# A class of MFG systems defined on a network

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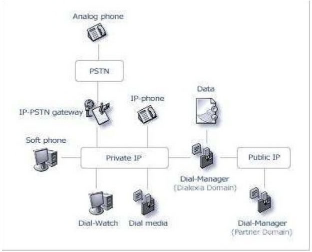
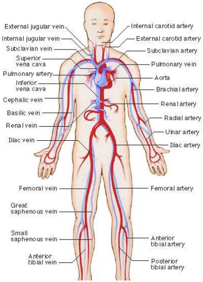
# Problems on Networks

Several phenomena in physics, engineering, chemistry and biology can be described by some physical quantities defined on the so-called ramified spaces, whose simplest example are the networks.

(a) traffic




(b) internet




(c) blood

The evolution of the physical quantities on the edges is described by some differential equations. The main problem is to find appropriate transition conditions describing the interactions of the physical quantities across the vertices

Aim is to define a MFG system on a network $\Gamma$

$$\begin{cases} -\nu\Delta u + H(x, Du) + \lambda = V[m](x) & x \in \Gamma \\ \nu\Delta m + \text{div}\left(\frac{\partial H}{\partial p}(x, Du)m\right) = 0 & x \in \Gamma \\ \int_\Gamma u(x)dx = 0,\ \int_\Gamma m(x)dx = 1,\ m \geq 0 \end{cases}$$

Inside the edges the equations are defined in standard sense (pointwise, weak, viscosity, etc.).

**Main point:**

Which transition conditions have to be imposed at the vertices in order to have a global well posed problem?

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# General references for PDE on networks

## Elliptic PDEs on network

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- J. von Below, Classical solvability of linear parabolic equations on networks. J. Diff. Equations 72 (1988), 316-337.

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## Stochastic process

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## Nonlinear problems

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### Hamilton-Jacobi equations

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## MFG on networks

- Discrete time, finite state space:
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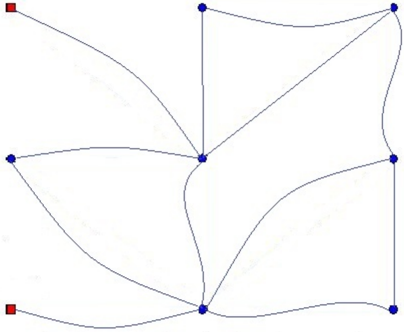
- Continuous time, finite state space:
  O.Gueant, An existence and uniqueness result for mean field games with congestion effect on graphs, preprint, arXiv:1110.3442

Here we consider a continuous time, continuous state space MFG model.

# Networks

A network is a connected set $\Gamma$ consisting of vertices $V := \{v_i\}_{i\in I}$ and edges $E := \{e_j\}_{j\in J}$ connecting the vertices. We assume that the network is embedded in the Euclidian space so that any two edges can only have intersection at a vertex.



(d) An example of network

# Some Notations

- $Inc_i := \{j \in J : \ e_j \text{ incident to } v_i\}$ is the set of edges incident the vertex $v_i$.
- A vertex $v_i$ is a boundary vertex if it has only one incident edge. We denote by $\partial\Gamma = \{v_i,\ i \in I_B\}$ the set of boundary vertices. A vertex $v_i$ is a transition vertex if it has more than one incident edge. We denote by $\Gamma_T = \{v_i,\ i \in I_T\}$ the set of transition vertices
- The graph is not oriented, but the parametrization of the arcs $e_j$ induces an orientation on the edges expressed expressed by the signed incidence matrix $A = \{a_{ij}\}_{i \in I, j \in J}$

$$a_{ij} := \begin{cases} 1 & \text{if } v_i \in \bar{e}_j \text{ and } \pi_j(0) = v_i, \\ -1 & \text{if } v_i \in \bar{e}_j \text{ and } \pi_j(l_j) = v_i, \\ 0 & \text{otherwise.} \end{cases}$$

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- Given $u : \Gamma \to \mathbb{R}$, $u^j : [0, l_j] \to \mathbb{R}$ denotes the restriction of $u$ to $\bar{e}_j$, i.e. $u^j(y) = u(\pi_j(y))$ for $y \in [0, l_j]$
- $u$ is **continuous** ($u \in C^0(\Gamma)$) if $u^j \in C([0, l_j])$ for any $j \in J$ and

$$u^j(\pi_j^{-1}(v_i)) = u^k(\pi_k^{-1}(v_i)) \qquad \text{for any } i \in I,\ j, k \in Inc_i$$

- Differentiation is defined with respect to the parameter variable inside the edges

$$\partial_j u(x) := \frac{d}{dy} u^j(y), \qquad \text{for } x \in e_j,\ y = \pi_j^{-1}(x)$$

and at a vertex $v_i$ by

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## Some functional spaces

- $u \in L^p(\Gamma)$, $p \geq 1$ if $u^j \in L^p(0, l_j)$ for each $j \in J$. We set
$$\|u\|_{L^p} := \left( \sum_{j \in J} \|u\|_{L^p(e_j)} \right)^{1/p}$$
$$\|u\|_{L^\infty} := \sup_{j \in J} \|u^j\|_{L^\infty(e_j)}.$$
- $u \in W^{k,p}(\Gamma)$, for $k \in \mathbb{N}$, $k \geq 1$ and $p \geq 1$ if $u \in C^0(\Gamma)$ and $u^j \in W^{k,p}(0, l_j)$ for each $j \in J$. We set
$$\|u\|_{W^{k,p}} := \left( \sum_{l=1}^{k} \|\partial^l u^j\|_{L^p}^p \right)^{1/p}$$
- $u \in C^k(\Gamma)$, $k \in \mathbb{N}$, if $u \in C^0(\Gamma)$ and $u^j \in C^k([0, l_j])$ for each $j \in J$. The space $C^k(\Gamma)$ is a Banach space with the norm
$$\|u\|_{C^k} = \max_{\beta \leq k} \|\partial^\beta u\|_{L^\infty}$$

**Remark:**

- The previous spaces are Banach and Hilbert space with the appropriate norms.
- It is important to observe that for a function in $W^{k,p}(\Gamma)$ is required continuity at the vertices.
- No continuity condition at the vertices is instead prescribed for the derivates of a function in $C^k(\Gamma)$.

# Elliptic PDEs on networks: the role of transition conditions

## Maximum Principle

Assume that the function $w \in C^2(\Gamma)$ satisfies

$$\boxed{\begin{array}{ll} \partial^2 w(x) \geq 0, & x \in \Gamma, \\ S(w) := \sum_{j \in Inc_i} a_{ij} \partial_j w(v_i) \geq 0, & i \in I_T \end{array}}$$

with $\{a_{ij}\}$ is the incidence matrix. Then $w$ cannot attain a maximum in the interior of $\Gamma$, i.e. in $\Gamma \setminus \partial\Gamma$.

**Remark:**
By $\partial^2 w \geq 0$ for $x \in \Gamma$, we intend $\partial_j^2 w(x) \geq 0$ for $x \in e_j$, $\forall j \in J$.

**Proof.:** Assume first $\partial^2 w > 0$ and $S(w) := \sum_{j \in Inc_i} a_{ij}\partial_j w(v_i) > 0$ and there exists $x_0 \in \Gamma \setminus \partial\Gamma$ such that $w$ attains a maximum at $x_0$.

If $x_0 \in e_j$ for some $j \in J$, then it follows that $\partial_j w(x_0) = 0$ and $\partial_j^2 w(x_0) \leq 0$, a contradiction to $\partial_j^2 w > 0$.

If $x_0 = v_i$ for some $i \in I_T$, then $a_{ij}\partial_j w(v_i) \leq 0$ for all $j \in Inc_i$, hence $Sw(v_i) \leq 0$, a contradiction to $S(w) > 0$.

For the general case, it can be proved that there exists $\varphi \in C^2(\Gamma)$ such that $\partial^2\varphi(x) > 0$ for $x \in e_j, j \in J$ and $S\varphi(v_i) > 0$ for $i \in I_T$. Consider $w_\delta = w + \delta\varphi$, $\delta > 0$, then by the previous argument $w_\delta$ cannot attain a maximum in $\Gamma \setminus \partial\Gamma$. The statement is obtained sending $\delta \to 0$.

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- It is clear that some sort of transition condition at the internal vertices is necessary for the validity of the maximum principle.
- The simplest form of a linear transition condition is the **Kirchhoff condition**

$$\boxed{\sum_{j \in Inc_i} a_{ij} \partial_j w(v_i) = 0}$$

from electrical circuits theory (the sum of the incoming currents is null).
- More general transition conditions are of the form

$$\sum_{j \in Inc_i} a_{ij} \beta_{ij} \partial_j w(v_i) + \gamma_i w(v_i) = 0 \qquad i \in I_T$$

with weights $\beta_{ij} > 0$ for $i \in I$, $j \in J$, and also nonlocal conditions (depending on the value of $w$ at the other vertices).

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# A formal derivation of the MFG system

We formally deduce the FP equation and the corresponding transition and boundary conditions using the fact that the FP equation is the adjoint of the linearized HJB equation.

Consider the HJB equation

$$\begin{cases} -\nu\partial_j^2 u + H(x,\partial u) + \lambda = V[m] & x \in e_j,\ j \in J \\ \sum_{j\in Inc_i} a_{ij}\nu_j\partial_j u(v_i) = 0 & i \in I_T \\ \partial_j u(v_i) = 0 & i \in I_B \end{cases}$$

and the linearized equation

$$\begin{cases} -\nu\partial^2 w + \partial_p H(x,\partial u)\partial w = 0 & x \in e_j,\ j \in J \\ \sum_{j\in Inc_i} a_{ij}\nu_j\partial_j w(v_i) = 0 & i \in I_T \\ \partial_j w(v_i) = 0 & i \in I_B \end{cases}$$

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Writing the weak formulation for a test function $m$, we get

$$
\begin{aligned}
0 &= \sum_{j\in J}\int_{e_j}\big(-\nu_j\partial^2 w+\partial_p H_j(x,\partial u)\partial w\big)m\,dx\\
&= \sum_{j\in J}\int_{e_j}\big(\nu_j\partial_j w\partial_j m-\partial(m(x)\partial_p H_j(x,\partial u))w\big)\,dx\\
&-\sum_{i\in I_T}\sum_{j\in Inc_i} a_{ij}\big(\nu_j m_j(v_i)\partial_j w(v_i)-m_j(v_i)\partial_p H_j(v_i,\partial u)w(v_i)\big)\\
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&= \sum_{j\in J}\int_{e_j}\big(-\nu_j\partial_j^2 m(x)-\partial(m(x)\partial_p H_j(x,\partial u))\big)w\,dx\\
&-\sum_{i\in I_T}\sum_{j\in Inc_i} m_j(v_i)\,a_{ij}\nu_j\partial_j w(v_i)-a_{ij}\big(\nu_j\partial_j m(v_i)+\partial_p H(v_i,\partial u)m_j(v_i)\big)w(v_i)\\
&-\sum_{i\in I_B}\sum_{j\in Inc_i} a_{ij}\,\big(\nu_j\partial_j m(v_i)+m_j(v_i)\partial_p H_j(x,\partial u)w(v_i)\big)
\end{aligned}
$$

Writing the weak formulation for a test function $m$, we get

$$0 = \sum_{j \in J} \int_{e_j} \big( - \nu_j \partial^2 w + \partial_p H_j(x, \partial u) \partial w \big) m \, dx$$

$$= \sum_{j \in J} \int_{e_j} \big( \nu_j \partial_j w \partial_j m - \partial(m(x) \partial_p H_j(x, \partial u)) w \big) \, dx$$

$$- \sum_{i \in I_T} \sum_{j \in Inc_i} a_{ij} \big( \nu_j m_j(v_i) \partial_j w(v_i) - m_j(v_i) \partial_p H_j(v_i, \partial u) w(v_i) \big)$$

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$$= \sum_{j \in J} \int_{e_j} \big( - \nu_j \partial_j^2 m(x) - \partial(m(x) \partial_p H_j(x, \partial u)) \big) w \, dx$$

$$- \sum_{i \in I_T} \sum_{j \in Inc_i} m_j(v_i) \, a_{ij} \nu_j \partial_j w(v_i) - a_{ij} \big( \nu_j \partial_j m(v_i) + \partial_p H(v_i, \partial u) m_j(v_i) \big) w(v_i)$$

$$- \sum_{i \in I_B} \sum_{j \in Inc_i} a_{ij} \, \big( \nu_j \partial_j m(v_i) + m_j(v_i) \partial_p H_j(x, \partial u) w(v_i) \big)$$

By the integral terms we obtain the FP equation

$$\nu\partial^2 m + \partial(m\,\partial_p H(x,\partial u)) = 0$$

Recalling the Kirchhoff condition satisfied by $w$, the first of three terms computed at the transition nodes vanishes since by the continuity of $m$

$$m_j(v_i) = m_k(v_i), \qquad j,k \in Inc_i,\ i \in I_T$$

The vanishing of the other two terms, i.e.

$$\sum_{j\in Inc_i} a_{ij}\Big(\nu_j\partial_j m(v_i) + \partial_p H(v_i,\partial u)m_j(v_i)\Big) = 0 \quad i \in I_T$$

gives the transition condition at the vertices $i \in I_T$, while the term computed at the boundary nodes gives the boundary condition

$$\nu_j\partial_j m(v_i) + m\partial_p H_j(v_i,\partial u) = 0 \quad i \in I_B,\ j \in Inc_i$$

Summarizing the MFG system on the network is

$$\begin{cases} -\nu\partial^2 u + H(x,\partial u) + \lambda = V[m](x) & x \in \Gamma \\ \nu\partial^2 m + \operatorname{div}(\partial_p H(x,\partial u)m) = 0 & x \in \Gamma \\ \\ \sum_{j\in Inc_i} a_{ij}\partial_j u(v_i) = 0 & i \in I_T \\ \sum_{j\in Inc_i} a_{ij}\big(\nu_j\partial_j m(v_i) + \partial_p H(v_i,\partial u)m_j(v_i)\big) = 0 & i \in I_T \\ \\ \partial_j u(v_i) = 0, & i \in I_B \\ \nu_j\partial_j m(v_i) + a_{ij} m\partial_p H_j(v_i,\partial u) = 0 & i \in I_B \\ \\ \int_\Gamma v(x)dx = 0,\ \int_\Gamma m(x)dx = 1,\ m \geq 0 \end{cases}$$

**Remarks:**

- Note that the continuity of $m$ at $v_i$ (i.e. $m_j(v_i) = m_k(v_i)$ for $j \neq k$) gives $(\#(Inc_i) - 1)$-conditions. The transition condition gives an additional condition which determine the unique value of $m$ at $v_i$.
- The condition

$$\sum_{j \in Inc_i} a_{ij}(\nu_j \partial_j m(v_i) + \partial_p H(v_i, \partial u) m_j(v_i)) = 0, \quad i \in I_T$$

gives the conservation of the total flux of the density $m$ at the transition vertices. A similar condition is also considered in traffic flow problems on networks (Coclite-Garavello '10).

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- Note that the continuity of $m$ at $v_i$ (i.e. $m_j(v_i) = m_k(v_i)$ for $j \neq k$) gives $(\#(Inc_i) - 1)$-conditions. The transition condition gives an additional condition which determine the unique value of $m$ at $v_i$.
- The condition

$$\sum_{j \in Inc_i} a_{ij}(\nu_j \partial_j m(v_i) + \partial_p H(v_i, \partial u) m_j(v_i)) = 0, \quad i \in I_T$$

gives the conservation of the total flux of the density $m$ at the transition vertices. A similar condition is also considered in traffic flow problems on networks (Coclite-Garavello '10).

# Existence and uniqueness for the Hamilton-Jacobi-Bellman equation

Consider the Hamilton-Jacobi-Bellman equation

$$\begin{cases} -\nu\partial^2 u + H(x,\partial u) + \lambda u = f(x), & x \in \Gamma \\ \sum_{j \in Inc_i} a_{ij}\nu_j\partial_j u(v_i) = 0 & i \in I_T \\ \partial u(x) = 0 & i \in I_B \end{cases} \qquad (\textbf{HJB})$$

where

• An Hamiltonian $H : \Gamma \times \mathbb{R} \to \mathbb{R}$ is a family $\{H^j\}_{j\in J}$ with $H^j : [0, l_j] \times \mathbb{R} \to \mathbb{R}$ continuous.

• The viscosity coefficient $\nu$ and the discount factor $\lambda$ may depend on the arc $e_j$, i.e. $\nu = \nu_j$ and $\lambda = \lambda_j$ for $j \in J$.

**Remark:** The Hamiltonian can be discontinuous across a vertex $v_i$, $i \in I_T$.

## Definition

- A function $u \in C^2(\Gamma)$ is said a strong solution of problem (HJB) if it satisfies the problem in point-wise sense
- A function $u \in H^1(\Gamma)$ is said a weak solution of the problem (HJB)

$$\sum_{j\in J} \int_{e_j} (\nu_j \partial_j u \partial_j \varphi + H^j(x, \partial_j u)\varphi + \lambda_j u \varphi) dx = 0 \qquad \text{for any } \varphi \in H^1(\Gamma)$$

**Remark:**

• In the definition of weak solution the transition and boundary conditions are included in the integral term.

• Integrating by parts it is easy to see that if $u \in C^2(\Gamma)$ is a weak solution of (HJB), then it is also a strong solution, i.e. it satisfies the transition and the boundary conditions.

In general semi-linear and fully nonlinear elliptic and parabolic equations on networks are not well investigated. We give an example of existence result based on

Boccardo, L. ; Murat, F. ; Puel, J.-P. Résultats d'existence pour certains problémes elliptiques quasilinéaires. Ann. Scuola Norm. Sup. Pisa Cl. Sci. 11 (1984), 213-235.

## Existence

Assume

- $H^j \in C^2([0, l_j] \times \mathbb{R})$, $H^j(x, \cdot)$ convex in $p$ for $x \in [0, l_j]$ and $H^j(x, p)| \leq C(1 + |p|^2)$ for $x \in [0, l_j]$;
- $\nu = \{\nu_j\}_{j \in J}$, $\lambda = \{\lambda_j\}_{j \in J}$ with $\nu_j, \lambda_j > 0\ \forall j \in J$;
- $f \in C^{0,\alpha}(\Gamma)$

Then there exists a weak solution to ($HJB$) and

$$\|u\|_{H^1} \leq C \tag{1}$$

with $C$ depending on $C_0$, $\lambda_0 = \inf_{j \in J} \lambda_j$ and $\nu_0 = \inf_{j \in J} \nu_j$.
Moreover $u \in C^{2,\alpha}(\Gamma)$ for $\alpha \in (0, 1)$ and

$$\|u\|_{C^{2,\alpha}} \leq C(\|f\|_{C^{0,\alpha}})$$

**Sketch of the proof:**

•Step 1, $H$ bounded: Show the existence of a bounded solution via a fixed point argument applied to the map $w \mapsto u$ defined by the solution of

$$\begin{cases} -\nu\partial^2 u + \lambda u = -H(x, \partial w) & x \in \Gamma \\ \sum_{e \in Inc_i} a_{ij}\nu_j \partial_j u(v_i) = 0 & i \in I_T \\ \partial u(v_i) = 0 & i \in I_B \end{cases}$$

•Step 2, $H$ with quadratic growth: Apply the Boccardo-Murat-Puel truncation argument to show the existence of a solution via approximation. It is important to observe that in the weak formulation the transition conditions are transparent hence it is possible to repeat the same argument to obtain the estimate on the $H^1$ norm.

•Step 3, Additional regularity: Regularity of the solution is based on the intrinsic 1-dimensionality of the problem and the Sobolev's embedding theorem. For example, by

$$-\nu_j \partial_j^2 u = -\lambda_j u - H(x, \partial_j u) + f(x) \quad x \in (0, l_j),$$

we get $u \in W^{2,1}((0, l_j))$, hence $\partial_j u \in L^p((0, l_j))$, $p \geq 1$.

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## Uniqueness

Under the same assumptions of the existence result, let $w_1, w_2 \in C^2(\Gamma)$ be such that

$$\begin{cases} -\nu\partial^2 w_1 + H(x, \partial_j w_1) + \lambda w_1 \geq -\nu\partial^2 w_2 + H(x, \partial_j w_2) + \lambda w_2 & x \in \Gamma, \\ \sum_{j\in Inc_i} a_{ij}\nu_j\partial_j w_1(v_i) \leq \sum_{j\in Inc_i} a_{ij}\nu_j\partial_j w_2(v_i) & i \in I_T \end{cases}$$

Then $w_1 \geq w_2$ on $\Gamma$.

For the proof, see

- F. Camilli, C. Marchi and D.Schieborn, The vanishing viscosity limit for Hamilton-Jacobi equation on networks, J.Diff. Equations 254 (2013).

# The control theoretic interpretation of the Kirchhoff condition

By the Dynamic Programming Principle the solution $u$ of the (HJB) equation is the value function of an optimal control problem for a controlled dynamics defined on $\Gamma$ which on each edge $e_j$ satisfies the stochastic differential equation

$$dX_s = \alpha_s\, ds + \sqrt{2\nu}\, dW_s,$$

where $\alpha_s = \partial_p H(X_s, \partial u(X_s))$ is the optimal control.

- The condition Kirchhoff condition implies that the process almost surely spends zero time at each transition vertex $v_i$, (Freidlin-Wentzell, Ann.Probab. '93)
- the term $\nu_j/(\sum_{j\in Inc_i} \nu_j)$ is the probability that the process $X_s$ enters in the edge $e_j$ when it is at the vertex $v_i$.

# An existence result for Fokker-Planck equation

$$\begin{cases} \nu\partial^2 m + \partial(b(x)\, m) = 0 & x \in \Gamma \\ \sum_{j \in Inc_i} a_{ij}[\nu_j \partial_j m(v_i) + b(x) m_j(v_i)] = 0 & i \in I_T \\ \nu_j \partial_j m(v_i) + b(x) m_j(v_i) = 0 & v_i \in I_B,\ j \in Inc_i \\ \int_\Gamma m(x)dx = 1,\ m \geq 0 \end{cases}$$

where $b = (b^j)_{j \in J}$ with $b^j : [0, l_j] \to \mathbb{R}$, $b^j \in C^1([0, l_j])$ for all $j \in J$.

## Definition

- A strong solution is a function $m \in C^2(\Gamma)$ which satisfies the equation in pointwise sense.
- A weak solution is a function $m \in H^1(\Gamma)$ such that

$$\sum_{j \in J} \int_{e_j} (-\nu_j \partial_j m + b(x) m) \partial_j \varphi \, dx = 0 \qquad \text{for any } \varphi \in H^1(\Gamma)$$

## Proposition

There exists a unique strong solution $m$ to the FP equation. Moreover

$$\|m\|_{H^1} \le C(\|b\|_{L^\infty}).$$

**Sketch of the Proof:**

Existence and uniqueness of a weak solution follows by the classical theory of bilinear forms (Nicaise-von Below '96).

Regularity of the solution is based on the intrinsic 1-dimensionality of the problem and the Sobolev's embedding theorem. □

# An existence result for MFG system

## Theorem

Assume

- $H^j \in C^2([0,l_j]\times\mathbb{R})$, $H^j(x,\cdot)$ convex in $p$ for $x\in[0,l_j]$ and $H^j(x,p)| \le C(1+|p|^2)$ for $x\in[0,l_j]$;
- $\nu=\{\nu_j\}_{j\in J},\ \lambda=\{\lambda_j\}_{j\in J}$ with $\nu_j,\lambda_j>0\ \forall j\in J$;
- $V[m(t,\cdot)](x)=F(m(t,x))$ where $F\in C^1(\mathbb{R}_+)$

Then there exists a classical solution $(u,m)$ to the MFG system

$$
\begin{cases}
-\nu\partial^2 u+H(x,\partial u)+\lambda u=V[m](x) & x\in\Gamma\\
\nu\partial^2 m+\operatorname{div}(\partial_p H(x,\partial u)m)=0 & x\in\Gamma\\
\sum_{j\in Inc_i} a_{ij}\partial_j u(v_i)=0 & i\in I_T\\
\sum_{j\in Inc_i} a_{ij}\left(\nu_j\partial_j m(v_i)+\partial_p H(v_i,\partial u)m_j(v_i)\right)=0 & i\in I_T\\
\partial_j u(v_i)=0,\quad \nu_j\partial_j m(v_i)+a_{ij}m\partial_p H_j(v_i,\partial u)=0 & i\in I_B\\
\int_\Gamma m(x)dx=1,\ m\ge 0
\end{cases}
$$

**Proof:**
Set $\mathcal{K} = \{\mu \in C^{0,\alpha}(\Gamma) : \int_\Gamma \mu dx = 1\}$ and define $\mathcal{T} : \mathcal{K} \to \mathcal{K}$ according to the scheme

$$\mu \to u \to m$$

as follows.

- Given $\mu \in \mathcal{K}$, solve (HJB) with $f(x) = V[\mu](x)$ for the unknown $u = u_\mu$ which is uniquely defined .
- Given $u = u_\mu$ by the previous step solve (FP) with $b(x) = \partial_p H(x, \partial u)$ for the unknown $m$ which is uniquely defined.

Set $m = T(\mu)$. By the estimate

$$\|u\|_{C^{2,\alpha}} \leq C(\|V[m]\|_{C^{0,\alpha}})$$
$$\|m\|_{H^1} \leq C\|\partial_p H(x, \partial u)\|_{L^\infty}$$

uniformly in $\mu$. It follows that the map $\mathcal{T}$ is continuous with compact image. Hence by Schauder's fixed point Theorem there exists a fixed point of $\mathcal{T}$, i.e. a solution of the (MFG) system. □

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## A stability result

With the same assumptions of the previous theorem, assume besides that

- $H^j(x,p) \geq 0 \qquad$ for any $(x,p) \in e_j \times \mathbb{R}$, for any $j \in J$.
- $\int_\Gamma (V[m_1] - V[m_2])(m_1 - m_2)dx \geq 0 \qquad$ for any $m_1$, $m_2 \in H^1(\Gamma)$.

For $\lambda > 0$, let $(u_\lambda, m_\lambda)$ be the solution of the (MFG) system. Then $\lambda u_\lambda$ converges to $\bar{\lambda} \in \mathbb{R}$ uniformly in $\Gamma$ and $(u_\lambda, m_\lambda)$ converges to the couple $(u, m)$ which is the unique solution of

$$
\begin{cases}
-\nu\Delta u + H(x, Du) + \bar{\lambda} = V[m](x) & x \in \Gamma \\
\nu\Delta m + \operatorname{div}\left(\frac{\partial H}{\partial p}(x, Du)m\right) = 0 & x \in \Gamma \\
\partial_j u(v_i) = 0, & i \in I_B \\
\nu_j \partial_j m(v_i) + a_{ij} m \partial_p H_j(v_i, \partial u) = 0 & i \in I_B \\
\sum_{j \in Inc_i} \partial_j u(v_i) = 0, & i \in I_T \\
\sum_{j \in Inc_i} \nu_j \partial_j m(v_i) + a_{ij} \partial_p H(v_i, \partial u) m_j(v_i) = 0 & i \in I_T \\
\int_\Gamma u(x)dx = 0,\ \int_\Gamma m(x)dx = 1,\ m \geq 0
\end{cases}
$$

### Sketch of the Proof.

The compactness of the sequence $(u_\lambda, m_\lambda, \lambda u_\lambda)$ follows by some a priori estimates and an argument similar to one used in homogenization of HJB equations.

Uniqueness can be proved exactly as in the classical case of MFG theory (see Lasry-Lions '06).

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# The evolutive problem

$$
\begin{cases}
u_t - \nu\partial^2 u + H(x, Du) = V[m], & (x,t) \in \Gamma \times (0,T) \\
m_t + \nu\partial^2 m + \partial(m\,\partial_p H(x,\partial u)) = 0, & (x,t) \in \Gamma \times (0,T) \\
m(x,0) = m_0(x), \quad u(x,T) = V_0[m(T)], & x \in \Gamma \\
\sum_{j \in Inc_i} a_{ij}\nu_j\partial_j u(v_i,t) = 0 & \forall i \in I_T,\ t \in (0,T), \\
\sum_{j \in Inc_i} a_{ij}[\partial_p H_j(v_i,\partial_j u)m_j(v_i,t) + \nu_j\partial_j m(v_i,t)] = 0 & \forall i \in I_T,\ t \in (0,T) \\
\partial_j u(v_i,t) = 0, & \forall i \in I_B,\ t \in (0,T) \\
\partial_p H_j(v_i,\partial_j u)m_j(v_i,t) + \nu_j\partial_j m(v_i,t) = 0 & \forall i \in I_B,\ t \in (0,T)
\end{cases}
$$

**Remark:** For the moment no theoretical result for the evolutive problem

# A numerical test

The data on the network are obtained by the restriction of functions defined in $\mathbb{R}^2$.

- Hamiltonian: $H(x,y,p) = |p|^2 - 5((x+1)^2 + y^2)$
- Coupling terms: $V[m](x) = m(x)$, $V_0[m] = 0$
- Initial density: $m_0(x,y) = 4(e^{-100(x-\frac{1}{2})^2} + e^{-100(y-\frac{1}{2})^2})$
- Final horizon: $T = 1$

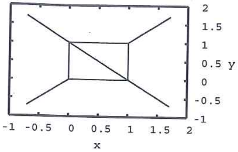


Figure 1: The network

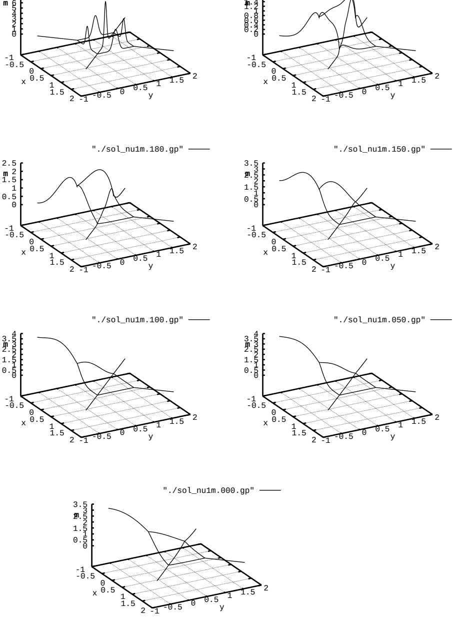


Figure 2: $\nu = 1$: Density at the times 1, 0.95, 0.90, 0.75,0.50, 0

2

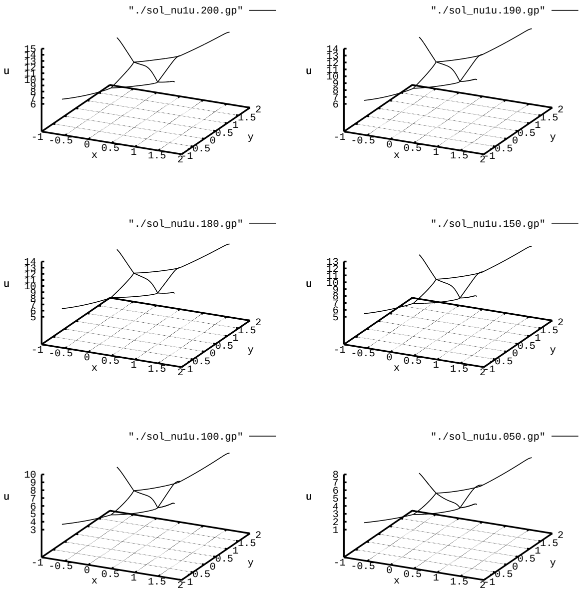


Figure 3: $\nu = 1$: Value function at the times 1, 0.95, 0.90, 0.75,0.50

**Comments and Remarks:**
The previous results for MFG system on network are very preliminary. There are several problems to study

- Interpretation of the problem in terms of Nash equilibria
- Existence result for the evolutive MFG system
- The planning problem on networks (an evolutive MFG system with initial and terminal conditions for $m$)
- $1^{st}$ MFG systems
- More general transitions conditions
- Approximation schemes for MFG systems on networks

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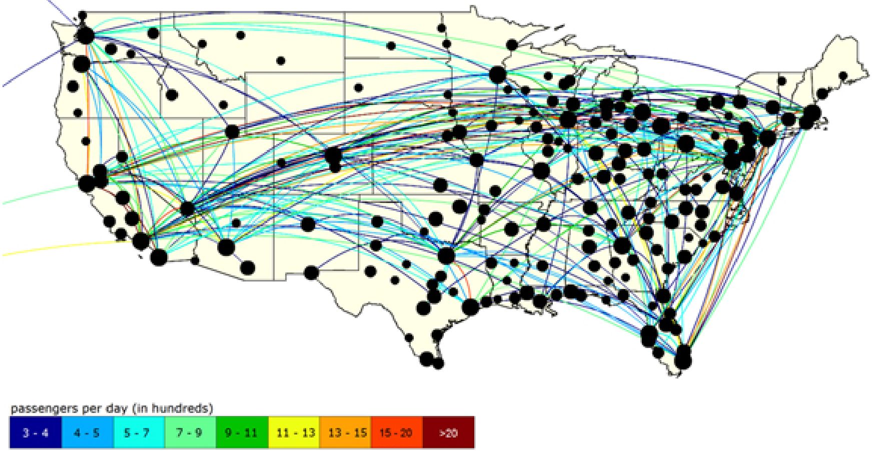
passengers per day (in hundreds)
3 - 4
4 - 5
5 - 7
7 - 9
9 - 11
11 - 13
13 - 15
15 - 20
>20

# Thank You!