Mean-Field (Sparse) Optimal Control

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Mean-Field Games and Related Topics - 2
Università degli Studi di Padova
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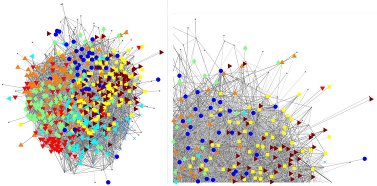
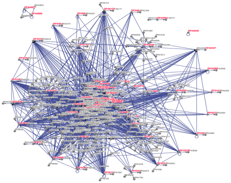
Introduction

Large particle systems arise in many modern applications:

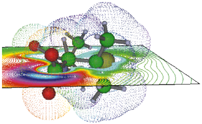
Image halftoning via variational dithering.



Large Facebook "friendship" network



Dynamical data analysis: R. palustris protein-protein interaction network.



Computational chemistry: molecule simulation.

Social dynamics

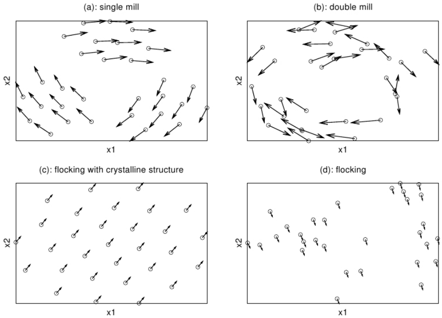
We consider large particle systems of form:

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = (H \star \mu_N)(x_i, v_i), i = 1, \dots N, \\ \text{where } , \mu_N = \frac{1}{N} \sum_{j=1}^{N} \delta_{(x_i, v_i)}, \end{cases}$$

Several "social forces" encoded in the interaction kernel H:

- Repulsion-attraction
- Alignment
- ...

Possible noise/uncertainty by adding stochastic terms.



Patterns related to different balance of social forces.

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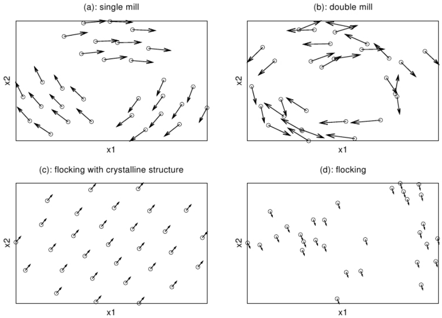
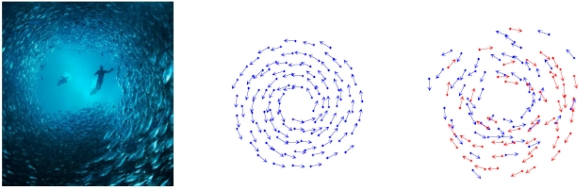


Patterns related to different balance of social forces.

Understanding how superposition of re-iterated binary "social forces" yields global self-organization.

An example inspired by nature



Mills in nature and in our simulations.

J. A. Carrillo, M. Fornasier, G. Toscani, and F. Vecil, *Particle, kinetic, hydrodynamic models of swarming*, within the book "Mathematical modeling of collective behavior in socio-economic and life-sciences", Birkhäuser (Eds. Lorenzo Pareschi, Giovanni Naldi, and Giuseppe Toscani), 2010.

Split coherence in homophilious societies: government?

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Question: can a government endowed with limited resources rescue/stabilize a society by minimal interventions? Which ones?

Consensus/flocking emergence

The Cucker-Smale model:

$$\begin{cases} \dot{x}_i = v_i \in \mathbb{R}^d \\ \dot{v}_i = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} a(\|x_j - x_i\|)(v_j - v_i) \in \mathbb{R}^d \end{cases},$$

where $a(t) := a_\beta(t) = \frac{1}{(1+t^2)^\beta}$, $\beta > 0$ governs the rate of communication.

[1] The *Laplacian* L of A is given by $L = D - A$, with $D = \mathrm{diag}(d_1, \dots, d_N)$ and $d_k = \sum_{j=1}^{N} a_{kj}$

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$$\begin{cases} \dot{x} = v \\ \dot{v} = -L_x v \end{cases}$$

where L_x is the Laplacian of the matrix[1] $(a(\|x_j - x_i\|)/N)_{i,j=1}^N$ and depends on x.

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- Mean-velocity conservation: $\frac{d}{dt}\bar{v}(t) = \frac{1}{N}\sum_{i=1}^N \dot{v}_i(t) = \frac{1}{N^2}\sum_{i=1}^N \sum_{j=1}^N \frac{v_j - v_i}{(1+\|x_j - x_i\|^2)^\beta} \equiv 0.$

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Without loss of generality $\bar{v} = 0$ and $\bar{x}(t) = \bar{x}(0) = \frac{1}{N}\sum_{i=1}^{N} x_i(0)$.

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Theorem (Cucker-Smale, Ha-Tadmor, Carrillo-F.-Rosado-Toscani)

Let $(x(t), v(t)) \in C^1([0, +\infty), \mathbb{R}^{2d\times N})$ be the solution of the Cucker-Smale system. We denote

$$\begin{cases} \mathcal{V}(t) = \max_{i=1,\dots N} \|v_i(t)\|, \quad \mathcal{V}_0 = \mathcal{V}(0), \\ \mathcal{X}(t) = \max_{i=1,\dots N} \|x_i(t) - x_i(0)\|, \quad \mathcal{X}_0 = \mathcal{X}(0). \end{cases}$$

If $0 < \beta < \frac{1}{2}$ then

$$\mathcal{V}(t) \leq \mathcal{V}_0 e^{-a(2\bar{\mathcal{X}})t} \to 0, t \to \infty, \quad \exists \bar{\mathcal{X}} > 0.$$

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Actually one has $\mathcal{V}(t) \to 0$ also for $\beta = 1/2$.

Conditional consensus emergence for a generic communication rate $a(\cdot)$

Consider the symmetric bilinear form

$$B(u,v) = \frac{1}{2N^2}\sum_{i,j}\langle u_i - u_j, v_i - v_j\rangle = \frac{1}{N}\sum_{i=1}^{N}\langle u_i, v_i\rangle - \langle \bar{u}, \bar{v}\rangle,$$

and

$$X(t) = B(x(t), x(t)), \quad V(t) = B(v(t), v(t)).$$

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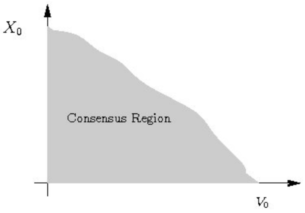

Theorem (Ha-Ha-Kim)

Let $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ be such that $X_0 = B(x_0, x_0)$ and $V_0 = B(v_0, v_0)$ satisfy

$$\sqrt{N}\int_{\sqrt{NX_0}}^{\infty} a(\sqrt{2}r)dr > \sqrt{V_0}\,.$$

Then the solution with initial data (x_0, v_0) tends to consensus.

Non-consensus events

If $\beta > 1/2$ then for $a(\cdot) = a_\beta(\cdot)$ the consensus condition is not satisfied by all $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$.

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Consider $\beta = 1$ and $x(t) = x_1(t) - x_2(t)$, $v(t) = v_1(t) - v_2(t)$ relative pos. and vel. of two agents on the line:

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\dfrac{v}{1+x^2} \end{cases}$$

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By direct integration

$$v(t) = -\arctan x(t) + \arctan x_0 + v_0.$$

Hence, if $\arctan x_0 + v_0 > \pi/2 + \varepsilon$ we have

$$v(t) > \pi/2 + \varepsilon - \arctan x(t) > \varepsilon, \quad \forall t \in \mathbb{R}_+.$$

Self-organization Vs organization by intervention

We introduce the notion of *organization via intervention*.

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Admissible controls: measurable functions $u = (u_1, \ldots, u_N) : [0, +\infty) \to \mathbb{R}^N$ such that $\sum_{i=1}^N \|u_i(t)\| \leq M$ for every $t > 0$, for a given constant M:

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = \dfrac{1}{N} \displaystyle\sum_{j=1}^N a(\|x_j - x_i\|)(v_j - v_i) + u_i \end{cases}$$

for $i = 1, \ldots, N$, and $x_i \in \mathbb{R}^d$, $v_i \in \mathbb{R}^d$.

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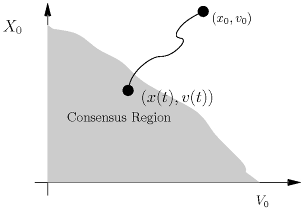

Our aim is then to find admissible controls steering the system to the consensus region.

Total control

Proposition (Caponigro-F.-Piccoli-Trélat)

For every initial condition $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ *and* $M > 0$ *there exist* $T > 0$ *and* $u : [0, T] \to (\mathbb{R}^d)^N$, *with* $\sum_{i=1}^N \|u_i(t)\| \leq M$ *for every* $t \in [0, T]$ *such that the associated solution tends to consensus.*

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Proof.

Consider a solution of system with initial data (x_0, v_0) associated with a feedback control $u = -\alpha(v - \bar{v})$, with $0 < \alpha \leq M/(N\sqrt{B(v_0, v_0)})$.

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\frac{d}{dt}V(t) &= \frac{d}{dt}B(v(t), v(t)) \\
&= -2B(L_x v(t), v(t)) + 2B(u(t), v(t)) \\
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Therefore $V(t) \leq e^{-2\alpha t} V(0)$ and $V(t)$ tends to 0 exponentially fast as $t \to \infty$. Moreover $\sum_{i=1}^N \|u_i\| \leq M$. □

More economical choices?

We wish to make

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the smallest possible and use the minimal amount of intervention: minimize $B(u(t), v(t))$ with additional sparsity constraints.

Greedy sparse control

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For every initial condition $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ *and* $M > 0$ *there exist* $T > 0$ *and a sparse control* $u : [0, T] \to (\mathbb{R}^d)^N$, *with* $\sum_{i=1}^N \|u_i(t)\| \leq M$ *for every* $t \in [0, T]$ *such that the associated AC solution tends to consensus.*

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$$\min B(v, u) + \frac{\gamma(x)}{N} \sum_{i=1}^N \|u_i\| \quad \textit{subject to} \sum_{i=1}^N \|u_i\| \leq M\,,$$

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$$\gamma(x) = \sqrt{N} \int_{\sqrt{NB(x,x)}}^{\infty} a(\sqrt{2}r)dr \quad \textit{threshold by Ha-Ha-Kim}$$

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The control $u(t)$ is a sparse vector with at most one nonzero coordinate, i.e., $u_i(t) \neq 0$ for a unique $i \in \{1, \ldots, N\}$ and

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Otherwise there exists a "best index" $i \in \{1, \ldots, N\}$ such that

$$\|v_{\perp_i}\| > \gamma(x) \quad \text{and} \quad \|v_{\perp_i}\| \geq \|v_{\perp_j}\| \quad \text{for every } j = 1, \ldots, N.$$

Therefore we can choose $i \in \{1, \ldots, N\}$ satisfying it, and a control law

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Hence the control acts on the most "stubborn". We may call this control the "shepherd dog strategy".

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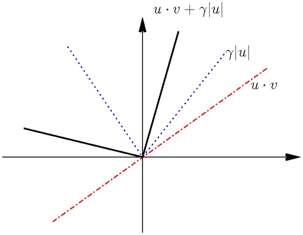
$$\|v_{\perp_i}\| > \gamma(x) \quad \text{and} \quad \|v_{\perp_i}\| \geq \|v_{\perp_j}\| \quad \text{for every } j = 1, \ldots, N.$$

Therefore we can choose $i \in \{1, \ldots, N\}$ satisfying it, and a control law

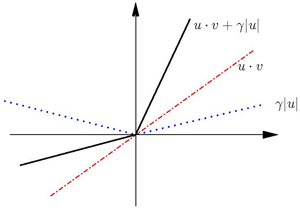
$$u_i = -M \frac{v_{\perp_i}}{\|v_{\perp_i}\|}, \quad \text{and} \quad u_j = 0, \quad \text{for every } j \neq i.$$

Hence the control acts on the most "stubborn". We may call this control the "shepherd dog strategy". This choice of the control makes $V(t) = B(v(t), v(t))$ vanishing in finite time, hence there exists T such that $B(v(t), v(t)) \leq \gamma(x)^2$, $t \geq T$.

Geometrical interpretation in the scalar case



For $|v| \leq \gamma$ the minimal solution $u \in [-M, M]$ is zero.



For $|v| > \gamma$ the minimal solution $u \in [-M, M]$ is $|u| = M$.

Instantaneous optimality of the greedy strategy

Consider generic control u (solution of the variation problem) of components

$$u_i(x, v) = \begin{cases} 0 & \text{if } v_{\perp_i} = 0 \\ -\alpha_i \dfrac{v_{\perp_i}}{\|v_{\perp_i}\|} & \text{if } v_{\perp_i} \neq 0 \end{cases}$$

where $\alpha_i \geq 0$ such that $\sum_{i=1}^{N} \alpha_i \leq M$.

Theorem (Caponigro-F.-Piccoli-Trélat)

The 1-sparse control is the minimizer of

$$\mathcal{R}(t, u) := \mathcal{R}(t) = \frac{d}{dt} V(t),$$

among all the control of the previous form.

⋆ *A policy maker, who is not allowed to have prediction on future developments, should always consider more favorable to intervene with stronger actions on the fewest possible instantaneous optimal leaders than trying to control more agents with minor strength.*

⋆ *Homophilious society can be stabilized by parsiminious interventions!*

Sparse optimal control

The problem is to minimize, for a given $\gamma > 0$

$$\mathcal{J}(u) = \int_0^T \frac{1}{N} \sum_{i=1}^N \left(\left(v_i(t) - \frac{1}{N} \sum_{j=1}^N v_j(t) \right)^2 \quad + \quad \frac{\gamma}{N} \sum_{i=1}^N \sum_{i=1}^N \|u_i(t)\| \right) dt,$$

$$\text{s.t. } \sum \|u_i\| \leq M$$

where the state is a trajectory of the control system

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = \dfrac{1}{N} \displaystyle\sum_{j=1}^N a(\|x_j - x_i\|)(v_j - v_i) + u_i \end{cases}$$

with initial constraint

$$(x(0), v(0)) = (x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N.$$

Beyond a greedy approach: sparse optimal control

Theorem (Caponigro-F.-Piccoli-Trélat)

For every (x_0, v_0) in $(\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$, for every $M > 0$, and for every $\gamma > 0$ the optimal control problem has an optimal solution. The optimal control $u(t)$ is "usually" instantaneously a vector with at most one nonzero coordinate.

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The PMP ensures the existence of $\lambda \geq 0$ and of a nontrivial covector $(p_x, p_v) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ satisfying the adjoint equations, for $i = 1, \dots, N$,

$$\begin{cases} \dot{p}_{x_i} = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} \dfrac{a(\|x_j - x_i\|)}{\|x_j - x_i\|} \langle x_j - x_i, v_j - v_i \rangle (p_{v_j} - p_{v_i}) \\ \dot{p}_{v_i} = -p_{x_i} - \dfrac{1}{N} \displaystyle\sum_{j \neq i} a(\|x_j - x_i\|)(p_{v_j} - p_{v_i}) - 2\lambda v_i + \dfrac{2\lambda}{N} \displaystyle\sum_{j=1}^{N} v_j. \end{cases}$$

The application of the PMP leads to minimize

$$\min \sum_{i=1}^{N} \langle p_{v_i}, u_i \rangle + \lambda\gamma \sum_{i=1}^{N} \|u_i\|, \qquad \text{subject to } \sum_{i=1}^{N} \|u_i\| \leq M.$$

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controlled by the minimizer of the cost functional

$$\mathcal{J}(u) := \int_0^T \int_{\mathbb{R}^{2d}} \left(L(x, v, \mu_N) d\mu_N(t, x, v) + \frac{1}{N} \sum_{i=1}^N |u_i| \right) dt,$$

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Which topology on $\mu_N = \frac{1}{N} \sum_{j=1}^N \delta_{(x_i, v_i)}$? Which topology on $\nu_N = \frac{1}{N} \sum_{j=1}^N u_i \delta_{(x_i, v_i)}$?

Too weak convergence

The compactness of the problem is way too weak

$$\nu_N = \frac{1}{N}\sum_{j=1}^{N} u_i \delta_{(x_i,v_i)} \rightharpoonup \nu, \quad \mu_N = \frac{1}{N}\sum_{j=1}^{N} \delta_{(x_i,v_i)} \rightharpoonup \mu,$$

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but $f \in L^1_\mu(\mathbb{R}^{2d}, \mathbb{R}^d)$ only and no stable solutions can be expected!

A natural relaxation: smoother controls

Definition

For a horizon time $T > 0$, and an exponent $1 \leq q < +\infty$ we fix a control bound function $\ell \in L^q(0,T)$. The class of admissible control functions $\mathcal{F}_\ell([0,T])$ is so defined: $f \in \mathcal{F}_\ell([0,T])$ if and only if

(i) $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}^d$ is a Carathéodory function,

(ii) $f(t,\cdot) \in W^{1,\infty}_{loc}(\mathbb{R}^n, \mathbb{R}^d)$ for almost every $t \in [0,T]$, and

(iii) $|f(t,0)| + \mathrm{Lip}(f(t,\cdot), \mathbb{R}^d) \leq \ell(t)$ for almost every $t \in [0,T]$.

Compactness

Theorem (F. and Solombrino)

Let $1 < p < \infty$. Assume that $(f_j)_{j\in\mathbb{N}}$ be a sequence of functions in $\mathcal{F}_\ell$ for a given function $\ell \in L^q(0,T)$, $1 \le q < +\infty$. Then there exist a subsequence $(f_{j_k})_{k\in\mathbb{N}}$ and a function $f \in \mathcal{F}_\ell$, such that

$$\lim_{k\to\infty} \int_0^T \langle \phi(t), f_{j_k}(t,\cdot) - f(t,\cdot)\rangle dt = 0, \tag{1}$$

for all $\phi \in L^{q'}([0,T], H^{-1,p'}(\mathbb{R}^n,\mathbb{R}^d))$ such that $\mathrm{supp}(\psi(t)) \Subset \Omega$ for all $t \in [0,T]$, where Ω is a relatively compact set in $\mathbb{R}^n$. Here the symbol $\langle\cdot,\cdot\rangle$ denotes the duality between $W^{1,p}$ and its dual $H^{-1,p'}$.

Product convergence

Corollary (F. and Solombrino)

For a given $\ell \in L^1(0, T)$*, let* $(f_k)_{k\in\mathbb{N}}$ *be a sequence of functions in* $\mathcal{F}_\ell$ *converging to* f *in the sense of* (1)*. Let* $\mu_k : [0, T] \to \mathcal{P}_1(\mathbb{R}^n)$ *be a sequence of functions taking values in the probability measures with finite first moment, and* $\mu : [0, T] \to \mathcal{P}_1(\mathbb{R}^n)$ *such that*

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^n} |x| d\mu_k(t, x) = M < \infty,$$

and

$$\lim_k \mathcal{W}_1(\mu_k(t), \mu(t)) = 0, \text{ for all } t \in [0, T].$$

Then

$$\lim_k \int_0^{\hat{t}} \langle \varphi, f_k(t, \cdot)\mu_k(t)\rangle dt = \int_0^{\hat{t}} \langle \varphi, f(t, \cdot)\mu(t)\rangle dt,$$

for all $\varphi \in C_c^1(\mathbb{R}^n, \mathbb{R}^d)$ *and for all* $\hat{t} \in [0, T]$*.*

Lower semi-continuity

Theorem (F. and Solombrino)

Consider a nonnegative convex function $\psi\colon \mathbb{R}^d \to [0,+\infty)$ *satisfying*

$$\mathrm{Lip}(\psi, B(0,R)) \leq CR^{q-1}.$$

Fix $\ell \in L^q(0,T)$ *and a sequence of functions* $(f_k)_{k\in\mathbb{N}}$ *in* $\mathcal{F}_\ell$ *converging to* f *in the sense of* (1)*. Let* $\mu_k : [0,T] \to \mathcal{P}_1(\mathbb{R}^n)$ *be a sequence such that*

$$\mathrm{supp}(\mu_k(t)) \Subset \Omega,$$

for a.e. $t \in [0,T]$ *and* $k \in \mathbb{N}$*, where* $\Omega \Subset \mathbb{R}^n$*. Let* $\mu : [0,T] \to \mathcal{P}_1(\mathbb{R}^n)$*, and assume that*

$$\lim_k \mathcal{W}_1(\mu_k(t), \mu(t)) = 0, \ \textit{for a.e. } t \in [0,T].$$

Then, we have

$$\liminf_{k\to+\infty} \int_0^T \langle \psi(f_k(t,\cdot)), \mu_k(t)\rangle \, dt \geq \int_0^T \langle \psi(f(t,\cdot)), \mu(t)\rangle \, dt.$$

Assumptions for the optimal control problem

We consider the phase space $\mathbb{R}^n$ where $n = 2d$ with state variables $z = (x, v)$, x, $v \in \mathbb{R}^d$. Assumptions:

(H) Let $H\colon \mathbb{R}^{2d} \to \mathbb{R}^d$ be a locally Lipschitz function such that

$$|H(z)| \leq C(1 + |z|), \quad \text{for all } z \in \mathbb{R}^{2d};$$

(L) Let $L : \mathbb{R}^{2d} \times \mathcal{P}_1(\mathbb{R}^{2d}) \to \mathbb{R}_+$ be a continuous function in the state variables (x, v) and such that if $(\mu_j)_{j\in\mathbb{N}} \subset \mathcal{P}_1(\mathbb{R}^{2d})$ is a sequence converging narrowly to μ in $\mathcal{P}_1(\mathbb{R}^{2d})$, then $L(x, v, \mu_j) \to L(x, v, \mu)$ uniformly with respect to (x, v) on compact sets of $\mathbb{R}^{2d}$;

(Ψ) Let $\psi : \mathbb{R}^d \to [0, +\infty)$ be a nonnegative convex function satisfying the following assumption: there exist $C \geq 0$ and $1 \leq q < +\infty$ such that

$$\mathrm{Lip}(\psi, B(0, R)) \leq CR^{q-1}$$

for all $R > 0$.

Finite dimensional optimal control problem

Theorem (F. and Solombrino)

Given $N \in \mathbb{N}$ and an initial datum $(x_1(0), \ldots, x_N(0), v_1(0), \ldots, v_N(0)) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$, The following (finite dimensional) optimal control problem:

$$\min_{f \in \mathcal{F}_\ell} \int_0^T \int_{\mathbb{R}^{2d}} \left[L(x, v, \mu_N(t, x, v)) + \psi(f(t, x, v)\right] d\mu_N(t, x, v) dt,$$

where

$$\mu_N(t, x, v) = \frac{1}{N} \sum_{j=1}^{N} \delta_{(x_i(t), v_i(t))}(x, v),$$

constrained by being the solution of the system

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = (H \star \mu_N)(x_i, v_i) + f(t, x_i, v_i), \quad i = 1, \ldots N, \quad t \in [0, T], \end{cases}$$

has solutions.

Mean-field solutions

Definition

Fix a function $f \in \mathcal{F}_\ell$. Given $H\colon \mathbb{R}^{2d} \to \mathbb{R}^d$ satisfying (H), we say that a map $\mu\colon [0,T] \to \mathcal{P}(\mathbb{R}^{2d})$ continuous with respect to $\mathcal{W}_1$ is a weak equi-compactly supported solution of the equation

$$\frac{\partial \mu}{\partial t} + v \cdot \nabla_x \mu = \nabla_v \cdot \left[(H \star \mu + f)\, \mu \right],$$

if there exists $R > 0$ such that $\operatorname{supp} \mu(t) \subset B(0,R)$, $t \in [0,T]$, and, defining

$$w_{H,\mu,f}(t,x,v) := (v, H \star \mu(t)(x,v) + f(t,x,v))$$

one has

$$\frac{d}{dt} \int_{\mathbb{R}^{2d}} \zeta(x,v)\, d\mu(t)(x,v) = \int_{\mathbb{R}^{2d}} \nabla \zeta(x,v) \cdot w_{H,\mu,f}(t,x,v)\, d\mu(t)(x,v)$$

for every $\zeta \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$.

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for every $\zeta \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$.

By particle approximations and characteristics method, given a compactly supported $\mu_0 \in \mathcal{P}(\mathbb{R}^{2d})$ and $f \in \mathcal{F}_\ell$, there exists always a unique solution of the equation.

Mean-field optimal control

Theorem (F. and Solombrino)

Assume that we are given maps H, L, and ψ as in assumptions (H), (L), and (Ψ). For $N \in \mathbb{N}$ and an initial datum $((x_N^0)_1, \dots, (x_N^0)_N, (v_N^0)_1, \dots, (v_N^0)_N) \in B(0, R_0) \subset (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$, for $R_0 > 0$ independent of N, we consider

$$\min_{f \in \mathcal{F}_\ell} \int_0^T \int_{\mathbb{R}^{2d}} \left[L(x, v, \mu_N(t, x, v)) + \psi(f(t, x, v))\right] d\mu_N(t, x, v) dt,$$

where $\mu_N(t, x, v) = \frac{1}{N} \sum_{j=1}^N \delta_{(x_j(t), v_j(t))}(x, v)$, constrained by being the solution of

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = (H \star \mu_N)(x_i, v_i) + f(t, x_i, v_i), \quad i = 1, \dots N, \quad t \in [0, T], \end{cases}$$

with initial datum $(x(0), v(0)) = (x_N^0, v_N^0)$ and, for consistency, we set

$$\mu_N^0 = \frac{1}{N} \sum_{i=1}^M \delta_{((x_N^0)_i, (v_N^0)_i)}(x, v).$$

For all $N \in \mathbb{N}$ let us denote the function $f_N \in \mathcal{F}_\ell$ as a solution of the finite dimensional optimal control problem.

Mean-field optimal control

If there exists a compactly supported $\mu_0 \in \mathcal{P}_1(\mathbb{R}^{2d})$ such that $\lim_{N\to\infty} \mathcal{W}_1(\mu_N^0, \mu^0) = 0$, then there exists a subsequence $(f_{N_k})_{k\in\mathbb{N}}$ and a function $f_\infty \in \mathcal{F}_\ell$ such that f_{N_k} converges to f_∞ in the sense of (1) and f_∞ is a solution of the infinite dimensional optimal control problem

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The proof is based on the simultaneous development of the mean-field limit for the equation and the Γ-limit for the optimization of the control.

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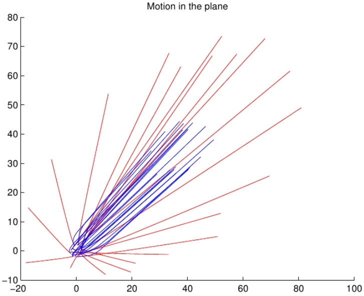
- Numerical methods to compute efficiently f_∞.

Conclusion



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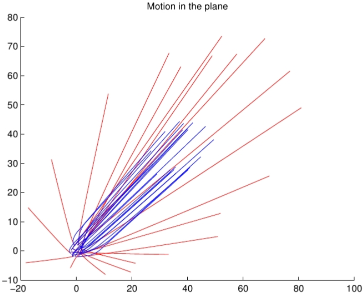

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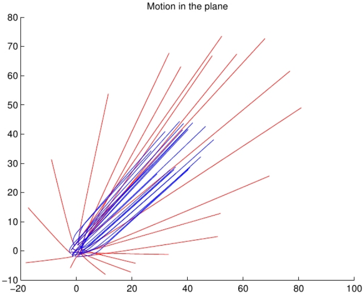

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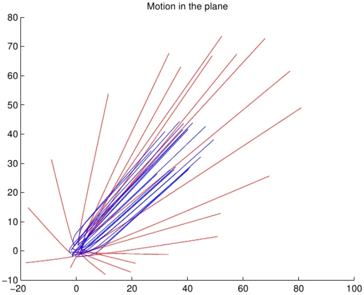

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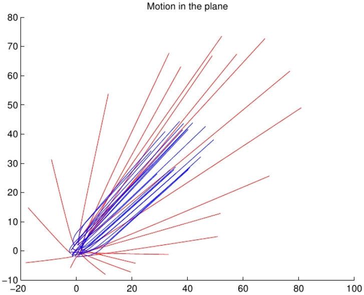

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- We presented a relaxation of the (sparse) finite dimensional optimal control problems and a general technique to derive their mean-field limits.

A few info

- **WWW:** http://www-m15.ma.tum.de/
- **References**:

  - M. Bongini and M. Fornasier, *Sparse stabilization of dynamical systems driven by attraction and avoidance forces*, preprint, 2013, pp. 30
  - M. Bongini, M. Fornasier, F. Fröhlich, and L. Haghverdi, *Sparse stabilization of dynamical systems driven by attraction and avoidance forces*, Proc. Conf. Mathematical Control in Trieste, 2013
  - M. Caponigro, M. Fornasier, B. Piccoli, and E. Trélat, *Sparse stabilization and optimal control of the Cucker-Smale model*, submitted to Nonlinearity, 2012, pp. 33
  - M. Fornasier and F. Solombrino, *Mean field optimal control*, submitted to ESAIM: Control, Optimization, and Calculus of Variations, June 2013, pp. 31