Mean Field Game Theory for Systems with Major and Minor Agents: Nonlinear Theory and Mean Field Estimation

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Background – Mean Field Game (MFG) Theory

Modeling Framework and Central Problem of MFG Theory (Huang, Malhamé, PEC ('03,'06,'07), Lasry-Lions ('06,'07)):

Framework Games over time with a large number of stochastic dynamical agents such that:

- Each agent interacts with a mass effect (e.g. average) of other agents via couplings in their individual cost functions and individual dynamics
- Each agent is minor in the sense that, asymptotically as the population size goes to infinity, it has a negligible influence on the overall system but the mass effect on the agent is significant

Problem Establish the existence and uniqueness of equilibria and the corresponding strategies of the agents

Background – Mean Field Game (MFG) Theory

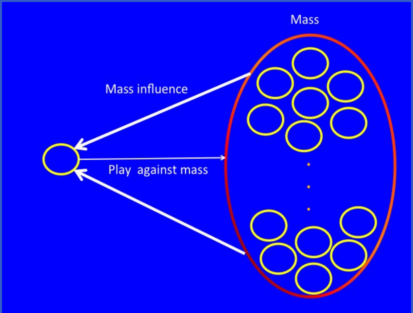
Solution Concepts for MFG Theory:

- The existence of Nash equilibria between the individual agents and the mass in the infinite population limit where

  (a) the individual strategy of each agent is a best response to the mass effect, and

  (b) the set of the strategies collectively replicate that mass effect

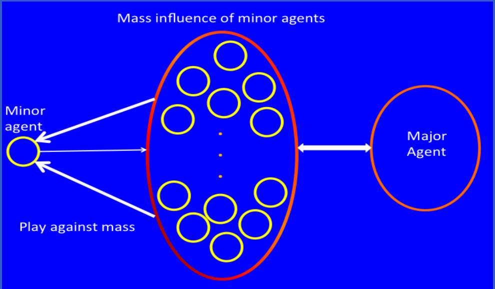
- The $\epsilon-$Nash Approximation Property: If agents in a finite population system apply the infinite population equilibrium strategies an approximation to the infinite population equilibrium results



Non-linear Major-Minor Mean Field Systems

MFG Theory Involving Major-Minor Agents

- Extension of the LQG MFG model for Major and Minor agents (Huang 2010, Huang-Ngyuan 2011) to the case of nonlinear dynamical systems
- Dynamic game models will involve nonlinear stochastic systems with (i) a major agent, and (ii) a large population of minor agents
- Partially observed systems become meaningful and hence estimation of major agent and mean field states becomes a meaningful problem
- Motivation and Applications:
  - Economic and social models with both minor and massive agents
  - Power markets with large consumers and large utilities together with many domestic consumers and generators using smart meters



MFG Nonlinear Major-Minor Agent Formulation

Problem Formulation:

- Notation: Subscript 0 for the major agent $\mathcal{A}_0$ and an integer valued subscript for minor agents $\{\mathcal{A}_i : 1 \leq i \leq N\}$.
- The states of $\mathcal{A}_0$ and $\mathcal{A}_i$ are $\mathbb{R}^n$ valued and denoted $z_0^N(t)$ and $z_i^N(t)$.

Dynamics of the Major and Minor Agents:

$$dz_0^N(t) = \frac{1}{N}\sum_{j=1}^{N} f_0(t, z_0^N(t), u_0^N(t), z_j^N(t))dt$$

$$+ \frac{1}{N}\sum_{j=1}^{N} \sigma_0(t, z_0^N(t), z_j^N(t))dw_0(t), \quad z_0^N(0) = z_0(0), \quad 0 \leq t \leq T,$$

$$dz_i^N(t) = \frac{1}{N}\sum_{j=1}^{N} f(t, z_i^N(t), u_i^N(t), z_j^N(t))dt$$

$$+ \frac{1}{N}\sum_{j=1}^{N} \sigma(t, z_i^N(t), z_j^N(t))dw_i(t), \quad z_i^N(0) = z_i(0), \quad 1 \leq i \leq N.$$

MFG Nonlinear Major-Minor Agent Formulation

Cost Functions for Major and Minor Agents: The objective of each agent is to minimize its finite time horizon cost function given by

$$J_0^N(u_0^N; u_{-0}^N) := E\int_0^T \Big(\frac{1}{N}\sum_{j=1}^N L_0[t, z_0^N(t), u_0^N(t), z_j^N(t)]\Big)dt,$$

$$J_i^N(u_i^N; u_{-i}^N) := E\int_0^T \Big(\frac{1}{N}\sum_{j=1}^N L[t, z_i^N(t), u_i^N(t), z_0^N(t), z_j^N(t)]\Big)dt.$$

- The major agent has non-negligible influence on the mean field (mass) behaviour of the minor agents due to presence of z_0^N in the cost function of each minor agent. A consequence is that the mean field is no longer a deterministic function of time.

Notation

- $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$: a complete filtered probability space
- $\mathcal{F}_t := \sigma\{z_j(s), w_j(s) : 0 \leq j \leq N, 0 \leq s \leq t\}$.
- $\mathcal{F}_t^{w_0} := \sigma\{z_0(0), w_0(s) : 0 \leq s \leq t\}$.

Assumptions

Assumptions: Let the empirical distribution of N minor agents' initial states be defined by $F_N(x) := \frac{1}{N}\sum_{i=1}^{N} 1_{\{z_i(0)<x\}}$.

(A1) The initial states $\{z_j(0) : 0 \leq j \leq N\}$ are $\mathcal{F}_0$-adapted random variables mutually independent and independent of all Brownian motions, and there exists a constant k independent of N such that $\sup_{0\leq j\leq N} E|z_j(0)|^2 \leq k < \infty$.

(A2) $\{F_N : N \geq 1\}$ converges weakly to the probability distribution F.

(A3) U_0 and U are compact metric spaces.

(A4) $f_0[t,x,u,y]$, $\sigma_0[t,x,y]$, $f[t,x,u,y]$ and $\sigma[t,x,y]$ are continuous and bounded with respect to all their parameters, and Lipschitz continuous in (x,y,z). In addition, their first order derivatives (w.r.t. x) are all uniformly continuous and bounded with respect to all their parameters, and Lipschitz continuous in y.

(A5) $f_0[t,x,u,y]$ and $f[t,x,u,y]$ are Lipschitz continuous in u.

Assumptions

Assunptions (ctd):

(**A6**) $L_0[t, x, u, y]$ and $L[t, x, u, y, z]$ are continuous and bounded with respect to all their parameters, and Lipschitz continuous in (x, y, z). In addition, their first order derivatives (w.r.t. x) are all uniformly continuous and bounded with respect to all their parameters, and Lipschitz continuous in (y, z).

(**A7**) (Non-degeneracy Assumption) There exists a positive constant α such that

$$\sigma_0[t, x, y]\sigma_0^T[t, x, y] \geq \alpha I, \ \sigma[t, x, y]\sigma^T(t, x, y) \geq \alpha I, \quad \forall(t, x, y).$$

Otherwise, a notion of viscosity like solutions seems necessary.

McKean-Vlasov Approximation for MFG Analysis

-Loop Major and Minor Agent Dynamics:

Assume $\varphi_0(\omega, t, x) \in L^2_{\mathcal{F}_t^{w_0}}([0,T]; U_0)$ and $\varphi(\omega, t, x) \in L^2_{\mathcal{F}_t^{w_0}}([0,T]; U)$ are two arbitrary $\mathcal{F}_t^{w_0}$-measurable stochastic processes, Lipschitz continuous in x, constituting the Major and Minor agent control laws. Then:

$$dz_0^N(t) = \frac{1}{N}\sum_{j=1}^{N} f_0(t, z_0^N(t), \varphi_0(t, z_0^N(t)), z_j^N(t))dt$$

$$+ \frac{1}{N}\sum_{j=1}^{N} \sigma_0(t, z_0^N(t), z_j^N(t))dw_0(t), \quad z_0^N(0) = z_0(0), \quad 0 \le t \le T,$$

$$dz_i^N(t) = \frac{1}{N}\sum_{j=1}^{N} f(t, z_i^N(t), \varphi(t, z_i^N(t), z_0^N(t)), z_j^N(t))dt$$

$$+ \frac{1}{N}\sum_{j=1}^{N} \sigma(t, z_i^N(t), z_j^N(t))dw_i(t), \quad z_i^N(0) = z_i(0), \quad 1 \le i \le N.$$

McKean-Vlasov Approximation for MFG Analysis

Major-Minor Agent McKean-Vlasov System:

For an arbitrary function g and a probability distribution μ_t in $\mathbb{R}^n$, set

$$g[t, z, \psi, \mu_t] = \int g(t, z, \psi, x)\mu_t(dx).$$

The pair of infinite population McKean-Vlasov (MV) SDE systems corresponding to the collection of finite population systems above is given by:

$$dz_0(t) = f_0[t, z_0(t), \varphi_0(t, z_0(t)), \mu_t]dt + \sigma_0[t, z_0(t), \mu_t]dw_0(t),$$
$$dz(t) = f[t, z(t), \varphi(t, z(t), z_0(t)), \mu_t]dt + \sigma[t, z(t), \mu_t]dw(t), \quad 0 \le t \le T$$

with initial conditions $(z_0(0), z(0))$.

- In using the MV system it is assumed that the (behaviour) of an infinite population of (parameter) uniform minor agents can be modelled by the collection of sample paths of agents with independent initial conditions and independent Brownian sample paths.
- In the above MV system $\big(z_0(\cdot), z(\cdot), \mu_{(\cdot)}\big)$ is a consistent solution if $\big(z_0(\cdot), z(\cdot)\big)$ is a solution to the above SDE system, and μ_t is the conditional law of $z(t)$ given $\mathcal{F}_t^{w_0}$ (i.e., $\mu_t := \mathcal{L}\big(z(t)|\mathcal{F}_t^{w_0}\big)$).

McKean-Vlasov Approximation for MFG Analysis

We shall use the notation:

$$dz_0(t) = f_0[t, z_0(t), \varphi_0(t, z_0(t)), \mu_t]dt + \sigma_0[t, z_0(t), \mu_t]dw_0(t), \quad 0 \le t \le T,$$
$$dz_i(t) = f[t, z_i(t), \varphi(t, z_i(t)), z_0(t), \mu_t]dt + \sigma[t, z_i(t), \mu_t]dw_i(t), \quad 1 \le i \le N,$$

with initial conditions $z_j(0) = z_j(0)$ for $0 \le j \le N$, to describe N independent samples of the MV SDE system.

Theorem (McKean-Vlasov Convergence Result)

*Assume (**A1**) and (**A3**)-(**A5**) hold. Then unique solutions exist for the finite and MV SDE equation schemes and we have*

$$\sup_{0 \le j \le N} \sup_{0 \le t \le T} E|\hat{z}_j^N(t) - z_j(t)| = O(\frac{1}{\sqrt{N}}),$$

where the right hand side may depend upon the terminal time T. □

The proof is based on the Cauchy-Schwarz inequality, Gronwall's lemma and the conditional independence of minor agents given $\mathcal{F}_t^{w_0}$.

Distinct Feature of the Major-Minor MFG Theory

- The non-standard nature of the SOCPs is due to the fact that the minor agents are optimizing with respect to the future stochastic evolution of the major agent's stochastically evolving state which is partly a result of that agent's future best response control actions. Hence the mean field becomes stochastic.

- This feature vanishes in the non-game theoretic setting of one controller with one cost function with respect to the trajectories of all the system components (the classical SOCP), moreover it also vanishes in the infinite population limit of the standard MFG models with no major agent.

- The nonstandard feature of the SOCPs here give rise to the analysis of systems with stochastic parameters and hence BSDEs enter the analysis.

An SOCP with Random Coefficients (after Peng '92)

Let $(W(t))_{t\geq 0}$ and $(B(t))_{t\geq 0}$ be mutually independent standard Brownian motions in $\mathbb{R}^m$. Denote

$$\mathcal{F}_t^{W,B} := \sigma\{W(s), B(s) : s \leq t\}, \quad \mathcal{F}_t^{W} := \sigma\{W(s) : s \leq t\}$$

$$\mathcal{U} := \big\{u(\cdot) \in U : u(t) \text{ is adapted to } \sigma\text{-field } \mathcal{F}_t^{W,B} \text{ and } E\int_0^T |u(t)|^2 dt < \infty\big\}.$$

Dynamics and optimal control problem for a single agent:

$$dz(t) = f[t,z,u]dt + \sigma[t,z]dW(t) + \varsigma[t,z]dB(t), \quad 0 \leq t \leq T,$$

$$\inf_{u\in\mathcal{U}} J(u) := \inf_{u\in\mathcal{U}} E\Big[\int_0^T L[t,z(t),u(t)]dt\Big],$$

where the coefficients f, σ, ς and L are are $\mathcal{F}_t^W$-adapted stochastic processes.

The value function $\phi(\cdot,\cdot)$ is defined to be the $\mathcal{F}_t^W$-adapted process

$$\phi(t,x_t) = \inf_{u\in\mathcal{U}} E_{\mathcal{F}_t^W} \int_t^T L[s,x(s),u(s)]ds,$$

where x_t is the initial condition for the process x.

Solution to the Optimal Control Problem with Random Coefficients

Let $u^o(\cdot)$ be the optimal control with corresponding closed-loop solution $x(\cdot)$
By the Principle of Optimality, the process

$$\zeta(t) := \phi\big(t, x(t)\big) + \int_0^t L[s, x(s), u^o(s, x(s))]ds,$$

is an $\{\mathcal{F}_t^W\}_{0\leq t\leq T}$-martingale. Next, by the martingale representation theorem along the optimal trajectory there exists an $\mathcal{F}_t^W$-adapted process $\psi\big(\cdot, x(\cdot)\big)$ such that

$$\begin{aligned}\phi\big(t, x(t)\big) &= \int_t^T L[s, x(s), u^o(s, x(s))]ds - \int_t^T \psi^T\big(s, x(s)\big)dW(s)\\ &=: \int_t^T \Gamma\big(s, x(s)\big)ds - \int_t^T \psi^T\big(s, x(s)\big)dW(s), \quad t \in [0, T].\end{aligned}$$

Extended Itô-Kunita formula

Theorem (Extended Itô-Kunita formula (after Peng'92)) Let $\phi(t,x)$ be a stochastic process represented by

$$d\phi(t,x) = -\Gamma(t,x)dt + \sum_{k=1}^{m} \psi_k(t,x)dW_k(t), \qquad (t,x) \in [0,T] \times \mathbb{R}^n,$$

where $\Gamma(t,x)$ and $\psi_k(t,x)$, $1 \leq k \leq m$, are $\mathcal{F}_t^W$-adapted stochastic processes. Let $x(\cdot) = \big(x^1(\cdot), \cdots, x^n(\cdot)\big)$ be a continuous semimartingale of the form

$$dx^i(t) = f_i(t)dt + \sum_{k=1}^{m} \sigma_{ik}(t)dW_k(t) + \sum_{k=1}^{m} \varsigma_{ik}(t)dB_k(t), \quad 1 \leq i \leq n,$$

where f_i, $\sigma_i = (\sigma_{i1}, \cdots, \sigma_{im})$ and $\varsigma_i = (\varsigma_{i1}, \cdots, \varsigma_{im})$, $1 \leq i \leq n$, are $\mathcal{F}_t^W$ adapted stochastic processes.

Extended Itô-Kunita formula

Then the composition map $\phi(\cdot, x(\cdot))$ is also a continuous semimartingale which has the form

$$
\begin{aligned}
d\phi\big(t,x(t)\big) &= -\Gamma\big(t,x(t)\big)dt + \sum_{k=1}^{m}\psi_k\big(t,x(t)\big)dW_k(t) + \sum_{i=1}^{n}\partial_{x_i}\phi\big(t,x(t)\big)f_i(t)dt \\
&+ \sum_{i=1}^{n}\sum_{k=1}^{m}\partial_{x_i}\phi\big(t,x(t)\big)\sigma_{ik}(t)dW_k(t) + \sum_{i=1}^{n}\sum_{k=1}^{m}\partial_{x_i}\phi\big(t,x(t)\big)\varsigma_{ik}(t)dB_k(t) \\
&+ \sum_{i=1}^{n}\sum_{k=1}^{m}\partial_{x_i}\psi_k\big(t,x(t)\big)\sigma_{ik}(t)dt + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{k=1}^{m}\partial^2_{x_ix_j}\phi\big(t,x(t)\big)\sigma_{ik}(t)\sigma_{jk}(t)dt \\
&+ \frac{1}{2}\sum_{i,j=1}^{n}\sum_{k=1}^{m}\partial^2_{x_ix_j}\phi\big(t,x(t)\big)\varsigma_{ik}(t)\varsigma_{jk}(t)dt.
\end{aligned}
$$

□

A Stochastic Hamilton-Jacobi-Bellman (SHJB) Equation

Using the extended Itô-Kunita formula and the Principle of Optimality, it may be shown that the pair $\big(\phi(s,x), \psi(s,x)\big)$ satisfies the following backward in time SHJB equation:

$$-d\phi(t,\omega,x) = \Big[H[t,\omega,x,D_x\phi(t,\omega,x)] + \big\langle \sigma[t,\omega,x], D_x\psi(t,\omega,x)\big\rangle + \frac{1}{2}\mathrm{Tr}\big(a[t,\omega,x]D^2_{xx}\phi(t,\omega,x)\big)\Big]dt - \psi^T(t,\omega,x)dW(t,\omega), \quad \phi(T,x) = 0,$$

in $[0,T]\times\mathbb{R}^n$, where $a[t,\omega,x] := \sigma[t,\omega,x]\sigma^T[t,\omega,x] + \varsigma[t,\omega,x]\varsigma^T(t,\omega,x)$, and the stochastic Hamiltonian H is given by

$$H[t,\omega,z,p] := \inf_{u\in\mathcal{U}} \big\{\big\langle f[t,\omega,z,u], p\big\rangle + L[t,\omega,z,u]\big\}.$$

Unique Solution of the SHJB Equation

Assumptions:

(H1)(Continuity Assumptions 1) $f[t,x,u]$ and $L[t,x,u]$ are a.s. continuous in (x,u) for each t, a.s. continuous in t for each (x,u), $f[t,0,0] \in L^2_{\mathcal{F}_t}([0,T];\mathbb{R}^n)$ and $L[t,0,0] \in L^2_{\mathcal{F}_t}([0,T];\mathbb{R}_+)$. In addition, they and all their first derivatives (w.r.t. x) are a.s. continuous and bounded.

(H2) (Continuity Assumptions 2) $\sigma[t,x]$ and $\varsigma[t,x]$ are a.s. continuous in x for each t, a.s. continuous in t for each x and $\sigma[t,0]$, $\varsigma[t,0] \in L^2_{\mathcal{F}_t}([0,T];\mathbb{R}^{n\times m})$. In addition, they and all their first derivatives (w.r.t. x) are a.s. continuous and bounded.

(H3) (Non-degeneracy Assumption) There exist non-negative constants α_1 and α_2, $\alpha_1+\alpha_2>0$, such that

$$\sigma[t,\omega,x]\sigma^T[t,\omega,x] \geq \alpha_1 I, \quad \varsigma[t,\omega,x]\varsigma^T(t,\omega,x) \geq \alpha_2 I, \quad a.s., \quad \forall (t,\omega,x).$$

Theorem (Peng'92)

Assume **(H1)**-**(H3)** *hold. Then the SHJB equation has a unique forward in time $\mathcal{F}_t^W$-adapted solution pair*

$$(\phi(t,\omega,x),\psi(t,\omega,x)) \in \left(L^2_{\mathcal{F}_t}([0,T];\mathbb{R}), L^2_{\mathcal{F}_t}([0,T];\mathbb{R}^m)\right)$$

□

Best Response Control Action and Verification Theorem

The optimal control process:

$$u^o(t,\omega,x) := \arg\inf_{u\in U} H^u[t,\omega,x,D_x\phi(t,\omega,x),u]$$
$$= \arg\inf_{u\in U} \big\{\langle f[t,\omega,x,u], D_x\phi(t,\omega,x)\rangle + L[t,\omega,x,u]\big\}.$$

is a forward in time $\mathcal{F}_t^W$-adapted process for any fixed x.

- By a verification theorem approach, Peng showed that if a unique solution $(\phi,\psi)(t,x)$ to the SHJB equation exists, and if it satisfies:

  (i) for each t, $(\phi,\psi)(t,x)$ is a $C^2(\mathbb{R}^n)$ map,

  (ii) for each x, $(\phi,\psi)(t,x)$ and $(D_x\phi, D^2_{xx}\phi, D_x\psi)(t,x)$ are continuous F_t^W-adapted stochastic processes,

  then $\phi(x,t)$ coincides with the value function of the optimal control problem.

The MFG Consistency Condition

The functional dependence loop of observation and control of the major and minor agents yields the following proof iteration loop, initiated with a nominal measure $\mu_t(\omega)$:

$$\begin{array}{ccccc} \mu_{(\cdot)}(\omega) & \overset{\text{M-SHJB}}{\longrightarrow} & (\phi_0(\cdot,\omega,x), \psi_0(\cdot,\omega,x)) & \overset{\text{M-SBR}}{\longrightarrow} & u_0^o(\cdot,\omega,x) \\ \uparrow \text{m-SMV} & & & & \downarrow \text{M-SMV} \\ u^o(\cdot,\omega,x) & \overset{\text{m-SBR}}{\longleftarrow} & (\phi(\cdot,\omega,x), \psi(\cdot,\omega,x)) & \overset{\text{m-SHJB}}{\longleftarrow} & z_0^o(\cdot,\omega) \end{array}$$

Closing the Loop: : By substituting u^o into the generic minor agent's dynamics we get the SMV dynamics:

$$\begin{aligned} dz^o(t,\omega) &= f[t, z^o(t,\omega), u^o(t,\omega,z), \mu_t(\omega)]dt \\ &\quad + \sigma[t, z^o(t,\omega), \mu_t(\omega)]dw(t), \quad z^o(0) = z(0), \end{aligned}$$

where f and σ are random processes via μ, and u^o which depend on the Brownian motion of the major agent w_0. Let $\tilde{\mu}_t(\omega)$ be the conditional law of $z^o(\cdot)$ with control u^o given $\mathcal{F}_t^{w_0}$. Then:

The MFG or Nash certainty equivalence (NCE) consistency condition: The "measure and control" mapping loop for the MM MFG equation schema is closed if $\tilde{\mu}_t(\omega) = \mu_t(\omega)$ a.s., $0 \le t \le T$, which consistitutes the measure valued part of the solution.

Major-Minor Agent Stochastic MFG System

Summary of the Major Agent's Stochastic MFG (SMFG) System:

MFG-SHJB
$$-d\phi_0(t,\omega,x) = \Big[\inf_{u\in\mathcal{U}_0} H_0[t,\omega,x,u,D_x\phi_0(t,\omega,x)] + \big\langle \sigma_0[t,x,\mu_t(\omega)], D_x\psi_0(t,\omega,x)\big\rangle + \frac{1}{2}\mathrm{Tr}\big(a_0[t,\omega,x]D^2_{xx}\phi_0(t,\omega,x)\big)\Big]dt - \psi_0^T(t,\omega,x)dw_0(t,\omega), \quad \phi_0(T,x)=0$$

MFG-SBR
$$u_0^o(t,\omega,x) = \arg\inf_{u\in U_0} H_0[t,\omega,x,u,D_x\phi_0(t,\omega,x)]$$

MFG-SMV
$$dz_0^o(t,\omega) = f_0[t,z_0^o(t,\omega),u_0^o(t,\omega,z_0),\mu_t(\omega)]dt + \sigma_0[t,z_0^o(t,\omega),\mu_t(\omega)]dw_0(t,\omega), \quad z_0^o(0)=z_0(0)$$

where $a_0[t,\omega,x] := \sigma_0[t,x,\mu_t(\omega)]\sigma_0^T[t,x,\mu_t(\omega)]$, and the stochastic Hamiltonian H_0 is

$$H_0[t,\omega,x,u,p] := \big\langle f_0[t,x,u,\mu_t(\omega)],p\big\rangle + L_0[t,x,u,\mu_t(\omega)].$$

Major-Minor Agent Stochastic MFG System

Summary of the Minor Agents' SMFG System:

MFG-SHJB $$-d\phi(t,\omega,x) = \Big[\inf_{u\in\mathcal{U}} H[t,\omega,x,u,D_x\phi(t,\omega,x)] + \frac{1}{2}\mathrm{Tr}\big(a[t,\omega,x]D^2_{xx}\phi(t,\omega,x)\big)\Big]dt - \psi^T(t,\omega,x)dw_0(t,\omega), \quad \phi(T,x)=0$$

MFG-SBR $$u^o(t,\omega,x) = \arg\inf_{u\in U} H[t,\omega,x,u,D_x\phi(t,\omega,x)]$$

MFG-SMV $$dz^o(t,\omega) = f[t,z^o(t,\omega),u^o(t,\omega,z),\mu_t(\omega)]dt + \sigma[t,z^o(t,\omega),\mu_t(\omega)]dw(t)$$

where $a[t,\omega,x] := \sigma[t,x,\mu_t(\omega)]\sigma^T[t,x,\mu_t(\omega)]$, and the stochastic Hamiltonian H is

$$H[t,\omega,x,p] := \big\langle f[t,x,u,\mu_t(\omega)],p\big\rangle + L[t,x,u,z_0^o(t,\omega),\mu_t(\omega)].$$

Major-Minor Agent Stochastic MFG System

Solution to the Major-Minor Agent SMFG System:

- The solution of the major-minor SMFG system consists of 8-tuple $\mathcal{F}_t^{w_0}$-adapted random processes

$$\big(\phi_0(t,\omega,x), \psi_0(t,\omega,x), u_0^o(t,\omega,x), z_0^o(t,\omega), \phi(t,\omega,x), \psi(t,\omega,x), u^o(t,\omega,x), z^o(t,\omega)\big)$$

  where $z^o(t,\omega)$ generates the random measure $\mu_t(\omega)$.

- The solution to the major-minor SMFG system is a stochastic mean field in contrast to the deterministic mean field of the standard MFG problems (HCM'03,HMC'06,LL'06).

Analysis of the MM-SMFG System

Existence and uniqueness of Solutions to the Major and Minor (MM) Agents' SMFG System: A fixed point argument with random parameters in the space of stochastic probability measures.

$$
\begin{array}{ccccc}
\mu_{(\cdot)}(\omega) & \overset{\text{M-SHJB}}{\longrightarrow} & (\phi_0(\cdot,\omega,x),\psi_0(\cdot,\omega,x)) & \overset{\text{M-SBR}}{\longrightarrow} & u_0^o(\cdot,\omega,x) \\
\uparrow\text{m-SMV} & & & & \downarrow\text{M-SMV} \\
u^o(\cdot,\omega,x) & \overset{\text{m-SBR}}{\longleftarrow} & (\phi(\cdot,\omega,x),\psi(\cdot,\omega,x)) & \overset{\text{m-SHJB}}{\longleftarrow} & z_0^o(\cdot,\omega)
\end{array}
$$

Theorem (Existence and Uniqueness of Solutions (Nourian and PEC. SIAM Jnl. Control and Optimization, 2013)

Under technical conditions including a contraction gain condition there exists a unique solution for the map Γ, and hence a unique solution to the major and minor agents' MM-SMFG system.

ϵ-Nash Equilibrium of the MFG Control Laws

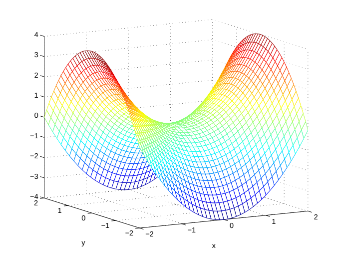
Given $\epsilon > 0$, the set of controls $\{u_j^o; 0 \leq j \leq N\}$ generates an ϵ-Nash equilibrium w.r.t. the costs $J_j^N, 1 \leq j \leq N\}$ if, for each j,

$$J_j^N(u_j^0, u_{-j}^0) - \epsilon \leq \inf_{u_j \in \mathcal{U}_j} J_j^N(u_j, u_{-j}^0) \leq J_j^N(u_j^0, u_{-j}^0).$$

Theorem (Nourian, PEC, SIAM Jnl Control and Optimization, 2013)

Subject to technical conditions, there exists a unique solution to the MM-MFG system such that the set of infinite population MF best response control processes in a finite N population system of minor agents $(u_0^o, \cdots, u_N^o)$ generates an ϵ_N-Nash equilibrium where $\epsilon_N = O(1/\sqrt{N})$. □

- Agent y is a maximizer
- Agent x is a minimizer



Major-Minor Agent LQG -MFG Systems

Major-Minor Agent LQG-MFG Systems

Application of non-linear theory to MM LQG-MFG systems: Yields retrieval of the MM MFG-LQG equations of [Nguyen-Huang'11] (given here in the uniform agent class case).

Dynamics:

$$\mathcal{A}_0: \; dz_0^N(t) = \big(a_0 z_0^N(t) + b_0 u_0^N(t) + c_0 z^{(N)}(t)\big)dt + \sigma_0 dw_0(t)$$

$$\mathcal{A}_i: \; dz_i^N(t) = \big(a z_i^N(t) + b u_i^N(t) + c z^{(N)}(t)\big)dt + \sigma dw_i(t), \; 1 \le i \le N$$

where $z^{(N)}(\cdot) := (1/N)\sum_{i=1}^N z_i^N(\cdot)$ is the average state of minor agents.

Costs:

$$\mathcal{A}_0: \; J_0^N(u_0^N, u_{-0}^N) = E\int_0^T \Big[\Big(z_0^N(t) - \big(\lambda_0 z^{(N)}(t) + \eta_0\big)\Big)^2 + r_0\big(u_0^N(t)\big)^2\Big]dt$$

$$\mathcal{A}_i: \; J_i^N(u_i^N, u_{-i}^N) = E\int_0^T \Big[\Big(z_i^N(t) - \big(\lambda z^{(N)}(t) + \lambda_1 z_0^N(t) + \eta\big)\Big)^2 + r\big(u_i^N(t)\big)^2\Big]dt$$

where $r_0, r > 0$.

Major-Minor Agent LQG-MFG Systems

The Major Agent's LQG-MFG System:

$$[\text{Backward SDE}]: \quad -ds_0(t) = \Big[\big(a_0 - (b_0^2/r_0)\Pi_0(t)\big)s_0(t) - \eta_0 + \big(c_0\Pi_0(t) - \lambda_0\big)z^o(t)\Big]dt - q_0(t)dw_0(t), \quad s_0(T) = 0$$

$$[\text{Best Response}]: \quad u_0^o(t) := -(b_0/r_0)\big(\Pi_0(t)z_0^o(t) + s_0(t)\big)$$

$$[\text{Forward SDE}]: \quad dz_0^o(t) = \big(a_0 z_0^o(t) + b_0 u_0^o(t) + c_0 z^o(t)\big)dt + \sigma_0 dw_0(t), \ z_0^o(0) = z_0(0)$$

where $\Pi_0(\cdot) \geq 0$ is the unique solution of the Riccati equation:

$$\partial_t \Pi_0(t) + 2a_0\Pi_0(t) - (b_0^2/r_0)\Pi_0^2(t) + 1 = 0, \ \Pi_0(T) = 0.$$

The Minor Agents' LQG-MFG System:

$$[\text{Backward SDE}]: \quad -ds_i(t) = \Big[\big(a - (b^2/r)\Pi(t)\big)s_i(t) - \eta - \lambda_1 z_0^o(t) + \big(c\Pi(t) - \lambda\big)z^o(t)\Big]dt - q_i(t)dw_0(t), \quad s_i(T) = 0$$

$$[\text{Best Response}]: \quad u_i^o(t) := -(b/r)\big(\Pi(t)z_i^o(t) + s_i(t)\big)$$

$$[\text{Forward SDE}]: \quad dz_i^o(t) = \big(az_i^o(t) + bu_i^o(t) + cz^o(t)\big)dt + \sigma dw_i(t), \quad z_i^o(0) = z_i(0)$$

where $\Pi(\cdot) \geq 0$ is the unique solution of the Riccati equation:

$$\partial_t \Pi(t) + 2a\Pi(t) - (b^2/r)\Pi^2(t) + 1 = 0, \ \Pi(T) = 0.$$

Major-Minor Agent LQG-MFG Systems

The Minor Agents' Mean Field $z^o(\cdot)$ Equations:

$$[\text{Backward SDE}]: \quad -ds(t) = \Big[\big(a - (b^2/r)\Pi(t)\big)s(t) - \eta - \lambda_1 z_0^o(t) + \big(c\Pi(t) - \lambda\big)z^o(t)\Big]dt - q(t)dw_0(t), \quad s(T) = 0$$

$$[\text{Best Response}]: \quad u_i^o(t) := -(b/r)\big(\Pi(t)z_i^o(t) + s(t)\big)$$

$$[\text{MF FDE}]: \quad dz^o(t) = \big((a+c)z^o(t) + bu^o(t)\big)dt.$$

- Key assumption for solution existence and uniqueness of MM-MFG system is that all drift and cost functions f and L and their derivatives are bounded which clearly does not hold for the MM-MFG LQG problem (as in classical LQG control), so particular methods must be used to deal with this.

The LQG-MFG Solution: The Generalized Four-Step Scheme

The Generalized Four-Step Scheme for the SMFG-LQG System: For given $z^o(\cdot)$ we set $s_0(t) = \theta(t, z_0^o(t))$ where the function θ is to be determined. By Itô's formula:

$$ds_0(t) = d\theta(t, z_0^o(t)) = \Big\{\theta_t(t, z_0^o(t)) + \theta_x(t, z_0^o(t))\big[(a_0 - \frac{b_0^2}{r_0})z_0^o(t) - \frac{b_0^2}{r_0}\theta(t, z_0^o(t)) + c_0 z^o(t)\big] + \frac{1}{2}\sigma_0^2\theta_{xx}(t, z_0^o(t))\Big\}dt + \sigma_0\theta_x(t, z_0^o(t))dw_0(t).$$

Comparing this to the Major agent's SDE implies that θ should satisfy the equations:

$$\theta_t(t, z_0^o(t)) + \theta_x(t, z_0^o(t))\big[(a_0 - \frac{b_0^2}{r_0})z_0^o(t) - \frac{b_0^2}{r_0}\theta(t, z_0^o(t)) + c_0 z^o(t)\big] + \frac{1}{2}\sigma_0^2\theta_{xx}(t, z_0^o(t)) = -\big[a_0 - \frac{b_0^2}{r_0}\Pi_0(t)\big]\theta(t, z_0^o(t)) + \eta_0 - \big[c_0\Pi_0(t) - \lambda_0\big]z^o(t)$$

$$\sigma_0\theta_x(t, z_0^o(t)) = -q_0(t).$$

We can get similar equations for the minor agent by setting

$$s(t) = \check{\theta}(t, z_0^o(t), z^o(t)).$$

The Four-Step Scheme for the LQG-MFG System

Step 1. For given $z^o(\cdot)$ solve the following parabolic PDE for $\theta(t,x)$:

$$\theta_t(t,x) + \theta_x(t,x)\big[(a_0 - \frac{b_0^2}{r_0}\Pi_0(t))x - \frac{b_0^2}{r_0}\theta(t,x) + c_0 z^o(t)\big] + \frac{1}{2}\sigma_0^2\theta_{xx}(t,x)$$
$$= -\big[a_0 - \frac{b_0^2}{r_0}\Pi_0(t)\big]\theta(t,x) + \eta_0 - \big[c_0\Pi_0(t) - \lambda_0\big]z^o(t), \quad \theta(T,x) = 0.$$

Step 2. Use θ in Step 1 to solve the following forward SDE:

$$dz_0^o(t) = \big[(a_0 - \frac{b_0^2}{r_0}\Pi_0(t))z_0^o(t) - \frac{b_0^2}{r_0}\theta(t, z_0^o(t)) + c_0 z^o(t)\big]dt + \sigma_0 dw_0(t), \; z_0^o(0) = z_0(0)$$

Step 3. Set

$$q_0(t) = -\sigma_0\theta_x(t, z_0^o(t))$$
$$s_0(t) = \theta(t, z_0^o(t)).$$

by the use of θ and $z_0^o(t)$ obtained in Steps 1 and 2.

The Four-Step Scheme for the LQG-MFG System (ctd')

Step 4. Use θ to solve the following PDE for $\check{\theta}(t,x,y)$:

$$\check{\theta}_t(t,x,y) + \check{\theta}_x(t,x,y)\big[(a_0 - \frac{b_0^2}{r_0})x - \frac{b_0^2}{r_0}\theta(t,x) + c_0 y\big] + \frac{1}{2}\sigma_0^2\check{\theta}_{xx}(t,x,y)$$

$$+ \check{\theta}_y(t,x,y)\big[(a + c - \frac{b^2}{r}\Pi(t))y - \frac{b^2}{r}\check{\theta}(t,x,y)\big]$$

$$= -\big[a - \frac{b^2}{r}\Pi(t)\big]\check{\theta}(t,x,y) + \eta - \lambda_1 x - \big(c\Pi(t) - \lambda\big)y, \quad \check{\theta}(T,x,y) = 0.$$

Step 5. Use $z_0^o(t)$ and $\check{\theta}$ obtained in Steps 2 and 4 to solve the following forward SDE:

$$dz^o(t) = \big[(a + c - \frac{b^2}{r}\Pi(t))z^o(t) - \frac{b^2}{r}\check{\theta}(t, z_0^o(t), z^o(t))\big]dt, \;\; z^o(0).$$

Step 6. Set

$$q(t) = -\sigma_0\check{\theta}_x(t, z_0^o(t), z^o(t))$$
$$s(t) = \check{\theta}(t, z_0^o(t), z^o(t)).$$

by the use of $z_0^o(t)$, $\check{\theta}$ and $z^o(t)$ obtained in Steps 2, 4 and 5.

Partially Observed Major-Minor Agent Mean Field Systems

Infinite Horizon Completely Observed MM MFG Problem Formulation (Huang 2010)

- Dynamics: Completely Observed Finite Population:

Major Agent $\rightarrow \quad dx_0 = [A_0 x_0 + B_0 u_0]dt + D_0 dw_0$

Minor Agents $\rightarrow \quad dx_i = [A(\theta_i)x_i + B(\theta_i)u_i + Gx_0]dt + Ddw_i,$

$$i \in \mathbf{N}$$

- The individual infinite horizon cost for the major agent:

$$J_0(u_0, u_{-0}) = \mathbb{E}\int_0^\infty e^{-\rho t}\left\{\left\|x_0 - \Phi(x^N)\right\|_{Q_0}^2 + \|u_0\|_{R_0}^2\right\}dt$$

$$\Phi(\cdot) = H_0 x^N + \eta_0 \qquad x^N = (1/N)\textstyle\sum_{i=1}^N x_i$$

- The individual infinite horizon cost for a minor agent i, $i \in \mathbf{N}$:

$$J_i(u_i, u_{-i}) = \mathbb{E}\int_0^\infty e^{-\rho t}\left\{\left\|x_i - \Psi(x^N)\right\|_{Q}^2 + \|u_i\|_{R}^2\right\}dt$$

$$\Psi(\cdot) = H_1 x_0 + H_2 x^N + \eta$$

Minor Agents Types

- Minor Agents' Types:

$$\mathcal{I}_k = \{i : \theta_i = k,\, i \in \mathbf{N}\}, \quad N_k = |\mathcal{I}_k|, \quad 1 \leq k \leq K$$

- $\pi^N = (\pi_1^N, ..., \pi_K^N)$, $\pi_k^N = N_k/N$, $1 \leq k \leq K$, denotes the empirical distribution of the parameters $(\theta_1, ..., \theta_N)$ of the agents $A_i \in \mathbf{N}$.

- Assumption: There exists π such that $\lim_{N\to\infty} \pi^N = \pi$ a.s.

Major Agent and Minor Agents

- Introduce the (auxiliary) state means:

$$x_k^N = \frac{1}{N_k} \sum_{i \in \mathcal{I}_k} x_i, \quad 1 \le k \le K$$

- If it exists, the L^2 limit of the system states' means $x^N = [x_1^N, ..., x_K^N]$ constitutes the system mean field.
- Subject to time invariant local state plus mean field plus major agent state feedback control, $x^N = [x_1^N, ..., x_K^N]$ satisfies the mean field equation

$$d\bar{x}_k = \sum_{j=1}^{K} \bar{A}_{k,j} \bar{x}_j dt + \bar{G}_k x_0 dt + \bar{m}_k dt, \quad 1 \le k \le K$$

$$\text{i.e.,} \quad d\bar{x}(t) = \bar{A}\bar{x}(t)dt + \bar{G}x_0(t)dt + \bar{m}(t)dt$$

where the quantities $\bar{G}_k$, $\bar{m}_k$ are to be solved for in the tracking solution.

Major Agent and Minor Agents LQG - MFG

- When MF plus x_0 plus local state dependent controls are applied, the MF-dependent extended state closes the system equations into state equations.
- Major Agent's Extended State:

  Major agent's state extended by the mean field: $\begin{bmatrix} x_0 \\ \bar{x} \end{bmatrix}$.

- Minor Agents' Extended States:
  Minor agent's state extended by major agent's state and the mean field: $\begin{bmatrix} x_i \\ x_0 \\ \bar{x} \end{bmatrix}$.

Major Agent and Minor Agents (Inf. Population)

- Major Agent's Dynamics (Infinite Population):

$$\begin{bmatrix} dx_0 \\ d\bar{x} \end{bmatrix} = \begin{bmatrix} A_0 & 0_{nK\times n} \\ \bar{G} & \bar{A} \end{bmatrix} \begin{bmatrix} x_0 \\ \bar{x} \end{bmatrix} dt + \begin{bmatrix} B_0 \\ 0_{nK\times m} \end{bmatrix} u_0 dt + \begin{bmatrix} 0_{n\times 1} \\ \bar{m} \end{bmatrix} dt + \begin{bmatrix} D_0 dw_0 \\ 0_{nK\times 1} \end{bmatrix}$$

$$\mathbb{A}_0 = \begin{bmatrix} A_0 & 0_{nK\times n} \\ \bar{G} & \bar{A} \end{bmatrix} \qquad \mathbb{B}_0 = \begin{bmatrix} B_0 \\ 0_{nK\times m} \end{bmatrix}$$

$$\mathbb{M}_0 = \begin{bmatrix} 0_{n\times 1} \\ \bar{m} \end{bmatrix} \qquad Q_0^\pi = \begin{bmatrix} Q_0 & -Q_0 H_0^\pi \\ -{H_0^\pi}^{\mathrm{T}} Q_0 & {H_0^\pi}^{\mathrm{T}} Q_0 H_0^\pi \end{bmatrix}$$

$$\bar{\eta}_0 = [I_{n\times n}, -H_0^\pi]^{\mathrm{T}} Q_0 \eta_0 \qquad H_0^\pi = \pi \otimes H_0 \triangleq [\pi_1 H_0 \ \pi_2 H_0 \ \ldots \ \pi_K H_0]$$

Major Agent and Minor Agents (Inf. Population)

- Minor Agents' Dynamics (Infinite Population):

$$\begin{bmatrix} dx_i \\ dx_0 \\ d\bar{x} \end{bmatrix} = \begin{bmatrix} A_k & [G \;\; 0_{n\times nK}] \\ 0_{(nK+n)\times n} & \mathbb{A}_0 \end{bmatrix} \begin{bmatrix} x_i \\ x_0 \\ \bar{x} \end{bmatrix} dt + \begin{bmatrix} B_k \\ 0_{(nK+n)\times m} \end{bmatrix} u_i dt + \begin{bmatrix} 0_{n\times 1} \\ \mathbb{M}_0 \end{bmatrix} dt$$

$$\begin{bmatrix} 0_{n\times m} \\ B_0 \\ 0_{nK\times m} \end{bmatrix} u_0 dt + \begin{bmatrix} D dw_i \\ D_0 dw_0 \\ 0_{nK\times 1} \end{bmatrix}$$

$$\mathbb{A}_k = \begin{bmatrix} A_k & [G \;\; 0_{n\times nK}] \\ 0_{(nK+n)\times n} & \mathbb{A}_0 - \mathbb{B}_0 R_0^{-1} \mathbb{B}_0^{\mathrm{T}} \Pi_0 \end{bmatrix}$$

$$\mathbb{B}_k = \begin{bmatrix} B_k \\ 0_{(nK+n)\times m} \end{bmatrix} \quad \mathbb{M} = \begin{bmatrix} 0_{n\times 1} \\ \mathbb{M}_0 - \mathbb{B}_0 R_0^{-1} \mathbb{B}_0^{\mathrm{T}} s_0 \end{bmatrix}$$

$$\bar{\eta} = [I_{n\times n}, -H, -H_2^{\pi}]^{\mathrm{T}} Q \eta \quad H_2^{\pi} = \pi \otimes H_2$$

Infinite Population Cost Function

- The individual cost for the major agent:

$$J_0^\infty(u_0, u_{-0}) = \mathbb{E}\int_0^\infty e^{-\rho t}\left\{\left\|x_0 - \Phi(\bar{x})\right\|_{Q_0}^2 + \|u_0\|_{R_0}^2\right\}dt$$

$$\Phi(\cdot) = H_0^\pi \bar{x} + \eta_0$$

- The individual cost for a minor agent i, $i \in \mathbf{N}$:

$$J_i^\infty(u_i, u_{-i}) = \mathbb{E}\int_0^\infty e^{-\rho t}\left\{\left\|x_i - \Psi(\bar{x})\right\|_{Q}^2 + \|u_i\|_{R}^2\right\}dt$$

$$\Psi(\cdot) = H_1 x_0 + H_2^\pi \bar{x} + \eta$$

Control Actions (Infinite Population)

- Major Agent Tracking Problem Solution:

$$\rho\Pi_0 = \Pi_0\mathbb{A}_0 + \mathbb{A}_0^{\mathrm{T}}\Pi_0 - \Pi_0\mathbb{B}_0R_0^{-1}\mathbb{B}_0^{\mathrm{T}}\Pi_0 + \mathbb{Q}_0^{\pi}$$

$$\rho s_0^* = \frac{ds_0^*}{dt} + (\mathbb{A}_0 - \mathbb{B}_0R_0^{-1}\mathbb{B}_0^{\mathrm{T}}\Pi_0)^{\mathrm{T}}s_0^* + \Pi_0\mathbb{M}_0 - \bar{\eta}_0$$

$$u_i^0 = -R_0^{-1}\mathbb{B}_0^{\mathrm{T}}\left[\Pi_0(x_0^{\mathrm{T}}, \bar{z}^{\mathrm{T}})^{\mathrm{T}} + s_0^*\right]$$

- Minor Agent Tracking Problem Solution:

$$\rho\Pi_k = \Pi_k\mathbb{A}_k + \mathbb{A}_k^{\mathrm{T}}\Pi_k - \Pi_k\mathbb{B}_kR^{-1}\mathbb{B}_k^{\mathrm{T}}\Pi_k + \mathbb{Q}$$

$$\rho s_k^* = \frac{ds_k^*}{dt} + (\mathbb{A}_k - \mathbb{B}_kR^{-1}\mathbb{B}_k^{\mathrm{T}}\Pi_k)^{\mathrm{T}}s_k^* + \Pi_k\mathbb{M} - \bar{\eta}$$

$$u_i^0 = -R^{-1}\mathbb{B}_k^{\mathrm{T}}\left[\Pi_k(x_i^{\mathrm{T}}, x_0^{\mathrm{T}}, \bar{z}^{\mathrm{T}})^{\mathrm{T}} + s_k^*\right]$$

Major-Minor MF Equations

$$\text{Define:} \quad \Pi_k = \begin{bmatrix} \Pi_{k,11} & \Pi_{k,12} & \Pi_{k,13} \\ \Pi_{k,21} & \Pi_{k,22} & \Pi_{k,23} \\ \Pi_{k,31} & \Pi_{k,32} & \Pi_{k,33} \end{bmatrix}, \quad 1 \le k \le K$$

$\mathbf{e}_k = [0_{n\times n}, ..., 0_{n\times n}, I_n, 0_{n\times n}, ..., 0_{n\times n}]$, where the $n \times n$ identity matrix I_n is at the kth block.

Major-Minor MF Equations for $\bar{A}$, $\bar{G}$, $\bar{m}$: Consistency Requirements

$$\begin{aligned}
\rho\Pi_0 &= \Pi_0\mathbb{A}_0 + \mathbb{A}_0^{\mathrm{T}}\Pi_0 - \Pi_0\mathbb{B}_0 R_0^{-1}\mathbb{B}_0^{\mathrm{T}}\Pi_0 + Q_0^{\pi}, \\
\rho\Pi_k &= \Pi_k\mathbb{A}_k + \mathbb{A}_k^{\mathrm{T}}\Pi_k - \Pi_k\mathbb{B}_k R^{-1}\mathbb{B}_k^{\mathrm{T}}\Pi_k + Q^{\pi}, \ \ \forall k, \\
\bar{A}_k &= [A_k - B_k R^{-1} B_k^{\mathrm{T}}\Pi_{k,11}]\mathbf{e}_k - B_k R^{-1} B_k^{\mathrm{T}}\Pi_{k,13}, \ \ \forall k, \\
\bar{G}_k &= -B_k R^{-1} B_k^{\mathrm{T}}\Pi_{k,12}, \ \ \forall k, \\
\rho s_0^* &= \frac{ds_0^*}{dt} + (\mathbb{A}_0 - \mathbb{B}_0 R_0^{-1}\mathbb{B}_0^{\mathrm{T}}\Pi_0)^{\mathrm{T}} s_0^* + \Pi_0\mathbb{M}_0 - \bar{\eta}_0, \\
\rho s_k^* &= \frac{ds_k^*}{dt} + (\mathbb{A}_k - \mathbb{B}_k R^{-1}\mathbb{B}_k^{\mathrm{T}}\Pi_k)^{\mathrm{T}} s_k^* + \Pi_k\mathbb{M} - \bar{\eta}, \ \ \forall k, \\
\bar{m}_k &= -B_k R^{-1} B_k^{\mathrm{T}} s_k^*, \ \ \forall k,
\end{aligned}$$

Assumptions

Define

$$M_1 = \begin{bmatrix} A_1 - B_1 R^{-1} B_1^{\mathrm{T}} \Pi_{1,11} & & \\ & \ddots & \\ & & A_K - B_K R^{-1} B_K^{\mathrm{T}} \Pi_{K,11} \end{bmatrix}$$

$$M_2 = \begin{bmatrix} B_1 R^{-1} B_1^{\mathrm{T}} \Pi_{1,13} \\ \vdots \\ B_K R^{-1} B_K^{\mathrm{T}} \Pi_{K,13} \end{bmatrix}$$

$$M_3 = \begin{bmatrix} A_0 & 0 & 0 \\ \bar{G} & \bar{A} & 0 \\ \bar{G} & -M_2 & M_1 \end{bmatrix}, \quad L_{0,H} = Q_0^{1/2}[I, 0, -H_0^{\pi}]$$

- H1: There exists a probability vector π such that $\lim_{N\to\infty} \pi^N = \pi$.
- H2: The initial states are independent, $\mathbb{E}x_i(0) = 0$ for each $i \geq 1$, $\sup_{j\geq 0} \mathbb{E}|x_j(0)|^2 \leq c$.
- H3: The pair $(L_{0,H}, M_3)$ is observable.
- H4: The pair $(L_a, \mathbb{A}_0 - (\rho/2)I)$ is detectable, and for each $k = 1, \ldots, K$, the pair $(L_b, \mathbb{A}_k - (\rho/2)I)$ is detectable, where $L_a = Q_0^{1/2}[I, -H_0^{\pi}]$ and $L_b = Q^{1/2}[I, -H, -\hat{H}^{\pi}]$. The pair $(\mathbb{A}_0 - (\rho/2)I, \mathbb{B}_0)$ is stabilizable and $(\mathbb{A}_k - (\rho/2)I, \mathbb{B}_k)$ is stabilizable for each $k = 1, \ldots, K$.

Major - Minor: MF Equilibrium

Theorem: (Huang, 2010)

Major and Minor Agents: MF Equilibrium: Subject to H1-H4 the MF equations generate a set of stochastic control laws $\mathcal{U}_{MF}^{N} \triangleq \{u_i^0; 0 \le i \le N\}$, $1 \le N < \infty$, such that

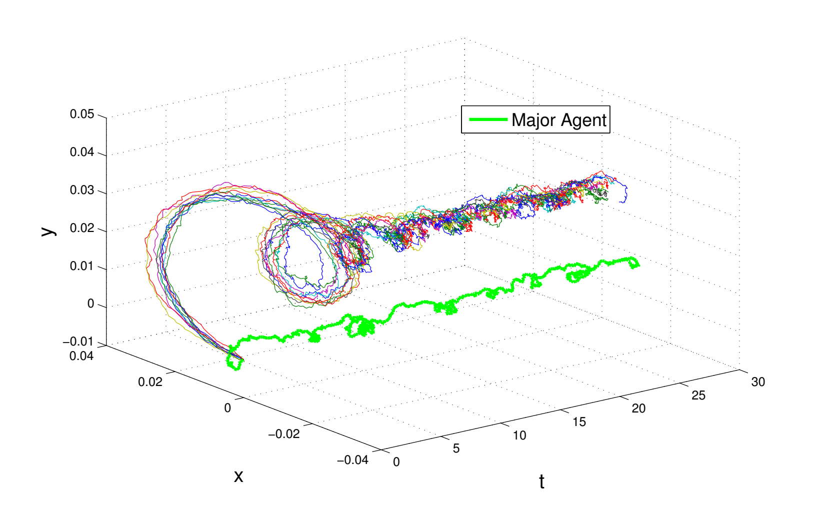
(i) All agent systems $S(A_i)$, $0 \le i \le N$, are second order stable.

(ii) $\{\mathcal{U}_{MF}^{N}; 1 \le N < \infty\}$ yields an ϵ-Nash equilibrium for all ϵ, i.e. for all $\epsilon > 0$, there exists $N(\epsilon)$ such that for all $N \ge N(\epsilon)$

$$J_i^N(u_i^0, u_{-i}^0) - \epsilon \le \inf_{u_i \in \mathcal{U}_g} J_i^N(u_i, u_{-i}^0) \le J_i^N(u_i^0, u_{-i}^0).$$

□

Simulation



State trajectories

Partially Observed Major-Minor Mean Field Systems

Partially Observed Major-Minor Agent Systems

Recall: Estimation of major agent and mean field states becomes a meaningful problem in MM case.

Major Agent $\rightarrow \quad dx_0 = [A_0 x_0 + B_0 u_0]dt + D_0 dw_0$

Minor Agents $\rightarrow \quad dx_i = [A(\theta_i)x_i + B(\theta_i)u_i + Gx_0]dt + Ddw_i,$

The observation process for minor agent A_i: $\quad i \in \mathbf{N}$

$$dy_i(t) = \mathbb{L}x_i^{0,\bar{x}}dt + dv_i(t) \equiv \mathbb{L}\begin{bmatrix} x_i \\ x_0 \\ \bar{x} \end{bmatrix} dt + dv_i(t)$$

where

$$\mathbb{L} = [L_1 \ L_2 \ 0].$$

Complete observations process for the major agent A_0:

$$dy_o(t) = dx_0(t)$$

Partially Observed Major-Minor Agent Systems

- The Major agent is assumed to have complete observations of its own state.
- This permits the Minor agents to form conditional expectations of the Major agent's MFG control action u_0 since it is a (linear) function of the Major agent's state.
- Such an estimate could not in general be generated by the Minor agents in the case where the Major agent's control action is a (linear) function of the conditional expectation of its state $\mathbb{E}[x_{0|\mathcal{F}^0}]$, where $\mathcal{F}^0$ is the Major agent's observation σ-field.

Estimation

The Riccati equation associated with the Kalman filtering equations for $x_i^{0,\bar{x}} \triangleq [x_i, x_0, \bar{x}]$:

$$\dot{V}(t) = \mathbb{A}_k V(t) + V(t)\mathbb{A}_k^{\mathrm{T}} - K(t) R_v K^{\mathrm{T}}(t) + Q_w,$$

where

$$Q_w = \begin{bmatrix} \Sigma_i & 0 & 0 \\ 0 & \Sigma_\zeta & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbb{A}_k = \begin{bmatrix} A_k & [G \;\; 0_{n \times nK}] \\ 0_{(nK+n)\times n} & \begin{bmatrix} A_0 & 0_{(nK \times n)} \\ \bar{G} & \bar{A} \end{bmatrix} \end{bmatrix}$$

and

$$V(0) = \mathbb{E}\left[x_i^{0,\bar{x}}(0) - \hat{x}_i^{0,\bar{x}}(0)\right]\left[x_i^{0,\bar{x}}(0) - \hat{x}_i^{0,\bar{x}}(0)\right]^{\mathrm{T}}.$$

Estimation

The innovation process is

$$d\nu_i = dy_i - \mathbb{L} \begin{bmatrix} \hat{x}_{i|\mathcal{F}_i^y} \\ \hat{x}_{0|\mathcal{F}_i^y} \\ \hat{\bar{x}}_{|\mathcal{F}_i^y} \end{bmatrix},$$

and the Kalman filter gain is given by

$$K(t) = V(t)\mathbb{L}^{\mathrm{T}} R_v^{-1}.$$

H5: The system parameter set Θ satisfies $[\mathbb{A}_k, Q_w]$ controllable and $[\mathbb{L}, \mathbb{A}_k]$ observable for $1 \le k \le K$. □

The Filtering Equations:

$$\begin{bmatrix} d\hat{x}_{i|\mathcal{F}_i^y} \\ d\hat{x}_{0|\mathcal{F}_i^y} \\ d\hat{\bar{x}}_{|\mathcal{F}_i^y} \end{bmatrix} = \begin{bmatrix} A_k & [G \;\; 0_{n\times nK}] \\ \begin{bmatrix} 0_{n\times n} \\ 0_{nK\times n} \end{bmatrix} & \begin{bmatrix} A_0 & 0_{nK\times n} \\ \bar{G} & \bar{A} \end{bmatrix} \end{bmatrix} \begin{bmatrix} \hat{x}_{i|\mathcal{F}_i^y} \\ \hat{x}_{0|\mathcal{F}_i^y} \\ \hat{\bar{x}}_{|\mathcal{F}_i^y} \end{bmatrix} dt$$

$$+ \begin{bmatrix} B_k \\ 0_{n\times m} \\ 0_{nK\times m} \end{bmatrix} u_i dt + \begin{bmatrix} 0_{n\times m} \\ B_0 \\ 0_{nK\times m} \end{bmatrix} \hat{u}_{0|\mathcal{F}_i^y} dt + \begin{bmatrix} 0_{n\times 1} \\ 0_{n\times 1} \\ \bar{m} \end{bmatrix} dt + K d\nu_i,$$

Separation Principle for MM-MFG Systems

- The control law dependent summand of the individual cost for the major agent A_0:

$$J_0^N(u_0, u_{-0}) = \mathbb{E} \int_0^\infty e^{-\rho t} \left\{ \left\| x_0 - H_0 \hat{x}^N_{|\mathcal{F}_0} - \eta_0 \right\|^2_{Q_0} + \| u_0 \|^2_{R_0} \right\} dt$$

$$\hat{x}^N_{|\mathcal{F}_0} = (1/N) \sum_{i=1}^N \hat{x}_{i|\mathcal{F}_0}$$

- The control law dependent summand of the individual cost for a minor agent A_i, $i \in \mathbf{N}$:

$$J_i^N(u_i, u_{-i}) = \mathbb{E} \int_0^\infty e^{-\rho t} \left\{ \left\| \hat{x}_{i|\mathcal{F}_i^y} - H_1 \hat{x}_{0|\mathcal{F}_i^y} - H_2 \hat{x}^N_{|\mathcal{F}_i^y} - \eta \right\|^2_Q + \| u_i \|^2_R \right\}$$

Separation Principle for PO MM-MFG Systems

Key Steps to Main Result:

(1) The major agent and individual minor agent state estimation recursive equations schemes are given by the MM KF-MF Equations (for size N finite populations and infinite populations).

(2) Apply the Separation Theorem strategy of reducing a partially observed SOC problem to a completely observed SOC problem for the controlled state estimate processes.

(3) Step 2 transforms the J_0 and J_i MM performance functions into LQG MM tracking performance functions on the controlled state estimate processes in the infinite and finite population cases.

Separation Principle for PO MM-MFG Systems

(4) The problem in (2) for the state estimate processes is solved using the completely observed LQG MM-MFG methodology which yields the $\hat{u}_0$ and $\hat{u}_i$ control laws and the J_0^{∞} and J_i^{∞} performance function values.

(5) The Major Agent performance value J_0^{∞} and Minor Agent performance value J_i^{∞} neccessarily correspond to infinite population Nash equilibria.

(6) Approximation Analysis gives $\epsilon-$Nash equilibria with respect to J_0^{∞} and J_i^{∞} for J_0^N and J_i^N in finite N populations.

Nash Equilibria for Partially Observed MM-MFG Systems

Theorem: (PEC, AK, 2013) ϵ-Nash Equilibria for PO MM-MF Systems

Subject to H1-H5, the KF-MF state estimation scheme plus MM-MFG equations generate the set of control laws $\hat{\mathcal{U}}_{MF}^{N} \triangleq \{\hat{u}_i^0; 1 \leq i \leq N\}$, $1 \leq N < \infty$, and u_0 given by

$$u_0^0 = -R_0^{-1}\mathbb{B}_0^{\mathrm{T}}\left[\Pi_0(x_0^{\mathrm{T}}, \bar{x}^{\mathrm{T}})^{\mathrm{T}} + s_0^*\right],$$

$$\hat{u}_i^0 = -R^{-1}\mathbb{B}_k^{\mathrm{T}}\left[\Pi_k(\hat{x}_{i|\mathcal{F}_i^y}^{\mathrm{T}}, \hat{x}_{0|\mathcal{F}_i^y}^{\mathrm{T}}, \hat{\bar{x}}_{|\mathcal{F}_i^y}^{\mathrm{T}})^{\mathrm{T}} + s_k^*\right]$$

such that

(i) All agent systems $S(A_i)$, $0 \leq i \leq N$, are second order stable.

(ii) $\{\hat{\mathcal{U}}_{MF}^{N}; 1 \leq N < \infty\}$ yields an ϵ-Nash equilibrium for all ϵ, i.e. for all $\epsilon > 0$, there exists $N(\epsilon)$ such that for all $N \geq N(\epsilon)$

$$J_i^N(\hat{u}_i^0, \hat{u}_{-i}^0) - \epsilon \leq \inf_{u_i \in \mathcal{U}_g} J_i^N(u_i, \hat{u}_{-i}^0) \leq J_i^N(\hat{u}_i^0, \hat{u}_{-i}^0).$$

□

Concluding Remarks

Future Work

- Building upon the MM-LQG-MFG theory for partially observed MM systems and the Nonlinear MM-MFG theory, the next step is to generate a MM-MFG theory for partially observed nonlinear MM systems.
- Investigate applications to power markets via the systematic application of MM-MFG Theory to markets where minor agents (customers and suppliers) receive intermittent observations on active major agents (such as utilities and international energy prices) and on passive major agents (such as wind and ocean behaviour).

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