Multi-population Ergodic Mean-Field Games with Neumann boundary conditions and a model of segregation

Mean Field Games and Related Topics - 2
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Outline

1. A Multi-population Ergodic MFG system with Neumann B.C.
2. Some existence results.
3. An Ergodic MFG from a Segregation model by T. C. Schelling.
4. Numerical Experiments.

Multipop. ergodic MFG system

We study the behavior of an "average" player, belonging to the i-th population ($i = 1, \dots, M$), which chooses α_t^i to drive the state variable X_t^i:

$$dX_t^i = \alpha_t^i dt + \sqrt{2\nu} dB_t^i + \boxed{L_t^i} \qquad \leftarrow \text{reflection at } \partial\Omega$$

aiming at minimizing a long-time average cost

$$J^i(X_0^i, \alpha_t^i, m_1, \dots, m_M) = \liminf_{T \to +\infty} \frac{1}{T} \mathbb{E}\left[\int_0^T L(X_t^i, \alpha_t^i) + V^i[m_1, \dots, m_M](X_t^k) dt\right]$$

where $m_i \in \mathcal{P}$ is the invariant measure of the optimal process $\bar{X}_t^i$ (optimal distribution of i-th population).
This leads to the system of $2M$ equations , $i = 1, \dots, M$

$$\begin{cases} -\nu_i \Delta u_i + H^i(x, Du_i) + \lambda_i = V^i[m], & \Omega \\ -\nu_i \Delta m_i - \operatorname{div}(D_p H^i(x, Du_i) m_i) = 0, & \\ \partial_n u_i = 0,\ \partial_n m_i + m_i D_p H^i(x, Du_i) \cdot n = 0 & \partial\Omega \end{cases} \tag{MFG}$$

Existence - Non-local case.

Theorem

Let Ω be a bounded C^2 domain of $\mathbb{R}^d$. If H^i are admissible* and V^i are regularizing:

1. $\|V^i[m]\|_{W^{1,\infty}(\Omega)} \leq C$ for all $m \in [W^{1,q}(\Omega)]^M \cap \mathcal{P}^M$, $q > d$,
2. $m^{(n)} \to m$ in $C(\overline{\Omega}) \Rightarrow V^i[m^{(n)}] \to V^i[m]$ in $C(\overline{\Omega})$.

Then, there exists a solution

$$(u, \lambda, m) \in [C^2(\overline{\Omega})]^M \times \mathbb{R}^M \times [W^{1,q}(\Omega)]^M$$

of (**??**).

* Have in mind: $H(x,p) = R(x)|p|^\gamma - H_0(x)$, $R(x) \geq r > 0, \gamma > 1$.

Idea of the proof.

Exploit the fixed point structure of (**??**). Given m solve

$$\begin{cases} -\nu_i \Delta u_i + H^i(x, Du_i) + \lambda_i = V^i[m], & \partial_n u_i = 0 \\ -\nu_i \Delta \mu_i - \mathrm{div}(D_p H^i(x, Du_i)\mu_i) = 0, & \partial_n \mu_i + \mu_i D_p H^i(x, Du_i) \cdot n = 0. \end{cases}$$

Prove that

$$m \mapsto \mu$$

is continuous from $[C(\overline{\Omega})]^M$ to $[C(\overline{\Omega})]^M$ and maps a ball of $[W^{1,q}]^M$ into itself (standard elliptic estimates) $\Rightarrow$

By Schauder there exists a fixed point.

Existence - Local (bounded) case.

Theorem

Let Ω be a bounded C^2 convex domain. If

$$H^i(x,p) = R^i|p|^{\gamma^i} - H_0^i(x),$$

$\gamma^i > 1$ and $V^i[m](x) = V^i(m(x))$, $V^i \in C(\mathbb{R}^M)$ are bounded:

$$|V^i(y)| \leq L \quad \forall y \in \mathbb{R}^M, i = 1, \dots, M$$

Then, there exists a solution

$$(u, \lambda, m) \in [C^{1,\delta}(\overline{\Omega})]^M \times \mathbb{R}^M \times [W^{1,q}(\Omega)]^M$$

of (**??**).

Idea of the proof.

Solve with the non-local existence theorem the approximating problems (as suggested by Lasry and Lions)

$$\begin{cases} -\nu_i \Delta u_i^\epsilon + H^i(x, Du_i^\epsilon) + \lambda_i^\epsilon = V^i(m^\epsilon) \star \varphi_\epsilon, \\ -\nu_i \Delta m^{\epsilon_i} - \mathrm{div}(D_p H^i(x, Du_i^\epsilon) m_i^\epsilon) = 0. \end{cases} \qquad (\text{MFG}\epsilon)$$

and pass to the limit as $\epsilon \to 0$ to obtain a solution (u, λ, m) for the original problem. Uniform estimates on solutions of (**??**) needed:

$\rightarrow$ again classic (see Liebermann, Ladyzhenskaya...).

A-Priori Estimates for the local Unbounded case. ($M = 2$)

Let $V^i(m_1, m_2) = W_{i1}(m_1) + W_{i2}(m_2)$, with entries $W_{ij} \in C^1((0, +\infty))$.

Theorem

Let Ω be convex. Suppose that (u, λ, m) is a (regular) solution of (**??**), H^i are *nice* and $\forall m_1, m_2 > 0$, $v, w \in \mathbb{R}^d$

A) $W_{i1}(m_1) + W_{i2}(m_2) \leq C(1 + m_1^\eta + m_2^\eta)$, $i = 1, 2$

B) $\epsilon(m_1^\gamma |v|^2 + m_2^\gamma |w|^2) \leq W'_{11}(m_1)|v|^2 + (W'_{12}(m_2) + W'_{21}(m_1)) v \cdot w + W'_{22}(m_2)|w|^2$,

for some $\gamma > -2$ and $\epsilon, \eta, \overline{W} > 0$ such that

$$\eta < \begin{cases} (\gamma + 2)/(d - 2) & \text{if } d \geq 3 \\ +\infty & \text{else} \end{cases}$$

Then,

$$\|Du\|_{L^q(\Omega)} \leq C, \quad \|m\|_{L^\infty(\Omega)} \leq C, \quad |\lambda| \leq C$$

Some examples.

Example 1 - Two populations (competition)

$$V^1(m) = W_{11}(m_1) + W_{12}(m_2) = Am_1 + Bm_2$$
$$V^2(m) = W_{21}(m_1) + W_{22}(m_2) = Cm_1 + Am_2,$$

Estimates for $d \leq 3$

$$A > 0, \quad B, C \geq 0, \quad B + C < 2A.$$

Example 2 - One population

$$V(m) = Am^\beta,$$

Estimates for $A > 0$ and

$$0 < \beta < \begin{cases} 1/(d-3) & \text{if } d \geq 4 \\ +\infty & \text{else} \end{cases}$$

Idea of the proof.

1. $\int_\Omega m^\gamma |Dm|^2 \leq C$ using the adjoint structure of (**??**). (see Gomes, Pires, Sanchez-Morgado) multi-pop. setting → condition B)

2. $\int_\Omega V^q(m) \leq C \Leftarrow \int_\Omega m^{\gamma'} \leq C, q > d$ by Sobolev imbeddings.

3. $\|Du\|_{L^r(\Omega)} \leq C$ integral Bernstein method. (see Lions)

4. 2. + 3. $\Rightarrow \|m\|_{L^\infty(\Omega)} \leq C$ estimates for Kolmogorov eq.

5. $|\lambda| \leq C$ using the maximum principle.

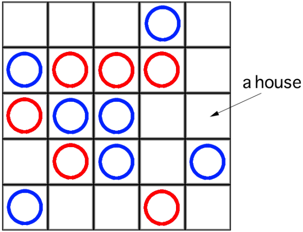
Existence: through continuity method (work in progress).

The model of Schelling

In the late '60 Schelling grew interested in segregated neighborhoods:
"It was easy in America to find neighborhoods that were mostly or entirely black or white, and correspondingly difficult to find neighborhoods where neither race made up more than, say, three fourths of the total."

BUT: Was that a result of widespread racism?

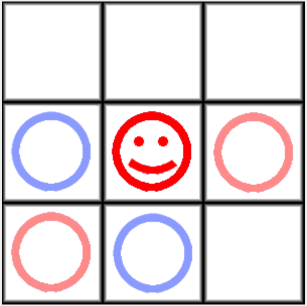
He had an intuition, and created the following model.
Suppose that blue people and red people live in a chessboard (the neighborhood).



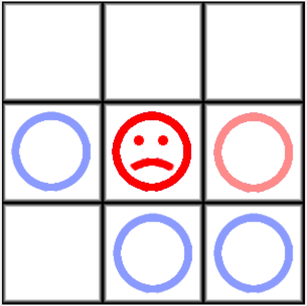
They start interacting. How?

Each individual wants to make sure that he lives near some of its own. He's happy if the percentage of same-color individuals among his neighbors is above some threshold a. If he's not happy he moves to another free house.

Example: $a = 30\%$



% of reds $= 50\% > a$

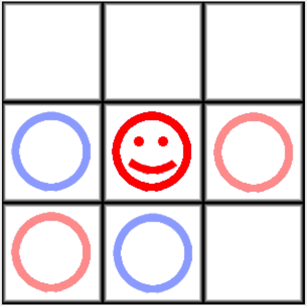


% of reds $= 25\% < a$

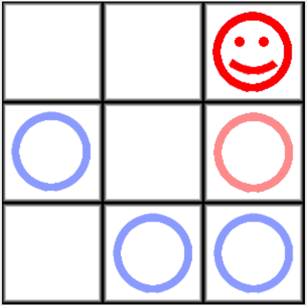
They start interacting. How?

Each individual wants to make sure that he lives near some of its own. He's happy if the percentage of same-color individuals among his neighbors is above some threshold a. If he's not happy he moves to another free house.

Example: $a = 30\%$

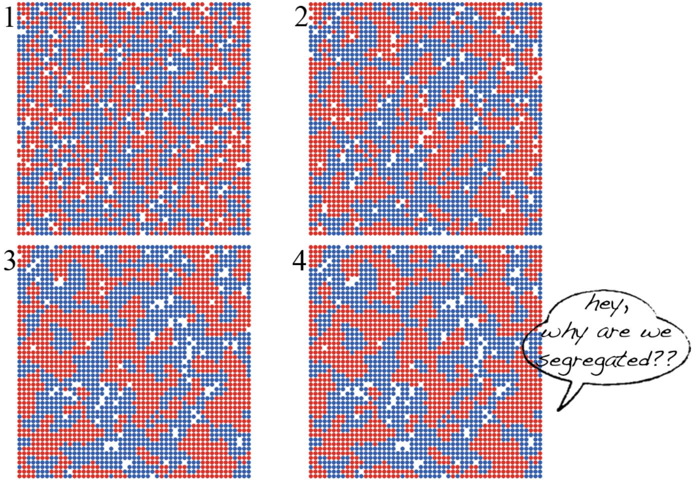


% of reds $= 50\% > a$



% of reds $= 100\% > a$

If we run a simulation starting from a random initial distribution of individuals (so, some of them are unhappy), after some time we always notice that ethnic clusters form.



In our test, the "happiness" threshold is $a = 30\%$, so the individual preference is "not to stay alone".

Everyone is initially surrounded by an average 50% of same-color neighbors, BUT, that average stabilizes at $\sim 75\%$.

So, segregation may come out from a mild ethnocentric attitude.

Some references:

- T. Schelling, "Dynamic model of segregation", 1971
- T. Schelling, "Micromotives and Macrobehavior, 1978

An $N + N$ players game.

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We try to design a simple game that fits the model of Schelling. Suppose that

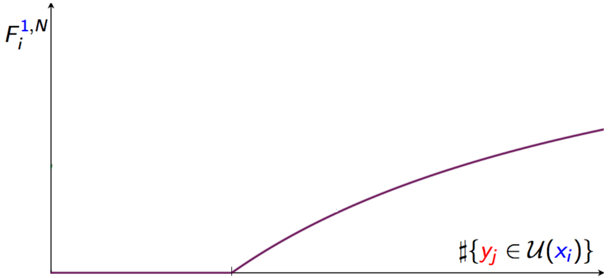
- $(x_1, \dots, x_N) \in (\overline{\Omega})^N$ are the players of the first population.
- $(y_1, \dots, y_N) \in (\overline{\Omega})^N$ are the players of the second population,

Ω a bounded domain of $\mathbb{R}^d$. The cost paid by every player x_k is

$$F_k^{1,N}(x_1, \dots, x_N, y_1, \dots, y_N) = \left(\frac{\sharp\{x_j \in \mathcal{U}(x_k) : j \neq k\}}{\sharp\{x_j \in \mathcal{U}(x_k) : j \neq i\} + \sharp\{y_j \in \mathcal{U}(x_k)\}} - a_1 \right)^{-},$$

with $a_1 \in [0, 1]$ and $\mathcal{U}(x)$ a neighborhood of x.

Cost paid by x_k:



It is zero if the number of same-color neighbors is sufficiently high, with respect to the individuals of the other population.

The players y_k pay costs of similar form but possibly with different thresholds a_2.

We notice that every player x_i and y_i behaves with the same criterion, i.e.

$$F_k^{1,N}(x_1, \ldots, x_N, y_1, \ldots, y_N) = V^{1,N}\left[\frac{1}{N-1}\sum_{k\neq j}\delta_{x_j}, \frac{1}{N}\sum \delta_{y_j}\right](x_k),$$

$$F_k^{2,N}(x_1, \ldots, x_N, y_1, \ldots, y_N) = V^{2,N}\left[\frac{1}{N}\sum \delta_{x_j}, \frac{1}{N-1}\sum_{k\neq j}\delta_{y_j}\right](y_k),$$

the functionals $V^{1,N}$, $V^{2,N}$ do not depend on $k = 1, \ldots, N$:

$$V^{1,N}[m_1, m_2](x) = \left(\frac{\int_{\mathcal{U}(x)} m_1}{\int_{\mathcal{U}(x)} m_1 + \frac{N}{N-1}\int_{\mathcal{U}(x)} m_2} - a_1\right)^-$$

Ergodic Schelling MFG

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Going back to our ergodic MFG setting with reflection, we consider the system, $i = 1, 2$,

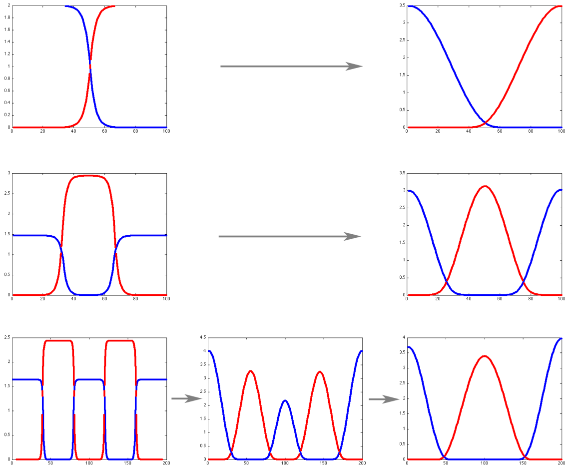
$$\begin{cases} -\nu_i \Delta u_i + |Du_i|^2 + \lambda_i = \left(\frac{m_i}{m_1 + m_2 + \eta_i} - a_i \right)^-, & \Omega \\ -\nu_i \Delta m_i - 2\mathrm{div}(Du_i m_i) = 0, & \\ \partial_n u_i = 0, \, \partial_n m_i = 0 \quad \int m_i = 1 & \partial\Omega \end{cases}$$

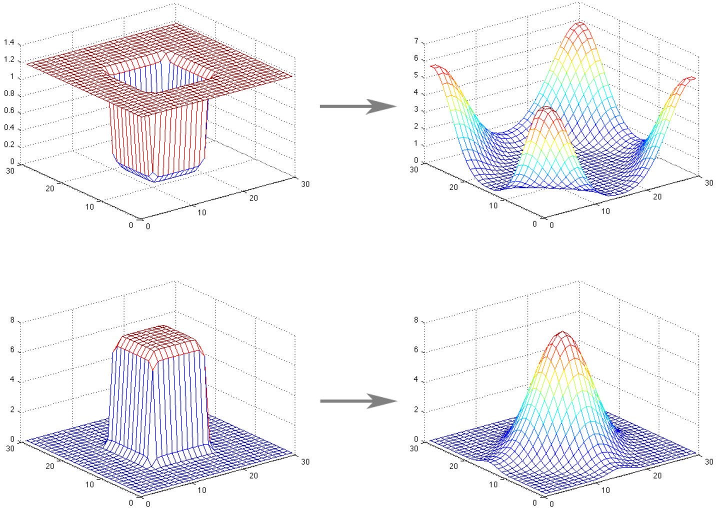
Continuous model, takes into account the effort of moving and randomness.

Solutions exist but are not unique.

Does segregation show up?

Numerical experiments in collaboration w Y. Achdou.
Long time approx., we fix initial $(m_i)_0$ at $T = 0$ and let $T \to \infty$ in the respective Forward-Forward non-stationary MFG system.





$$\nu = 0.03,\ T = 5,\ \alpha_{1,2} = 0.4$$

Thanks for your attention !