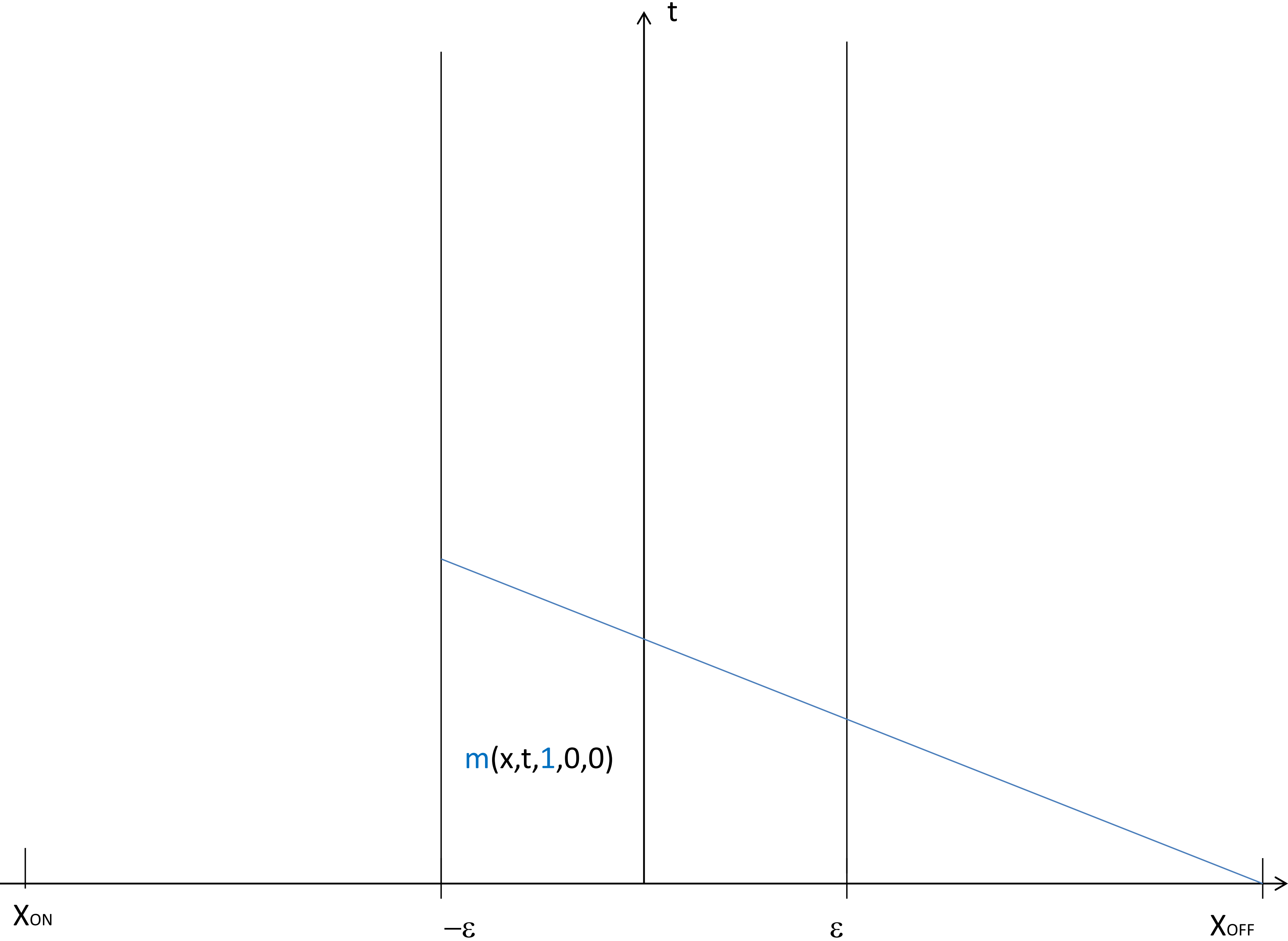

t
m(x,t,0,0,0)
X ON
−ε
ε
X OFF

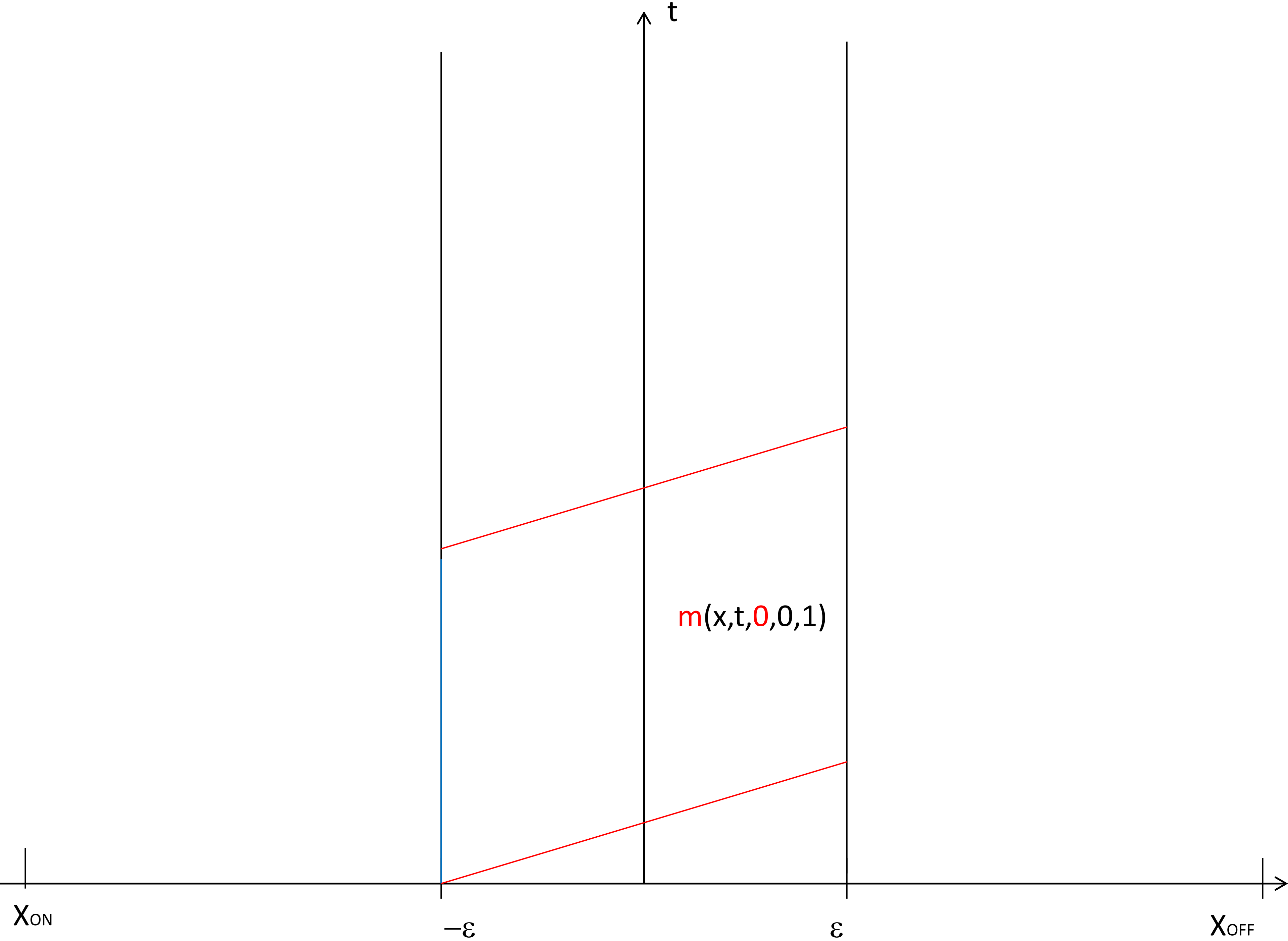


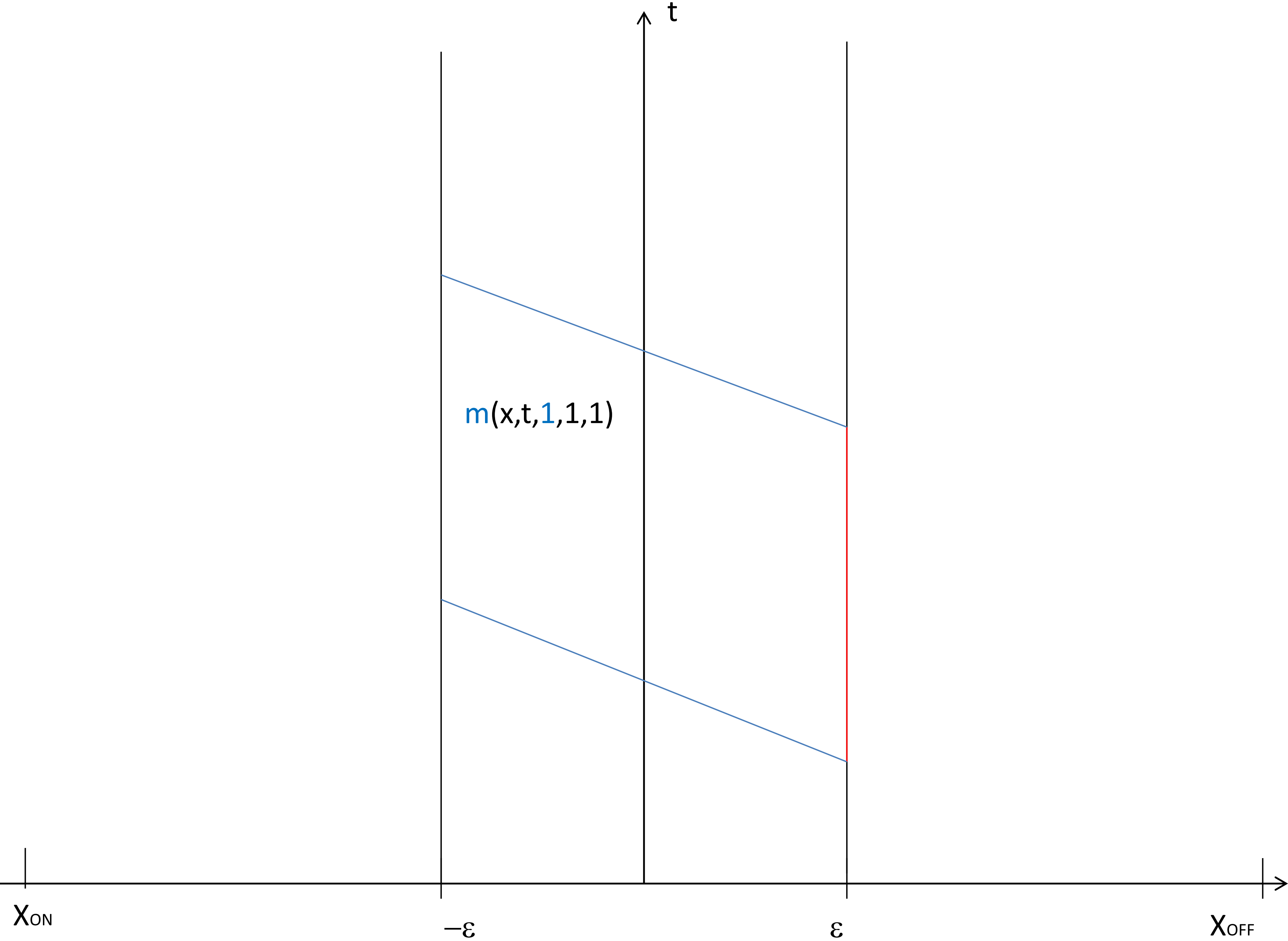
t
m(x,t,1,1,0)
XON
−ε
ε
XOFF



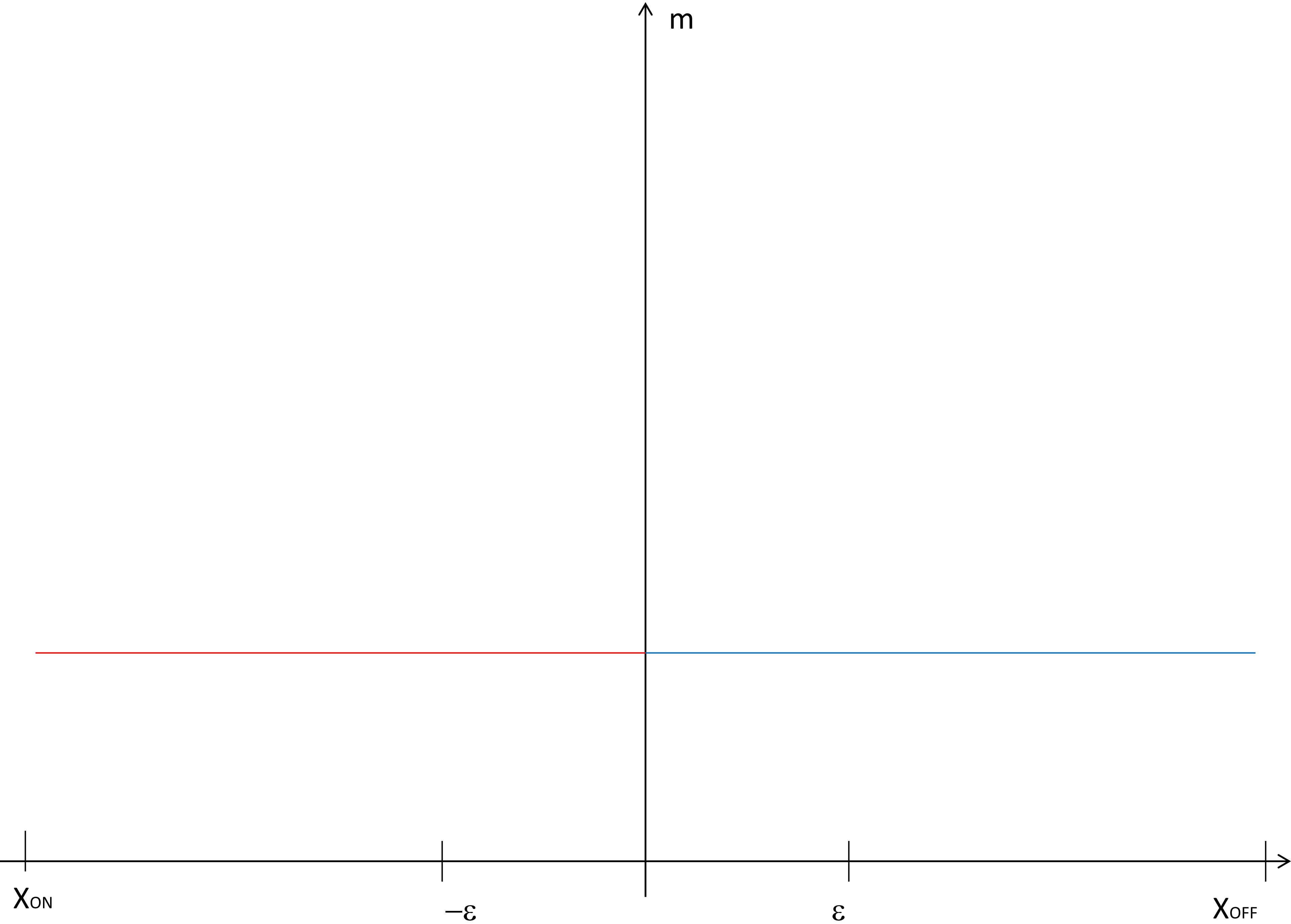
t
m(x,t,0,1,1)
XON
−ε
ε
XOFF

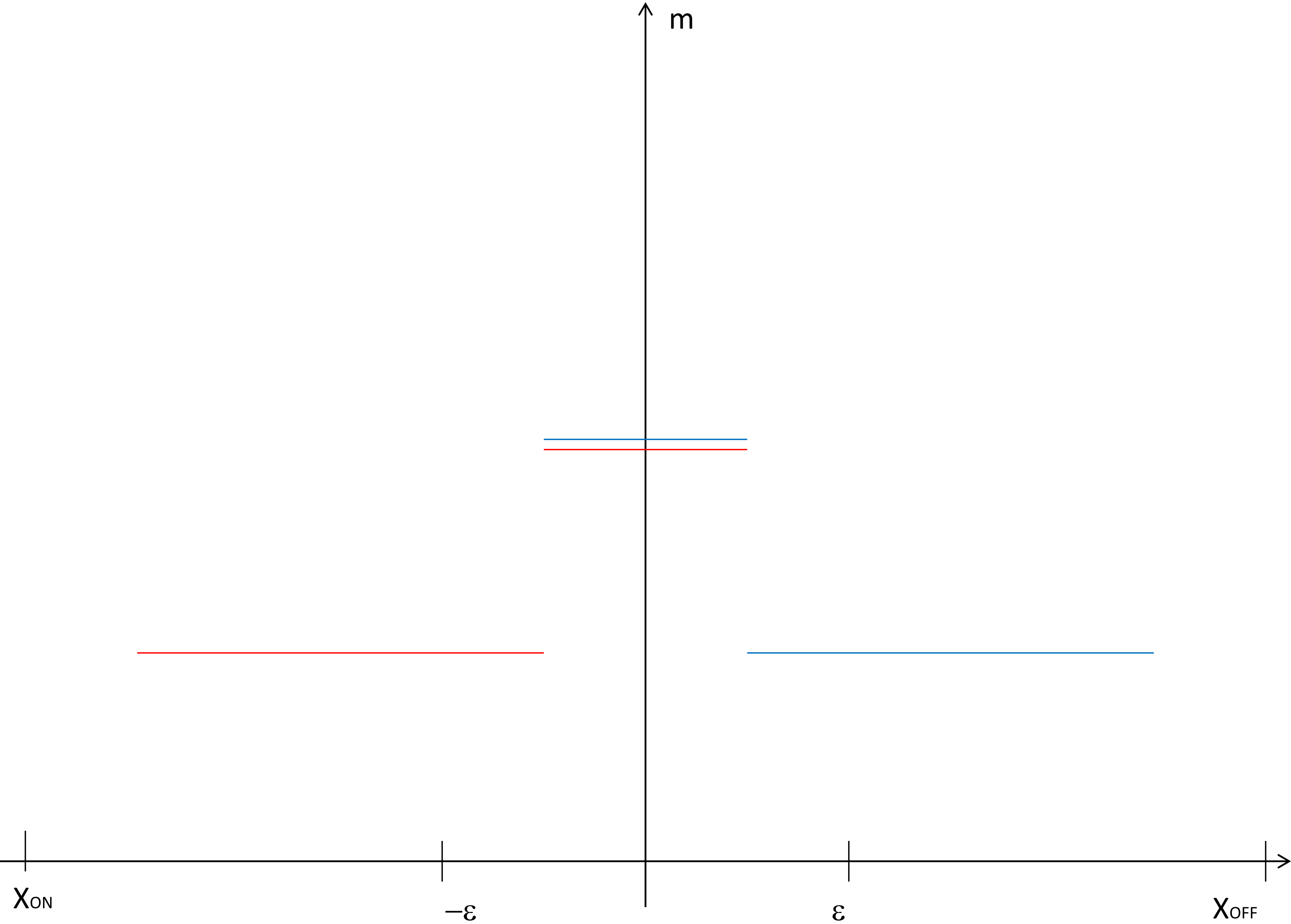

t
m(x,t,1,0,0)
XON
−ε
ε
XOFF


t
m(x,t,0,0,1)
XON
−ε
ε
XOFF

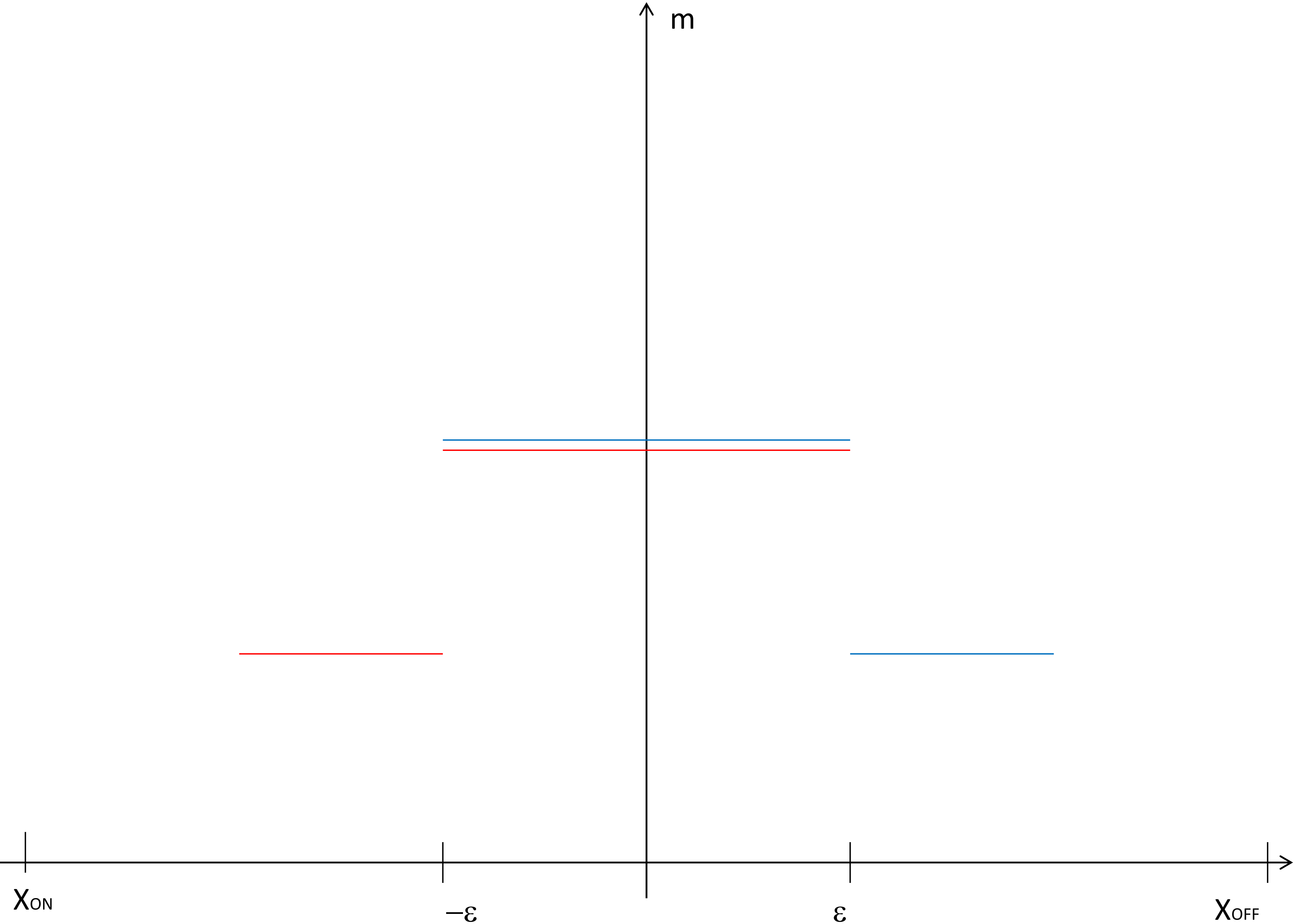


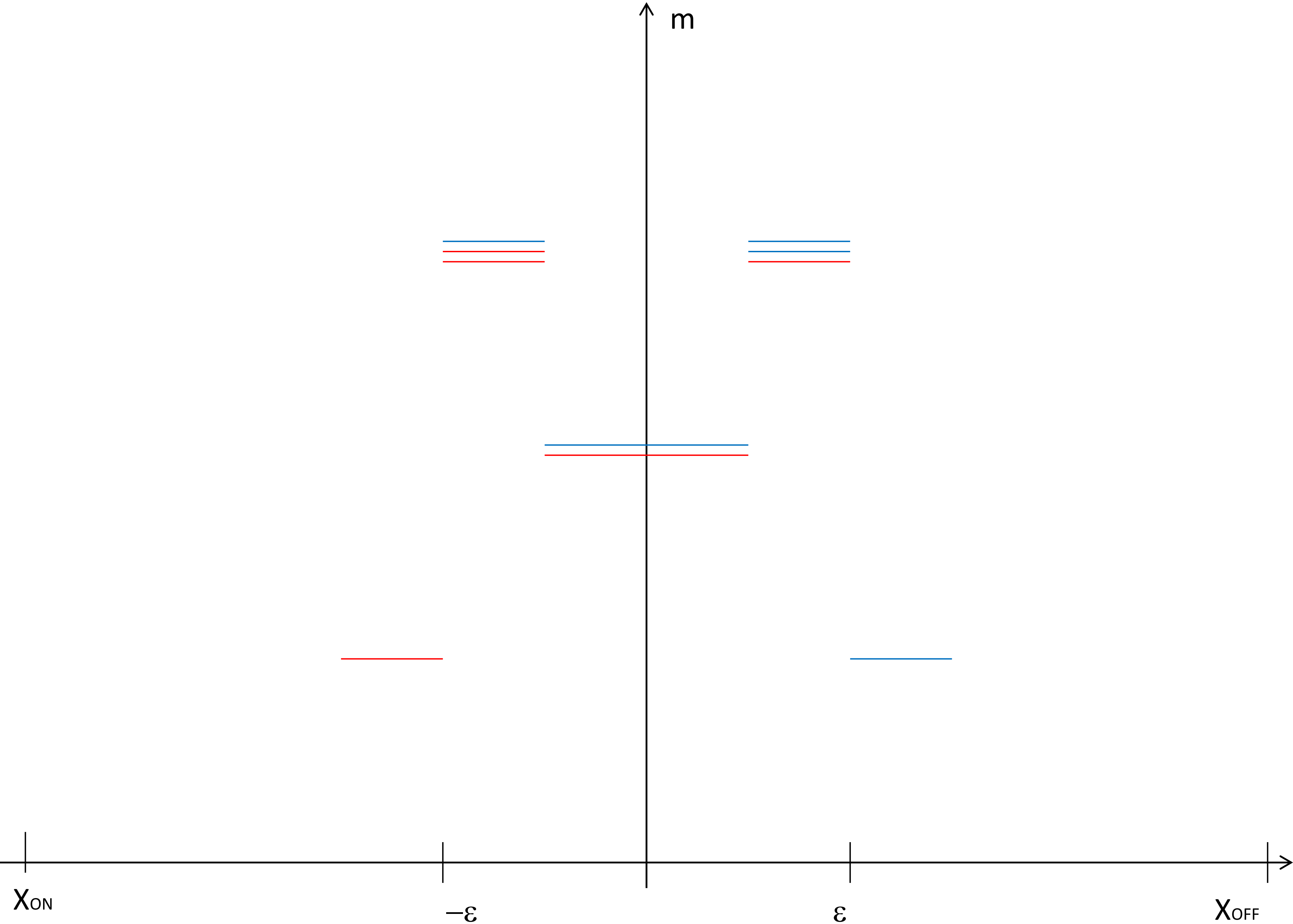
t
m(x,t,1,1,1)
X_{ON}
−ε
ε
X_{OFF}


m
XON
−ε
ε
XOFF

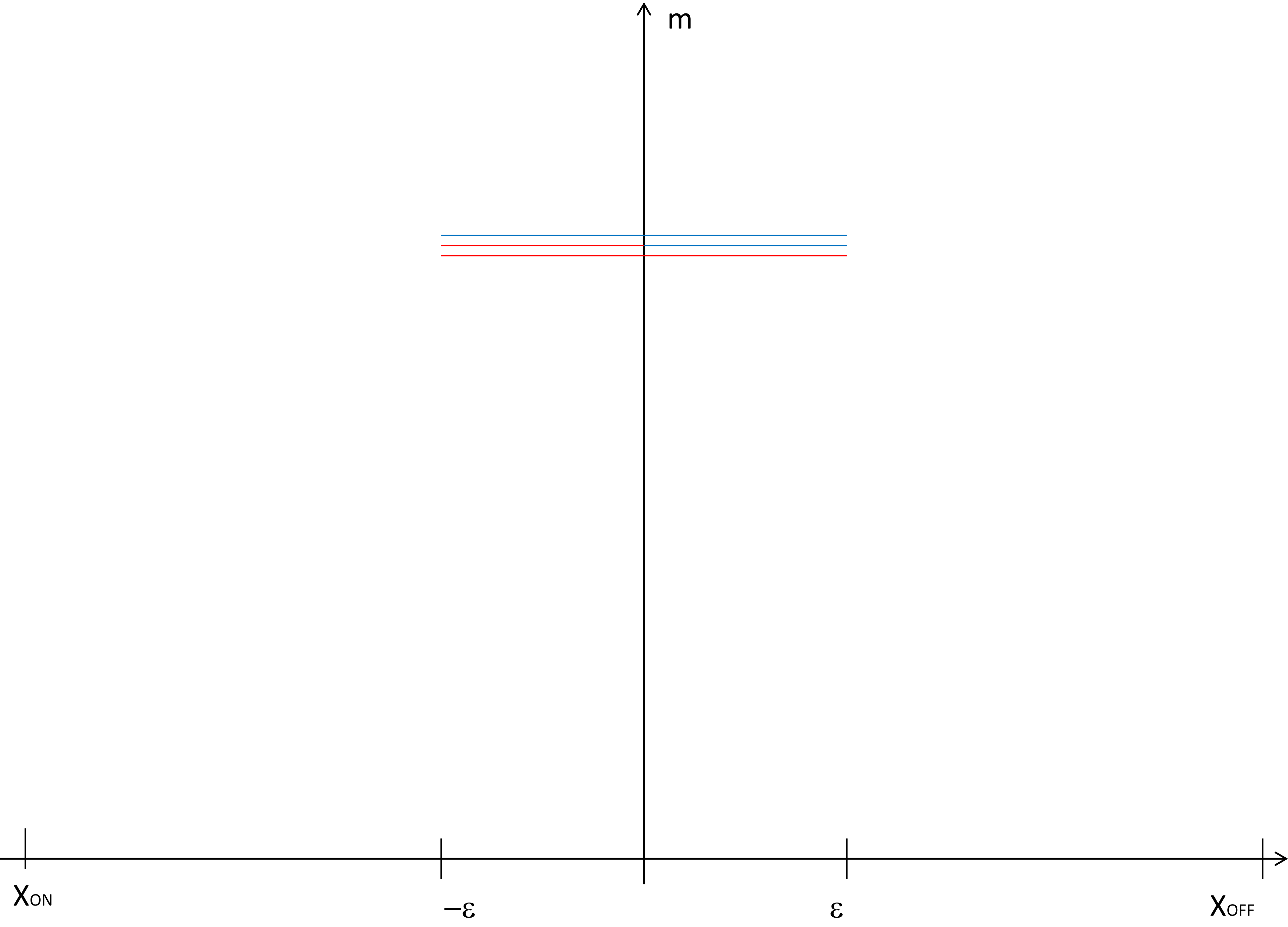


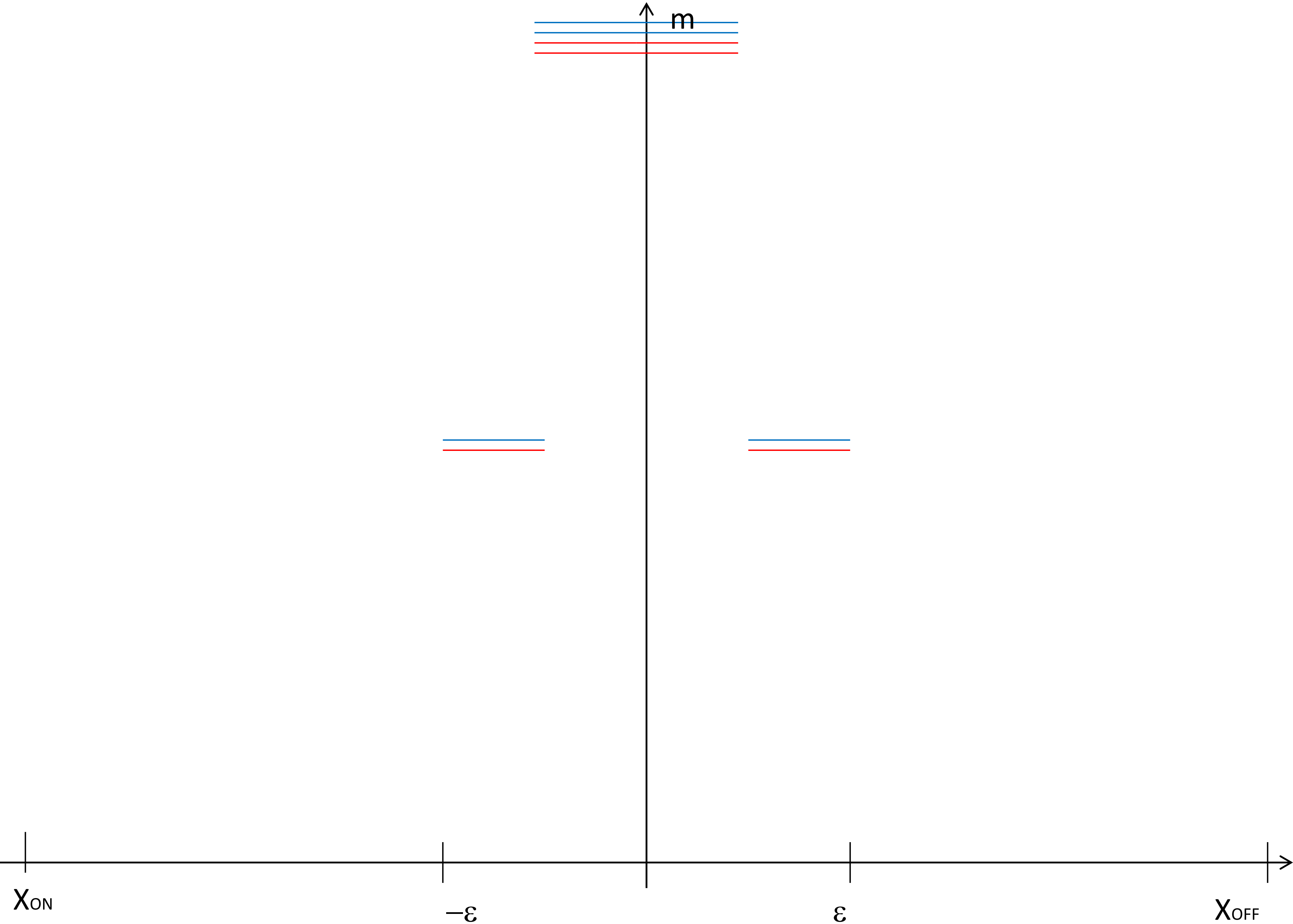
m
X_{ON}
$-\varepsilon$
ε
X_{OFF}

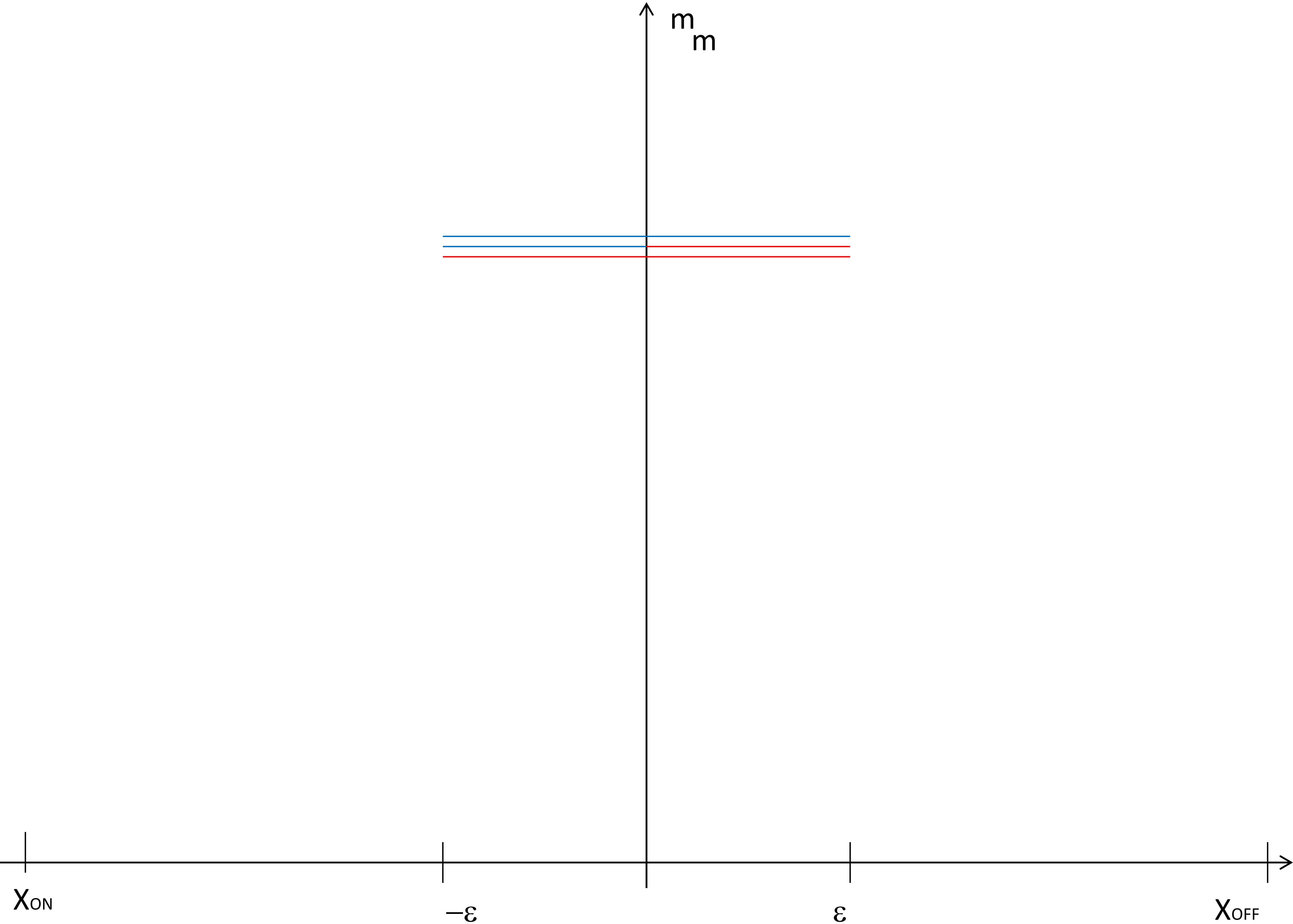

m
X_{ON}
$-\varepsilon$
ε
X_{OFF}



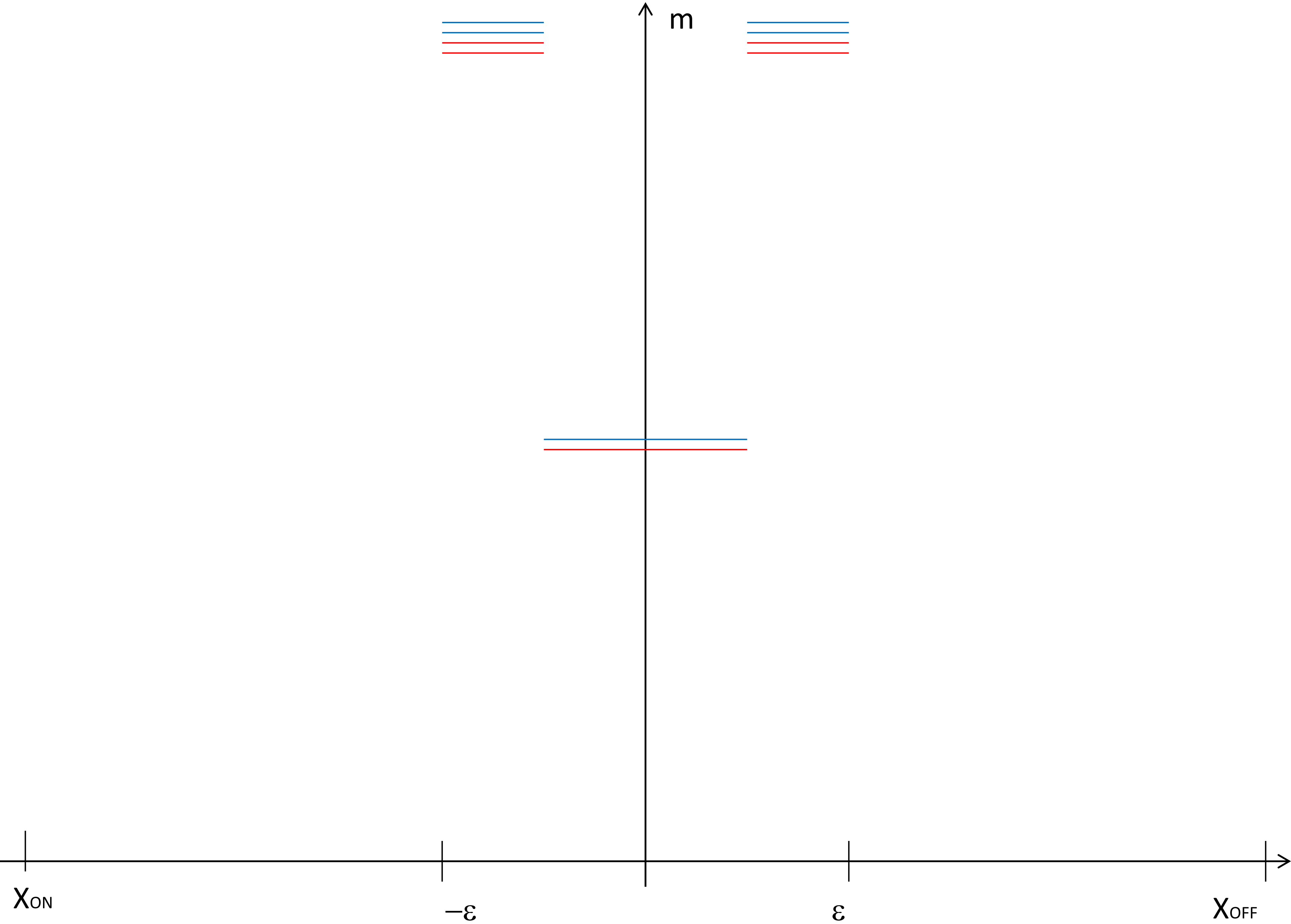
m
XON
−ε
ε
XOFF

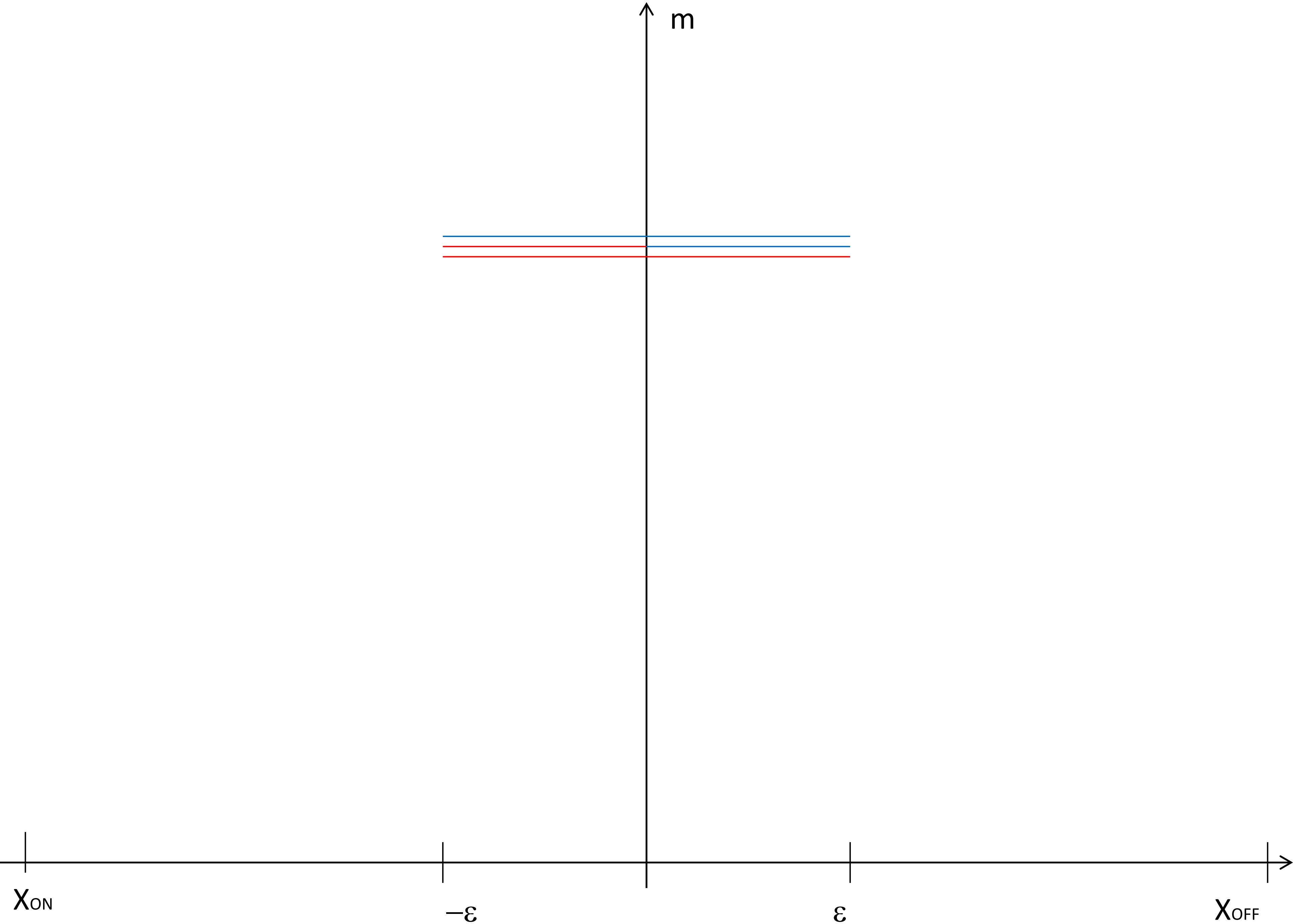

m
X_{ON}
$-\varepsilon$
ε
X_{OFF}


m
X_{ON}
$-\varepsilon$
ε
X_{OFF}

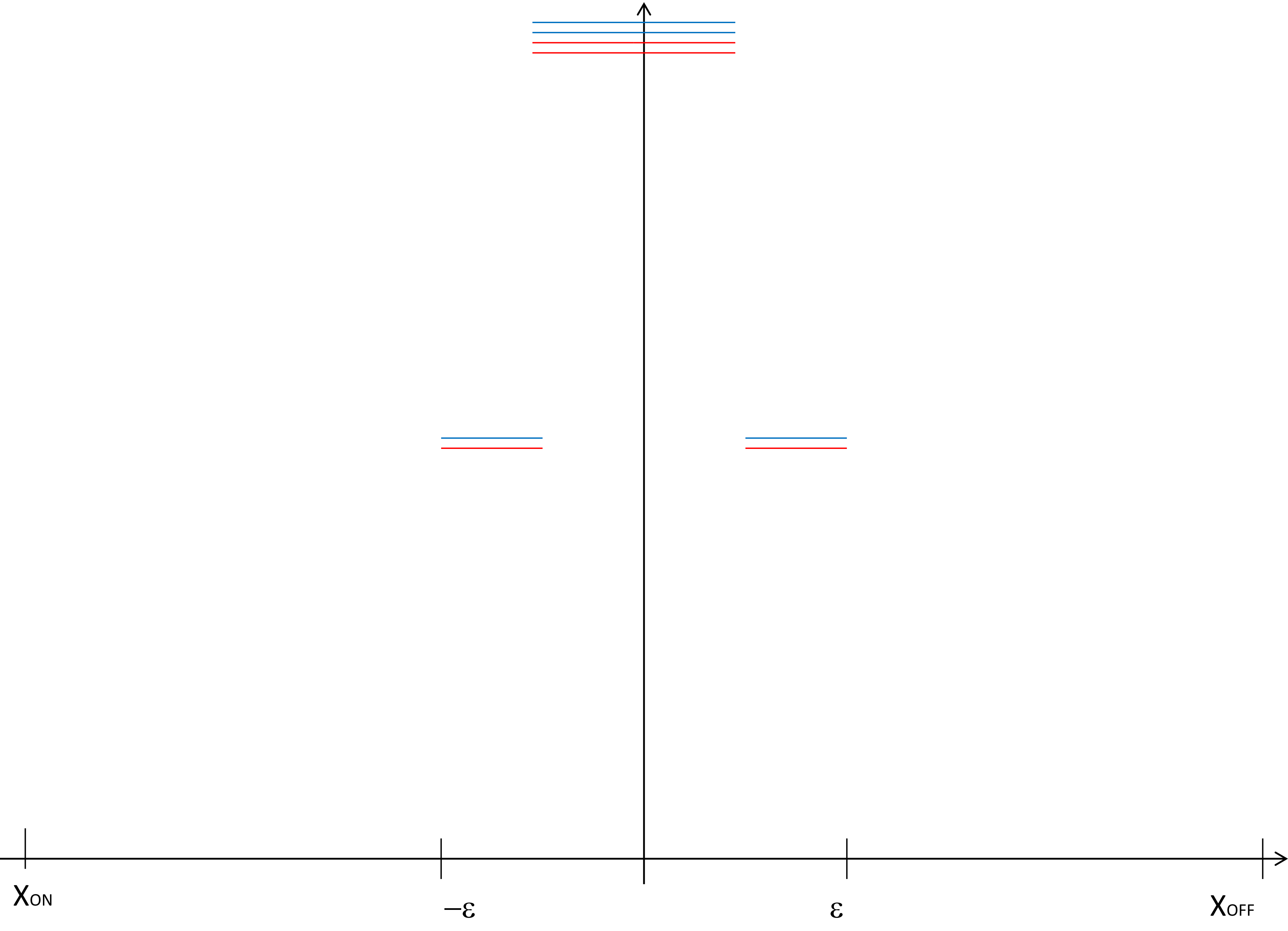


m
m
X ON
−ε
ε
X OFF

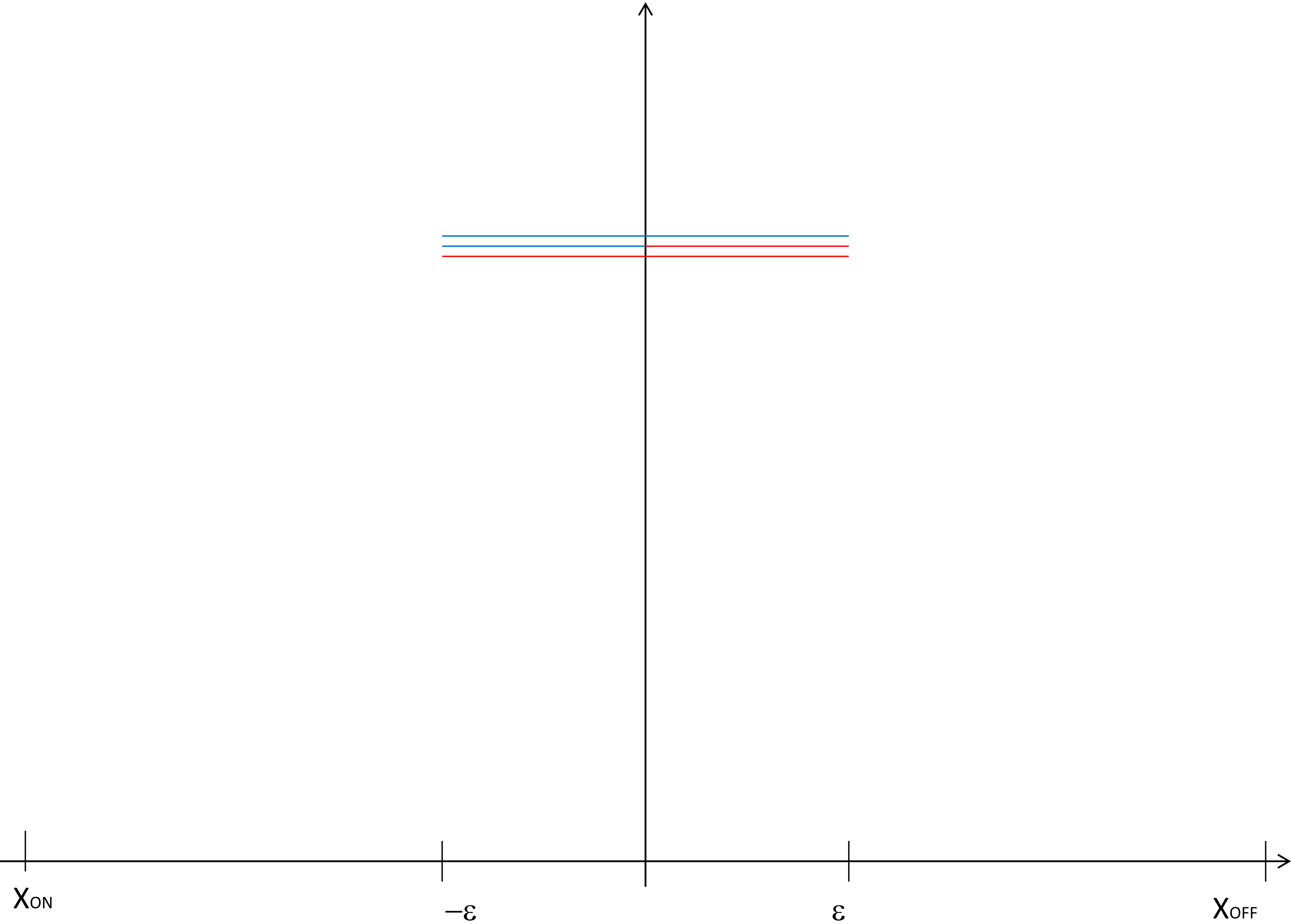

m
X ON
−ε
ε
X OFF



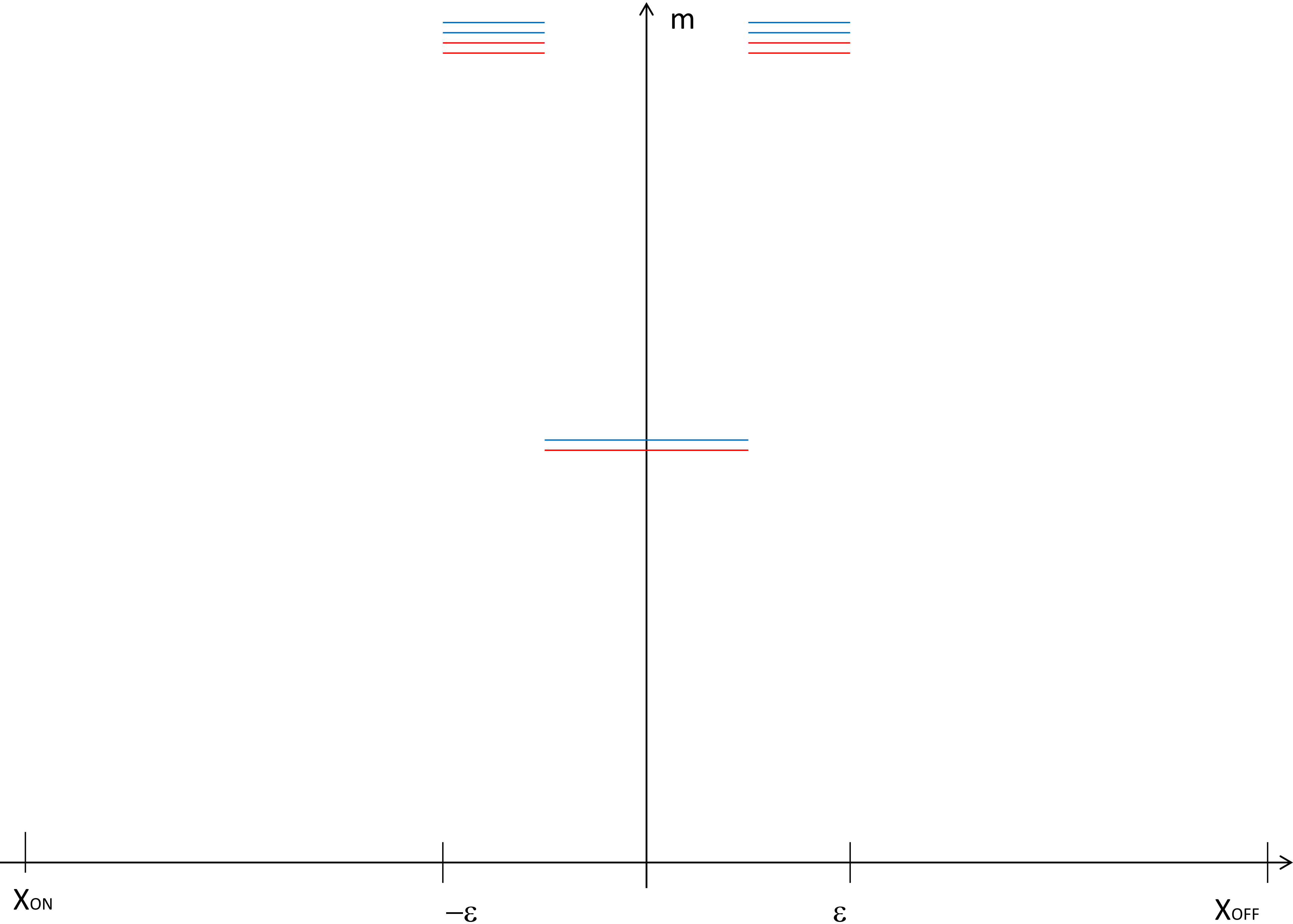
m
XON
−ε
ε
XOFF



XON
−ε
ε
XOFF



X_{ON}
$-\varepsilon$
ε
X_{OFF}



m
XON
−ε
ε
XOFF