On first order mean field game systems

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The 1rst order MFG system with local coupling

In this talk we concentrate on the system :

$$(MFG) \quad \left\{ \begin{array}{ll} (i) & -\partial_t u + H(x, Du) = f(x, m(t,x)) \\ & \qquad\qquad\qquad\qquad\qquad \text{in } [0,T] \times \mathbb{T}^d \\ (ii) & \partial_t m - \operatorname{div}(m\, D_p H(x, Du)) = 0 \\ & \qquad\qquad\qquad\qquad\qquad \text{in } [0,T] \times \mathbb{T}^d \\ (iii) & m(0,x) = m_0(x),\ u(T,x) = u_T(x) \qquad \text{in } \mathbb{T}^d \end{array} \right.$$

where

- $H = H(x,p)$ is convex in p, periodic in x,
- $f = f(x,m)$ is a local coupling, increasing in m, periodic in x
- $u_T = u_T(x)$ is a periodic terminal cost,
- m_0 is a probability density on $\mathbb{T}^d$.

Because of the lack of regularity, the usual fixed-point method does not work.

Two approaches :

- Reduction to a quasi-linear equation $\rightsquigarrow$ smooth solutions
- Variational methods $\rightsquigarrow$ weak solutions.

Reduction to a quasi-linear equation

Assume for simplicity that $H = H(p)$, $f = f(m)$.
The MFG system becomes

$$(MFG) \qquad \left\{ \begin{array}{ll} (i) & -\partial_t u + H(Du) = f(m(t,x)) \\ (ii) & \partial_t m - \mathrm{div}(m D_p H(Du)) = 0 \\ (iii) & m(0) = m_0, \; u(T,x) = u_T(x) \end{array} \right.$$

Writing

$$m(t,x) = f^{-1}\left(-\partial_t u + H(Du)\right),$$

the MFG systems reduces to the quasilinear elliptic equation

$$\left\{ \begin{array}{l} -Tr\left(A(D_{t,x}u)D^2_{t,x}u\right) = 0 \text{ in } [0,T] \times \mathbb{T}^d \\ \\ -\partial_t u + H(Du) = f(m_0) \text{ at } t = 0 \\ \\ u(T,\cdot) = u_T \text{ at } t = T \end{array} \right.$$

A priori estimates (Lasry-Lions)

1. The map $\Phi(t,x) := -\partial_t u + H(Du)$ is bounded above.

   Indeed :

   - Φ satisfies an equation of the form

   $$-Tr(A(D^2_{t,x}\Phi) - B.D\Phi \leq 0 \text{ in } (0,T)\times\mathbb{T}^d$$

     So max Φ reached at the boundary,
   - $\Phi(0,\cdot) = f(m_0)$ is bounded,
   - $\Phi(T,x)$ is bounded (barrier argument).

2. $|Du|$ is bounded (Bernstein method)

Consequences : (Lasry-Lions)

1. A priori estimates for the solution : $u \in W^{1,\infty}$ and $m \in L^\infty$,
2. If $f(m) \sim \log(m)$ at 0, the system is uniformly elliptic and the solution is smooth.

Variational approach

Following Lasry-Lions, the MFG system can be formally viewed as a system of optimality conditions for

- an optimal control problem of a continuity equation.
- an optimal control problem of a Hamilton-Jacobi equation

Reminiscent of

- Benamou-Brenier formulation of the Wasserstein distance,
- A new class of transport problems introduced by Dolbeault-Nazaret-Savaré (2009) : optimality conditions studied in C.-Carlier-Nazaret (2012).

THE OPTIMAL CONTROL OF CONTINUITY EQUATION

$$\inf_{(m,w)} \left\{ \int_0^T \int_Q mH^*(x,-v) + F(x,m)\; dxdt + \int_Q u_T(x)m(T,x)dx \right\}$$

where the infimum is taken over the pairs (m, v) such that

$$\partial_t m + \mathrm{div}(mv) = 0,\;\; m(0) = m_0$$

in the sense of distributions.

We have set :

$$F(x,m) = \begin{cases} \displaystyle\int_0^m f(x,m')dm' & \text{if } m \geq 0 \\ +\infty & \text{otherwise} \end{cases}$$

and $H^*(x,v) = \sup_{p\in\mathbb{R}^d} p.v - H(x,p)$.

The optimal control of HJ equation

$$\inf_{\alpha}\left\{\int_0^T\int_Q F^*(x,\alpha(t,x))\;dxdt-\int_Q u(0,x)m_0(x)dx\right\}$$

where u is the solution to the HJ equation

$$\left\{\begin{array}{ll} -\partial_t u+H(x,Du)=\alpha & \text{in } (0,T)\times\mathbb{T}^d\\ u(T,\cdot)=u_T \quad \text{in } \mathbb{T}^d \end{array}\right.$$

We have set $F^*(x,a)=\sup_{m\in\mathbb{R}}(am-F(x,m))$.

Heuristic argument for the link with (MFG)

Assume (u, α) is optimal in the problem

$$\inf \left\{ \int_0^T \int_Q F^*(x, \alpha(t,x)) \, dxdt - \int_Q u(0,x) m_0(x) dx \right\}$$

where u is the solution to the HJ equation

$$\left\{ \begin{array}{l} -\partial_t u + H(x, Du) = \alpha \qquad \text{in } (0,T) \times \mathbb{T}^d \\ u(T, \cdot) = u_T \qquad \text{in } \mathbb{T}^d \end{array} \right.$$

Necessary condition : take the derivative in the direction β,

$$\int_0^T \int_Q D_\alpha F^*(x, \alpha) \beta \, dxdt - \int_Q w(0,x) m_0(x) dx = 0$$

where w is the solution to the linearized HJ equation

$$\left\{ \begin{array}{l} -\partial_t w + D_p H(x, Du).Dw = \beta \qquad \text{in } (0,T) \times \mathbb{T}^d \\ w(T, \cdot) = 0 \qquad \text{in } \mathbb{T}^d \end{array} \right.$$

Set $m(t,x) = D_\alpha F^*(x, \alpha(t,x))$, i.e., $\alpha(t,x) = f(x, m(t,x))$. Then

$$\int_0^T \int_Q m\left(-\partial_t w + D_pH(x,Du).Dw\right)\, dxdt - \int_Q w(0,x)m_0(x)dx = 0.$$

Integrate by parts :

$$\int_0^T \int_Q \left(\partial_t m - \mathrm{div}\left(mD_pH(x,Du)\right)\right)\, w\, dxdt + \int_Q w(0,x)(m(0,x) - m_0(x))dx = 0.$$

This holds for any w with $w(T,x) = 0$: hence m solves

$$\left\{\begin{array}{l} \partial_t m - \mathrm{div}\left(mD_pH(x,Du)\right) = 0 \\ m(0,x) = m_0(x) \end{array}\right.$$

By definition, u is the solution to the HJ equation

$$\left\{\begin{array}{l} -\partial_t u + H(x,Du) = \alpha = f(x,m) \\ u(T,\cdot) = u_T \qquad \text{in } \mathbb{T}^d \end{array}\right.$$

So (u,m) solves (MFG).

Aim :

- Provide a framework in which both problems are well-posed and in duality,
- derive from these problems the existence of a weak solution for the MFG system, as optimality conditions,
- discuss properties of the weak solution.

Outline

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Assumptions

- $f : \mathbb{T}^d \times [0, +\infty) \to \mathbb{R}$ is smooth and increasing w.r.t. to m with $f(x, 0) = 0$, and

$$-\bar{C} + \frac{1}{\bar{C}}|m|^{q-1} \leq f(x, m) \leq \bar{C}(1 + |m|^{q-1}) \qquad \text{(where } q > 1) .$$

- There is $r > d(q-1) \vee 1$ such that

$$\frac{1}{\bar{C}}|\xi|^r - \bar{C} \leq H(x, \xi) \leq \bar{C}(|\xi|^r + 1) \qquad \forall (x, \xi) \in \mathbb{T}^d \times \mathbb{R}^d .$$

- + technical conditions of $D_x H$...

Analysis of the optimal control of HJB

We study the optimal control of the HJ equation :

$$(\mathbf{HJ}-\mathbf{Pb}) \qquad \inf_{\alpha} \left\{ \int_0^T \int_Q F^*(x,\alpha(t,x))\; dxdt - \int_Q u(0,x)dm_0(x) \right\}$$

where u is the solution to the HJ equation

$$\begin{cases} -\partial_t u + H(x,Du) = \alpha & \text{in } (0,T)\times\mathbb{T}^d \\ u(T,\cdot) = u_T & \text{in } \mathbb{T}^d \end{cases}$$

Recall the notation : $F^*(x,a) = \sup_{m\in\mathbb{R}}(am - F(x,m))$ where

$$F(x,m) = \begin{cases} \displaystyle\int_0^m f(x,m')dm' & \text{if } m \geq 0 \\ +\infty & \text{otherwise} \end{cases}$$

Note that $F^*(x,a) = 0$ for $a \leq 0$.

Let (u_n, α_n) be a minimizing sequence for

$$(\mathbf{HJ} - \mathbf{Pb}) \qquad \inf\left\{\int_0^T \int_Q F^*(x, \alpha(t,x))\, dxdt - \int_Q u(0,x) dm_0(x)\right\}$$

where u is the solution to the HJ equation

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Proposition

- the (α_n) are bounded in L^p (with $p = q'$), with $\alpha_n \geq 0$.
- the (u_n) are uniformly continuous in $[0,T] \times \mathbb{T}^d$.
- the Du_n are bounded in L^r and the $(\partial_t u_n)$ are bounded in L^1.

Consequence : There exists a minimizer (u, α) of the optimal control of HJB such that :

- $\alpha \in L^p$,
- u is continuous in $[0, T] \times \mathbb{T}^d$,
- $u \in BV((0, T) \times \mathbb{T}^d)$ and $Du \in L^r((0, T) \times \mathbb{T}^d)$,
- u solves in the sense of distribution

$$\left\{ \begin{array}{ll} -\partial_t u + H(x, Du) \leq \alpha & \text{in } (0, T) \times \mathbb{T}^d \\ u(T, \cdot) = u_T & \text{in } \mathbb{T}^d \end{array} \right.$$

One key ingredient of proof of the proposition :

Theorem (Hölder estimates, C.-Silvestre, 2012)

Let u be a bounded viscosity solution of

$$\begin{cases} -\partial_t u + H(x, Du) = \alpha & \text{in } (0,T)\times\mathbb{T}^d \\ u(T,x) = u_T(x) & \text{in } \mathbb{T}^d \end{cases}$$

where $\alpha \geq 0$, $\alpha \in L^p$ with $p > 1 + d/r$, $r > 1$.
Then, for any $\delta > 0$, u is Hölder continuous in $[0, T-\delta]\times\mathbb{T}^d$:

$$|u(t,x) - u(s,y)| \leq C|(t,x) - (s,y)|^\gamma$$

where $\gamma = \gamma(\|u\|_\infty, \|\alpha\|_p, d, r)$, $C = C(\|u\|_\infty, \|\alpha\|_p, d, r, \delta)$.

Related results (2nd order results)

- Capuzzo Dolcetta-Leoni-Porretta (2010), Barles (2010) : stationary equations, bounded RHS,
- C. (2009), Cannarsa-C. (2010), C., Rainer (2011) : evolution equations, bounded RHS,
- Dall'Aglio-Porretta (preprint) : stationary setting, unbounded RHS.

The dual of the optimal control of HJ eqs

Proposition

The dual of the optimal control of HJ (**HJ-Pb**) equation is given by optimal control problem for the continuity equation :

$$(\mathbf{K}-\mathbf{Pb}) \qquad \inf\left\{\int_0^T\int_Q mH^*(x,-\frac{w}{m}) + F(x,m)\ dxdt + \int_Q u_T(x)m(T,x)dx\right\}$$

where the infimum is taken over the pairs $(m,w)\in L^1((0,T)\times\mathbb{T}^d)\times L^1((0,T)\times\mathbb{T}^d,\mathbb{T}^d)$ such that

$$\partial_t m + \mathrm{div}(w) = 0,\ \ m(0) = m_0$$

in the sense of distributions.

Moreover the dual problem has a unique minimum (m,w) and $m\in L^q$.

Definition of weak solution

We say that a pair $(m, u) \in L^q((0,T)\times\mathbb{T}^d)\times BV((0,T)\times\mathbb{T}^d)$ is a weak solution of (MFG) if

(i) u is continuous in $[0,T]\times\mathbb{T}^d$, $Du \in L^r((0,T)\times\mathbb{T}^d)$, $mD_pH(x,Du)\in L^1$,

(ii) Equality $-\partial_t u^{ac}(t,x) + H(x, Du(t,x)) = f(x, m(t,x))$ holds a.e. in $\{m>0\}$ and inequality $-\partial_t u + H(x,Du) \leq f(x,m)$ holds in the sense of distribution, with $u(T,x) = u_T(x)$ in the sense of trace,

(iii) $\partial_t m - \mathrm{div}(mD_pH(x,Du)) = 0$ holds in the sense of distribution in $(0,T)\times\mathbb{T}^d$ and $m(0) = m_0$,

(iv) Equality $\displaystyle\int_0^T\int_Q m\left(\partial_t u^{ac} - \langle Du, D_pH(x,Du)\rangle\right) = \int_Q m(T)u_T - m_0 u(0)$ holds.

(where $\partial_t u^{ac}$ is the a.c. part of the measure $\partial_t u$).

Existence/uniqueness of weak solutions

Theorem

There exists a weak solution (m, u) of (MFG) such that u is locally Hölder continuous in $[0, T) \times \mathbb{T}^d$ and which satisfies in the viscosity sense

$$-\partial_t u + H(x, Du) \geq 0 \qquad \text{in } (0, T) \times \mathbb{T}^d \; .$$

Idea of proof :

- Let (m, w) is a minimizer of (**K-Pb**) and (u, α) is a minimizer of (**HJ-pb**) such that u is continuous. Then one can show that (m, u) is a solution of mean field game system (MFG) and $w = -mD_pH(\cdot, Du)$ while $\alpha = f(\cdot, m)$ a.e..
- Conversely, if (u, m) is a solution of (MFG), then the pair $(m, -mD_pH(\cdot, Du))$ is the minimizer of (**K-Pb**) while $(u, f(\cdot, m))$ is a minimizer of (**HJ-pb**).

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Uniqueness for the *MFG* system

Theorem

Let (m, u) and (m', u') be two weak solutions of (MFG). Then $m = m'$ and $u = u'$ in $\{m > 0\}$.

Moreover, if u satisfies the additional condition

$$(*) \qquad -\partial_t u + H(x, Du) \geq 0 \qquad \text{in } (0,T) \times \mathbb{T}^d ,$$

in the viscosity sense, then $u \geq u'$.

Remark : In particular, if we add condition (*) to the definition of weak solution of (*MFG*), then the weak solution exists and is unique.

Outline

Hamiltonian structure

Formally the (MFG) system can be rewritten as the Hamiltonian system

$$\left\{\begin{array}{l} \partial_t u = \dfrac{\partial \mathcal{E}}{\partial m}(u(t,\cdot), m(t,\cdot)) \\ \partial_t m = -\dfrac{\partial \mathcal{E}}{\partial u}(u(t,\cdot), m(t,\cdot)) \\ m(0,x) = m_0(x),\ u(0,x) = u_T(x) \end{array}\right.$$

where

$$\mathcal{E}(u,m) = \int_{\mathbb{T}^d} m(x)H(x,Du(x)) - F(x,m(x))\,dx$$

Proposition

Let (u,m) be a weak solution of the (MFG) system. Then there exists $C \in \mathbb{R}$ with

$$\mathcal{E}(u(t,\cdot), m(t,\cdot)) = C \qquad \text{for a.e. } t \in (0,T).$$

Link with the quasilinear elliptic equation

Proposition

If (u, m) is a weak solution of the MFG system, then u is a viscosity solution of

$$\left\{\begin{array}{l} \mathcal{G}\left(x, \partial_t u, Du, \partial_{tt} u, D\partial_t u, D^2 u\right) = 0 \text{ in } (0,T)\times\mathbb{T}^d \\ u(T,\cdot) = u_T \text{ in } \mathbb{T}^d \\ -\partial_t u + H(x, Du) = f(m_0) \text{ in } \mathbb{T}^d \end{array}\right.$$

where

$$\mathcal{G}(x, p_t, p_x, a, b, C) = -Tr\left(\mathcal{A}(x,p_t,p_x)\left(\begin{array}{cc} a & b^T \\ b & C \end{array}\right)\right) - F^*_{\alpha,\alpha}\langle H_p, H_x\rangle - \langle F^*_{x,\alpha}, H_p\rangle - F^*_\alpha Tr(H_{x,p})$$

with

$$\mathcal{A}(x,p_t,p_x) = F^*_{\alpha,\alpha}\left(\begin{array}{cc} 1 & -H_p^T \\ -H_p & H_p\otimes H_p \end{array}\right) + F^*_\alpha\left(\begin{array}{cc} 0 & 0 \\ 0 & H_{pp} \end{array}\right) \geq 0$$

Other properties of the solutions

- Stability of the solution with respect to the data.
- Application to differential games with finitely many players.

  $\longrightarrow$ Well-known for 2nd order MFG system with nonlocal coupling : Huang, Caines, Malhamé (2006), Kolokoltsov, Li, Yang (2011), Carmona, Delarue (2012).

  $\longrightarrow$ Specific difficulties : only weak solutions to the MFG system, local coupling.

Outline

We study the link between the solution (u^T, m^T) of the finite horizon problem

$$(MFG_T) \qquad \begin{cases} (i) & -\partial_t u^T + H(x, Du^T) = f(x, m^T(x,t)) \\ (ii) & \partial_t m^T - \mathrm{div}(m^T D_p H(x, Du^T)) = 0 \\ (iii) & m^T(0) = m_0, \; u^T(x,T) = u_f(x) \end{cases}$$

and the solution $(\overline{\lambda}, \overline{u}, \overline{m})$ of the ergodic problem

$$(MFG - ergo) \qquad \begin{cases} (i) & \overline{\lambda} + H(x, D\overline{u}) = f(x, \overline{m}(x)) \\ (ii) & -\mathrm{div}(\overline{m} D_p H(x, D\overline{u})) = 0 \\ (iii) & \overline{m} \geq 0, \; \int_{\mathbb{T}^d} \overline{m} = 1 \end{cases}$$

References :

- Gomes, Mohr, Souza (2010) : discrete setting,
- C., Lasry, Lions, Porretta (2010, 2013) : 2nd order MFG systems,
- C. (2013) : 1rst order MFG system with nonlocal coupling.

The ergodic problem

A triple $(\lambda, m, u) \in \mathbb{R} \times L^q(\mathbb{T}^d) \times W^{1,pr}(\mathbb{T}^d)$ is a solution of (MFG-ergo) if

(i) $m \geq 0$, $\int_{\mathbb{T}^d} m = 1$ and $mD_pH(x, Du) \in L^1(\mathbb{T}^d)$,

(ii) Equation (MFG-ergo)-(i) holds in the following sense :
$\lambda + H(x, Du(x)) = f(x, m(x)) \quad \text{a.e. in } \{m > 0\}$
and $\lambda + H(x, Du) \leq f(x, m) \quad \text{a.e. in } \mathbb{T}^d$,

(iii) Equation (MFG-ergo)-(ii) holds :
$-\mathrm{div}(mD_pH(x, Du(x))) = 0 \quad \text{in } \mathbb{T}^d$,
in the sense of distribution.

Theorem

There exists at least one solution $(\overline{\lambda}, \overline{m}, \overline{u})$ to the ergodic MFG system (MFG-ergo). Moreover, the pair $(\overline{\lambda}, \overline{m})$ is unique.

Idea of proof.

As for the time-dependent problem, the existence relies on two optimization problems :
- Optimization of an ergodic cost

$$\inf_{(\lambda,u)} \int_{\mathbb{T}^d} F^*\left(x, \lambda + H(x, Du(x))\right)\, dx - \lambda.$$

- Optimization of a cost on invariant measures.

$$\inf_{(m,w)} \int_{\mathbb{T}^d} m(x) H^*\left(x, -\frac{w(x)}{m(x)}\right) + F(x, m(x))\, dx,$$

where $m \in L^1$ is a measure and $\quad \mathrm{div}(w) = 0$ in $\mathbb{T}^d$.

$\longrightarrow$ Claim : the two problems are in duality and have optimal solutions $(\overline{\lambda}, \overline{u})$ and $(\overline{m}, \overline{w})$. Moreover $(\overline{\lambda}, \overline{u}, \overline{m})$ is a solution to $(MFG - ergo)$.

The convergence result

Let (u^T, m^T) and $(\overline{\lambda}, \overline{u}, \overline{m})$ be the solution to (MFG_T) and $(MFG - ergo)$ respectively.

Set
$$v^T(s,x) = u^T(Ts,x) \text{ and } \mu^T(s,x) = m^T(Ts,x)$$
for $(s,x) \in (0,1) \times \mathbb{T}^d$.

Theorem

As $T \to +\infty$,

- *(v^T/T) converges to $-\overline{\lambda}$ in L^θ for any $\theta > 0$,*
- *(μ^T) converges to $\overline{m}$ in L^θ for $\theta \in [1,p)$.*

Ingredients of proof

1) Lasry-Lions usual estimate is still valid :

Proposition (Lasry-Lions key estimate)

$$\int_0^T \int_{\mathbb{T}^d} m^T \left(H(x, D\overline{u}) - H(x, Du^T) - \langle D_pH(x, Du^T), D(\overline{u} - u^T)\rangle\right) \, dxdt$$

$$+ \int_0^T \int_{\mathbb{T}^d} \overline{m} \left(H(x, Du^T) - H(x, D\overline{u}) - \langle D_pH(x, D\overline{u}), D(u^T - \overline{u})\rangle\right) \, dxdt$$

$$+ \int_0^T \int_{\mathbb{T}^d} (f(x, m^T) - f(x, \overline{m}))(m^T - \overline{m}) \, dxdt$$

$$\leq - \left[\int_{\mathbb{T}^d} (m^T(t) - \overline{m})(u^T(t) - \overline{u}) \, dx \right]_0^T$$

Problem : show that the RHS is a $o(T)$.

2) The optimal control of HJB equations has a limit :

Lemma

$$\lim_{T\to+\infty} \inf_u \frac{1}{T}\int_0^T \int_{\mathbb{T}^d} F^*(x, -\partial_t u + H(x, Du))\, dxdt - \frac{1}{T}\int_{\mathbb{T}^d} u(x,0)m_0(x)\, dx$$
$$= \inf_{(\lambda,u)} \int_{\mathbb{T}^d} F^*\left(x, \lambda + H(x, Du(x))\right)\, dx - \lambda.$$

Moreover

$$\lim_{T\to+\infty} \frac{1}{T} u^T(0,\cdot) = -\bar\lambda \qquad \text{in} \qquad L^\theta(\mathbb{T}^d)$$

for any $\theta \geq 1$ (where $\bar\lambda$ is the ergodic constant).

By the Lasry-Lions key estimate,

$$\int_0^1 \int_{\mathbb{T}^d} (f(x,\mu^T) - f(x,\overline{m}))(\mu^T - \overline{m})\, dxdt \\ \leq -\frac{1}{T}\left[\int_{\mathbb{T}^d} (\mu^T(t) - \overline{m})(v^T(t) - \overline{u})\, dx\right]_0^1$$

where, at $t = 1$,

$$\lim_{T\to+\infty} \int_{\mathbb{T}^d} (\mu^T(1) - \overline{m})\frac{(u_f - \overline{u})}{T}\, dx = 0$$

while, at $t = 0$,

$$\lim_{T\to+\infty} \int_{\mathbb{T}^d} (\mu_0 - \overline{m})(\frac{v^T(0) - \overline{u}}{T})\, dx = \int_{\mathbb{T}^d} (\mu_0 - \overline{m})(-\overline{\lambda})\, dx = 0$$

So

$$\lim_{T\to+\infty} \int_0^1 \int_{\mathbb{T}^d} (f(x,\mu^T) - f(x,\overline{m}))(\mu^T - \overline{m})\, dxdt = 0,$$

which proves the convergence of μ^T to $\overline{m}$.

Conclusion

Summary

- Existence/uniqueness of weak solutions in to 1rst order MFG system with local coupling,
- Link with optimal control problems and with a quasilinear elliptic system,
- Application to games with finitely many players.
- Long time-average.

Open problems

- Regularity of solutions for 1rst order, local MFG systems in full generality,
- Vanishing viscosity limit,
- Existence/uniqueness for the MFG system of congestion type ($\alpha \in (0,2)$)

$$\left\{ \begin{array}{ll} (i) & -\partial_t u + \dfrac{|Du|^2}{2m^\alpha} = 0 \\ (ii) & \partial_t m - \mathrm{div}(m^{1-\alpha} Du)) = 0 \\ (iii) & m(0) = m_0,\ u(T,x) = u_T(x) \end{array} \right.$$