On the 1/N convergence rates for mean-field approximations [1]

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Highlights

A steady increase in complexity of technological development $\Longrightarrow$ appropriate (or better optimal) management of complex stochastic systems consisting of large number of interacting components (agents, mechanisms, vehicles, subsidiaries, species, police units, etc), which may have competitive or common interests.

Traditional Markov decision analysis is often unfeasible.

Sometimes: the limiting problem (LLN limit): better manageable deterministic evolution: measure-valued (a nonlinear Markov process).

Controlled version: nonlinear Markov control process or a nonlinear Markov game.

In case of finite initial state space, space of measures is $= \mathbf{R}^d_+$), so the limiting is a differential game in $\mathbf{R}^d$.

Aim: identification of deterministic limit and proof of convergence with explicit rates.

Plan

Part 1: Finite state space:
1) Dynamic law of large numbers for interacting Markov chains, 2) Control without competition, 3) Two player zero-sum games with mean -field interaction,
Part II: Infinite state space.
Part III: Applications to mean-field games.

Mean-field interacting particles, I

Part I

State space $\{1, ..., d\}$ (types of a particle, say, opinions, fitness, etc), $\{Q(t,x)\} = \{(Q_{ij})(t,x)\}$ a family of Q-matrices or Kolmogorov matrices, piecewise continuous in $t \geq 0$ and Lipshitz in x from

$$\Sigma_d = \{x = (x_1, ..., x_d) \in \mathbf{R}_+^d : \sum_{j=1}^d x_j = 1\}.$$

For any x, the family $\{Q(., x)\}$ specifies a Markov chain:

$$(Q(t,x)f)_n = \sum_{m \neq n} Q_{nm}(t,x)(f_m - f_n).$$

Transition matrices $P(s,t,x) = (P_{ij}(s,t,x))_{i,j=1}^d$ satisfy the Kolmogorov equations

$$\frac{d}{dt} P_{ij}(s,t,x) = \sum_{l=1}^d Q_{lj}(t,x) P_{il}(s,t,x), \quad s \leq t.$$

Mean-field interacting particles, II

State space S for many particles: sequences $N = (n_1, ..., n_d)$, $|N| = n_1 + ... + n_d$. For $i \neq j$ denote by N^{ij} the state obtained from N by removing one particle of type i and adding a particle of type j, that is n_i and n_j are changed to $n_i - 1$ and $n_j + 1$ respectively.

The *mean-field interacting particle system*: Markov process on S with generator

$$L_t f(N) = \sum_{i,j=1}^{d} n_i Q_{ij}(t, N/|N|)[f(N^{ij}) - f(N)]. \qquad (1)$$

Probabilistic description and master equation.

Mean-field interacting particles, III

Let $h = 1/|N|$. Normalized state $N/|N|$ live in $\Sigma_d^h = \{hN \in \Sigma_d\}$:

$$L_t^h f(\frac{N}{|N|}) = \sum_{i=1}^d \sum_{j=1}^d \frac{n_i}{|N|}|N|Q_{ij}(t, \frac{N}{|N|})[f(\frac{N^{ij}}{|N|}) - f(\frac{N}{|N|})], \quad (2)$$

$$L_t^h f(x) = \sum_{i=1}^d \sum_{j=1}^d x_i Q_{ij}(t,x)\frac{1}{h}[f(x-he_i+he_j)-f(x)], \quad x \in h\mathbf{Z}_+^d, \quad (3)$$

The transition operators:

$$\Psi_{s,t}^h f(hN) = \mathbf{E}_{s,hN} f(hN^{t,h}), \quad s \le t. \quad (4)$$

LLN, I

Main question: asymptotic behavior as $h \to 0$. Formally:

$$\lim_{|N|\to\infty,\, N/|N|\to x} |N|[f(N^{ij}/|N|) - f(N/|N|)] = \frac{\partial f}{\partial x_j}(x) - \frac{\partial f}{\partial x_i}(x),$$

$$\lim_{|N|\to\infty,\, N/|N|\to x} L_t^h f(N/|N|) = \Lambda_t f(x),$$

$$= \sum_{i=1}^{d} \sum_{j\neq i} x_i Q_{ij}(t,x)[\frac{\partial f}{\partial x_j} - \frac{\partial f}{\partial x_i}](x)$$

$$= \sum_{k=1}^{d} \sum_{i\neq k} [x_i Q_{ik}(t,x) - x_k Q_{ki}(t,x)] \frac{\partial f}{\partial x_k}(x).$$

LLN, II

The limiting operator $\Lambda_t f$ is a first-order PDO with characteristics solving the equation

$$\dot{x}_k = \sum_{i \neq k} [x_i Q_{ik}(t,x) - x_k Q_{ki}(t,x)] = \sum_{i=1}^{d} x_i Q_{ik}(t,x), \quad k = 1, ..., d, \tag{5}$$

called the *kinetic equations* for the process of interaction described above. They define a deterministic Feller process with transition operators:

$$\Phi_{s,t} f(x) = f(X_{s,x}(t)), \quad s \leq t, \tag{6}$$

where $X_{s,x}(t)$ is the solution to (5) with the initial condition x at time s.

LLN, Theorem I

Let $Q(t,.) \in C^{1,\alpha}(\Sigma)$, $\alpha \in (0,1]$. If for $s > 0$ and $x \in \mathbf{R}^d$, the initial data hN_s converge to x in $\mathbf{R}^d$, as $h \to 0$, the Markov chains $hN^{t,h}$ with the initial data hN_s converge in distribution and in probability to the deterministic characteristic $X_{s,x}(t)$.
Rates:

$$\sup_{0 \le s \le t \le T} \left[\mathbf{E}_{s,hN} f(hN^{t,h}) - f(X_{s,x}(t)) \right]$$

$$\le C(T) \left((t-s) h^\alpha \|f\|_{C^{1,\alpha}(\Sigma_d)} + \|f\|_{C^1(\Sigma_d)} |hN - x| \right).$$

LLN, Theorem II

If $Q_{ij}(t,.) \in C^1(\Sigma)$, the convergence of Markov chains to the deterministic characteristics still holds, but weaker rates:

$$\sup_{0 \le s \le t \le T} \sup_{N \in \mathbf{Z}_+^d : |N| = 1/h} \left[\Psi_{s,t}^h f(hN) - \Phi_{s,t} f(hN) \right]$$

$$\le C(T)(t-s) \left(w_{hC(T)}(\nabla f) + w_{hC(T)}(\nabla Q) \|f\|_{C^1(\Sigma_d)} \right),$$

where $C(T)$ depends on the $C^1(\Sigma)$-norm of Q.

Mean field Markov control, I

$\{Q(t,u,x)\} = \{(Q_{ij})(t,u,x),\ i,j = 1,\cdots d\}$, $x \in \Sigma_d$, $t \geq 0$, $u \in U$.
Main assumption: $Q \in C^{1,\alpha}(\Sigma_d)$ as a function of x.
Controlled Markov chain on Σ_d^h:

$$L_{t,u(t)} f\left(\frac{N}{|N|}\right) = \sum_{i,j}^{d} n_i Q_{ij}\left(t, u(t), \frac{N}{|N|}\right)\left[f(\frac{N^{ij}}{|N|}) - f(\frac{N}{|N|})\right], \tag{7}$$

or equivalently

$$L^h_{t,u(t)} f(x) = \sum_{i=1}^{d}\sum_{j=1}^{d} x_i Q_{ij}(t,u(t),x)\frac{1}{h}[f(x - he_i + he_j) - f(x)]. \tag{8}$$

Mean field Markov control, II

Again for $f \in C^1(\Sigma_d)$,

$$\lim_{h=1/|N|\to 0,\, N/|N|\to x} L^h_{t,u(t)} f(N/|N|) = \Lambda_{t,u(t)} f(x),$$

where

$$\Lambda_{t,u(t)} f(x) = \sum_{k=1}^d \sum_{i\neq k} [x_i Q_{ik}(t,u(t),x) - x_k Q_{ki}(t,u(t),x)] \frac{\partial f}{\partial x_k}(x),$$

with the corresponding controlled characteristics governed by the equations

$$\dot{x}_k = \sum_{i=1}^d x_i Q_{ik}(t,u(t),x), \quad k = 1, ..., d. \tag{9}$$

Mean field Markov control, III

We are looking for

$$V^h(t,x) = \sup_{u(.)\in C_{pc}[t,T]} \mathbf{E}^{u(.)}_{t,x} \left[\int_t^T (J(s,u(s),hN(s))ds + V_T(hN(T)) \right]$$

where $E^{u(.)}_{t,x}$ denotes the expectation with respect to the Markov chain on Σ_d^h started at $x = hN$ at time t.
We expect $V^h(t,x)$ to be approximated by

$$V(t,x) = \sup_{u(.)\in C_{pc}[t,T]} \left[\int_t^T J(s,u(s),x(s))\, ds + V_T(x(T)) \right] \tag{10}$$

for the controlled deterministic dynamics above.

Mean field Markov control, IV

A function $\gamma(t,x)$ is called an optimal synthesis (adaptive policy) for $\Gamma(T,h)$ if

$$V^h(t,x) = \mathbf{E}^{\gamma}_{t,x}\left[\int_t^T (J(s,\gamma(s,hN(s)),hN(s))ds + V_T(hN(T))\right] \tag{11}$$

for all $t \leq T$ and $x \in \Sigma^h_d$. A function $\gamma(t,x)$ is called an ϵ-optimal synthesis (ϵ-adaptive policy), if the r.h.s. of (11) differs from its l.h.s. by not more than ϵ. Similarly an optimal synthesis or an adaptive policy are defined for the limiting deterministic system.

Control LLN: Theorem I

Assume that Q, J depend continuously on t, u and $Q, J, V \in C^{1,\alpha}(\Sigma_d)$, $\alpha \in (0,1]$, as functions of x, with the norms bounded uniformly in t, u. Then

$$\sup_{0\le t\le T} [V^h(t,hN) - V(t,x)] \le C(T)((T-t)h^\alpha + |hN - x|)$$

$$\times \left(\|V_T\|_{C^{1,\alpha}(\Sigma_d)} + \sup_{s,u} \|J(t,u,.)\|_{C^{1,\alpha}(\Sigma_d)} \right).$$

Moreover, if $u(t)$ is an ϵ-optimal control for deterministic dynamics (9), then $u(.)$ is an $(\epsilon + C(T)h^\alpha)$-optimal control for $|N| = 1/h$ particle system.

Control LLN: Theorem II

Suppose additionally that u belong to a convex subset of a Euclidean space and that $Q(t,u,x)$ depends Lipschitz continuously on u. Let $\epsilon \geq 0$, and let $\gamma(t,x)$ be a Lipschitz continuous function of x uniformly in t that represents an ϵ-optimal synthesis for the limiting deterministic control problem. Then, for any $\delta > 0$, there exists h_0 such that, for $h \leq h_0$, $\gamma(t,x)$ is an $(\epsilon+\delta)$-optimal synthesis for the approximate optimal problem $\Gamma(T,h)$ on Σ_d^h.

Interpretation in terms of viscosity solutions of HJB-Isaacs equations.

Two players with mean-field interaction, I

$\{Q(t,u,x) = (Q_{ij})(u,x)\}$ and $\{P(t,v,x) = (P_{ij})(v,x)\}$,
$i,j = 1,\cdots d$, $x \in \Sigma_d$, $u \in U$, $v \in V$.
Controlled Markov chains:

$$L_{t,u(t),v(t)} f(\frac{N}{|N|}, \frac{M}{|M|})$$

$$= \sum_{i,j}^{d} n_i Q_{ij}(t, u(t), \frac{N}{|N|})[f\left(\frac{N^{ij}}{|N|}, \frac{M}{|M|}\right) - f\left(\frac{N}{|N|}, \frac{M}{|M|}\right)]$$

$$+ \sum_{i,j}^{d} m_i P_{ij}(t, v(t), \frac{M}{|M|})[f\left(\frac{N}{|N|}, \frac{M^{ij}}{M}\right) - f\left(\frac{N}{|N|}, \frac{M}{|M|}\right)],$$

where $N = (n_1, \cdots, n_d)$, $M = (m_1, \cdots, m_d)$.
We shall assume for simplicity that $|N| = |M| = 1/h$.

Two players with mean-field interaction, II

$$L^h_{t,u(t),v(t)} f(x,y)$$

$$= \sum_{i=1}^{d}\sum_{j=1}^{d} x_i Q_{ij}(t,u(t),x)\frac{1}{h}[f(x-he_i+he_j,y)-f(x,y)]$$

$$+\sum_{i=1}^{d}\sum_{j=1}^{d} y_i P_{ij}(t,v(t),y)\frac{1}{h}[f(x,y-he_i+he_j)-f(x,y)].$$

$$\lim_{h\to 0,\, N/|N|\to x, M/|M|\to y} L^h_{t,u(t),v(t)} f(N/|N|, M/|M|) = \Lambda_{t,u(t),v(t)} f(x,y)$$

$$= \sum_{k=1}^{d}\sum_{i\neq k}[x_i Q_{ik}(t,u(t),x) - x_k Q_{ki}(t,u(t),x)]\frac{\partial f}{\partial x_k}(x)$$

$$+\sum_{k=1}^{d}\sum_{i\neq k}[y_i P_{ik}(t,u(t),x) - y_k P_{ki}(t,v(t),y)]\frac{\partial f}{\partial y_k}(y).$$

Two players with mean-field interaction, III

The corresponding controlled characteristics are governed by the equations

$$\dot{x}_k = \sum_{i=1}^{d} x_i Q_{ik}(t, u(t), x), \quad k = 1, ..., d,$$

$$\dot{y}_k = \sum_{i=1}^{d} y_i P_{ik}(t, v(t), y), \quad k = 1, ..., d.$$

Mean field competitive control

Denote $\Gamma(T,h)$ stochastic zero-sum game with player I (controlling Q via u) maximizing

$$\int_0^T J\left(s, u(s), v(s), \frac{N(s)}{|N|}, \frac{M(s)}{|M|}\right) ds + V_T\left(\frac{N(T)}{|N|}, \frac{M(T)}{|M|}\right). \tag{12}$$

As previously we want to approximate it by the deterministic zero-sum differential game $\Gamma(T)$ with the payoff of player I given by

$$\int_0^T J(s, u(s), v(s), x(s), y(s))\, ds + V_T(X(T), Y(T)). \tag{13}$$

Upper and lower values, I

A progressive strategy of player I is a mapping β from $C_{pc}([0,T];V)$ to $C_{pc}([0,T];U)$:
if $v_1(s) = v_2(s)$ for $s \in [0,t]$, $t < T$, then
$u_1 = (\beta(v_1)(s) = (\beta(v_2)(s))$ for $s \in [0,t]$. Similarly progressive strategies are defined for player II. The sets of progressive strategies for players I and II by $S_p([0,T];U)$ and $S_p([0,T];V)$.

$$V_+(t,x,y) = \sup_{\beta \in S_p([0,T];U)} \inf_{v(.) \in C_{pc}([0,T];V)}$$

$$\left[\int_0^T J\left(s, (\beta(v))(s), v(s)\frac{N(s)}{|N|}, \frac{M(s)}{|M|}\right) ds + V_T\left(\frac{N(T)}{|N|}, \frac{M(T)}{|M|}\right)\right] \quad (14)$$

$$V_-(t,x,y) = \inf_{\beta \in S_p([0,T];V)} \sup_{u(.) \in C_{pc}([0,T];U)}$$

$$\left[\int_0^T J\left(s, u(s), (\beta(u))(s), \frac{N(s)}{|N|}, \frac{M(s)}{|M|}\right) ds + V_T\left(\frac{N(T)}{|N|}, \frac{M(T)}{|M|}\right)\right]$$

Upper and lower values, II

If the so called Isaac's condition holds, that is, for any V,

$$\max_u \min_v \left[J(t,u,v,x,y) + \Lambda_{t,u,v} V\right]$$

$$= \min_v \max_u \left[J(t,u,v,x,y) + \Lambda_{t,u,v} V\right],$$

then the upper and lower values coincide:
$V_+(t,x,y) = V_-(t,x,y)$.
Similarly the upper and the lower values $V_+^h(t,x,y)$ and $V_-^h(t,x,y)$ for the stochastic game $\Gamma(T,h)$ are defined.
HJB-Isaac's equations.

Mean field competitive control

Theorem Assume that Q, P, J depend continuously on t, u and $Q, P, J, V_T \in C^{1,\alpha}(\Sigma_d)$, $\alpha \in (0, 1]$, as functions of x, with the norms bounded uniformly in t, u, v. Then

$$\sup_{0 \le t \le T} [V^h_\pm(t, hN) - V_\pm(t, x)] \le C(T)((T - t)h^\alpha + |hN - x|)$$

$$\times \left(\|V_T\|_{C^{1,\alpha}(\Sigma_d)} + \sup_{s,u} \|J(t, u, v, .)\|_{C^{1,\alpha}(\Sigma_d)} \right),$$

with $C(T)$ depending only on the bounds of the norms of Q in $C^{1,\alpha}(\Sigma_d)$. Moreover, if $\beta \in S_p([0, T]; U)$ and $v(.) \in C_{pc}([0, T]; V)$ are ϵ-optimal for the minimax problem (14), then this pair is also $(\epsilon + C(T)h^\alpha)$-optimal for the corresponding stochastic game $\Gamma(T, h)$.

Two players with binary interaction

$$L_{t,u(t),v(t)}f(N,M)$$

$$= \sum_{i,j,l,r=1}^{d} n_i m_j Q_{ij}^{lr}(t,u(t),v(t),\frac{N}{|N|},\frac{M}{|M|})[f(N^{il},M^{jr})-f(N,M)].$$

Assume $|M| = |N|$ and let $h = 1/|N| = 1/|M|$. To get a reasonable scaling limit, it is necessary to scale time by factor h leading to the generators

$$L^h_{t,u(t),v(t)}f(\frac{N}{|N|},\frac{M}{|M|})$$

$$= h\sum_{i,j,l,r=1}^{d} n_i m_j Q_{ij}^{lr}(t,u(t),v(t),\frac{N}{|N|},\frac{M}{|M|})[f(N^{il},M^{jr})-f(N,M)],$$

Two players with binary interaction

This generator for $x = hN$, $y = hM$ and $h \to 0$, tends to

$$\Lambda_{t,u(t),v(t)} f(x,y)$$

$$= \sum_{i,j,l,r=1}^{d} x_i y_j Q_{ij}^{lr}(t,u(t),v(t),x,y) \left[\frac{\partial f}{\partial x_l} + \frac{\partial f}{\partial y_r} - \frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial y_j} \right](x,y).$$

The limiting kinetic equations:

$$\dot{x}_k = \sum_{i,j,r=1}^{d} y_j \left[x_i Q_{ij}^{kr}(t,u(t),v(t)) - x_k Q_{kj}^{ir}(t,u(t),v(t)) \right],$$

$$\dot{y}_k = \sum_{i,j,l=1}^{d} x_i \left[y_j Q_{ij}^{lk}(t,u(t),v(t)) - y_k Q_{ik}^{lj}(t,u(t),v(t)) \right],$$

Results are the same.

K players with coupled mean-field interaction

$$
\begin{aligned}
& L_{t,u(t)} f\left(\frac{N_1}{|N_1|}, \cdots, \frac{N_K}{|N_K|}\right) \\
= & \sum_{k=1}^{K} \sum_{i_k, j_k=1}^{d} n_{i_k}^k Q_{i_k j_k}^k \left(t, u_k(t), \frac{N_1}{|N_1|}, \cdots, \frac{N_K}{|N_K|}\right) \\
\times & \left[f\left(\cdots, \frac{N_{k-1}}{|N_{k-1}|}, \frac{N_k^{i_k j_k}}{|N_k|}, \frac{N_{k+1}}{|N_{k+1}|}, \cdots\right) - f\left(\frac{N_1}{|N_1|}, \cdots, \frac{N_K}{|N_K|}\right) \right],
\end{aligned}
$$

where $N_k = (n_1^k, \cdots, n_d^k)$ with elements totting up to $|N_k|$.

K players with coupled mean-field interaction

The limiting deterministic evolution

$$\frac{d}{dt}x_k^j = \sum_{i=1}^{d} x_i Q_{ij}(t, u_k(t), x_1, \cdots, x_K), \quad j = 1, ..., d, \quad k = 1, \cdots, K$$

$$V_k^h(t, x_1, \cdots, x_k, u(.))$$
$$= \mathbf{E}_{t,x_1,\cdots,x_k}^{u(.)}\Big[\int_0^T J_k\left(s, u_k(s), \frac{N_1(s)}{|N_1|}, \cdots, \frac{N_K(s)}{|N_K|}\right) ds$$
$$+V_T^k\left(\frac{N_1(T)}{|N_1|}, \cdots, \frac{N_K(T)}{|N_K|}\right)\Big]$$

with given functions J_k (current payoffs) and V_T^k (terminal payoffs).
Results are the same.

Part II: setting

X is a locally compact separable metric space;
$X^j = X \times \cdots \times X$ (j times) $\mathcal{X} = \cup_{j=0}^{\infty} X^j$, $C_{sym}(\mathcal{X}) = C(S\mathcal{X})$.
Pairing

$$(f,\rho) = \int f(\mathbf{x})\rho(d\mathbf{x})$$

$$= f^0\rho_0 + \sum_{n=1}^{\infty} \int f(x_1,...,x_n)\rho(dx_1 \cdots dx_n),$$

$$f \in C_{\mathrm{sym}}(\mathcal{X}),\ \rho \in \mathcal{M}(\mathcal{X}).$$

Inclusion $S\mathcal{X}$ to $\mathcal{M}(X)$:

$$\mathbf{x} = (x_1,...,x_l) \mapsto \delta_{x_1} + \cdots + \delta_{x_l} = \delta_{\mathbf{x}},$$

Interacting particles: generators I

Interactions preserving the number of particles (mean field and binary for simplicity):

$$\sum_{i=1}^{n}(B_\mu^1)_i f(x_1, \dots, x_n) + \sum_{\{i,j\}\subset\{1,\dots,n\}} (B_\mu^2)_{ij} f(x_1, \dots, x_n), \quad (15)$$

where $(B_\mu^1)_i$ and $(B_\mu^2)_{ij}$ denote the action of the operators B_μ^1 and B_μ^2 on the variables x_i and x_i, x_j respectively. Here B_μ^1 and B_μ^2 are the Lévy-Khintchine type operators in $C(x)$ and $C^{sym}(X^2)$ respectively depending on a measure μ as on a parameter, allowing for additional mean field interaction:

$$B_\mu^1 f(x) = \frac{1}{2}(G_\mu(x)\nabla, \nabla) f(x) + (b_\mu(x), \nabla f(x))$$

$$+ \int (f(x+y) - f(x) - (\nabla f(x), y)\mathbf{1}_{B_1}(y))\nu_\mu(x, dy),$$

Interacting particles: generators II

$$B^2_\mu f(x,y) = \big[(\gamma_\mu(x,y)\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$$

$$+\frac{1}{2}(G_\mu(x,y)\frac{\partial}{\partial x}, \frac{\partial}{\partial x}) + \frac{1}{2}(G_\mu(y,x)\frac{\partial}{\partial y}, \frac{\partial}{\partial y})\big] f(x,y)$$

$$+\left[(b_\mu(x,y), \frac{\partial}{\partial x}) + (b_\mu(y,x), \frac{\partial}{\partial y}) f(x,y)\right]$$

$$+\int_{X^2} \nu_\mu(x,y,dv_1 dv_2)\big[f(x+v_1, y+v_2) - f(x,y)$$

$$-(\frac{\partial f}{\partial x}(x,y), v_1)\mathbf{1}_{B_1}(v_1) - (\frac{\partial f}{\partial y}(x,y), v_2)\mathbf{1}_{B_1}(v_2)\big],$$

where $G(x,y)$ and $\gamma(x,y)$ are symmetric matrices such that $\gamma(x,y) = \gamma(y,x)$ and the Lévy kernels ν enjoy the relation

$$\nu_\mu(x,y,dv_1 dv_2) = \nu_\mu(y,x,dv_2 dv_1).$$

Interacting particles: scaling

Scaling: $B^2 \to hB^2$, function $f(\mathbf{x})$ on $S\mathcal{X}$ to the functional $F(h\delta_{\mathbf{x}})$ on measures:

$$(\Lambda_h^1 + \Lambda_h^2)F(h\delta\mathbf{x}) = \sum_{i=1}^{n} (B^1_{h\delta_{\mathbf{x}}})_i F(h\delta_{\mathbf{x}}) + h \sum_{\{i,j\}\subset\{1,\dots,n\}} (B^2_{h\delta_{\mathbf{x}}})_{ij} F(h\delta_{\mathbf{x}}).$$

Proposition If F is smooth enough, then

$$\Lambda_h^1 F(Y) = \int_X \left(B_Y^1 \frac{\delta F}{\delta Y(.)} \right)(x) + O(h)$$

$$\Lambda_h^2 F(Y) = \frac{1}{2} \int_{X^2} \left(B_Y^2 \left(\frac{\delta F}{\delta Y(.)} \right)^{\oplus} \right)(x, y) Y(dx) Y(dy) + O(h),$$

where $Y = h\delta_{\mathbf{x}}$, $x = (x_1, \dots, x_n)$, and $O(h)$ has an explicit representation in terms of $\frac{\delta^2 F}{\delta Y(x)\delta Y(y)}$.

Interacting particles: Example: diffusions

$$A[t,\mu]f(z) = (h(t,z,\mu), \nabla f(z)) + \frac{1}{2}(G(t,z,\mu)\nabla, \nabla)f(z), \tag{16}$$

and

$$\widehat{A}_t^N[\gamma]f(\mathbf{x}) = \widehat{A}_t^N[\gamma]f(x_1, \cdots, x_N) := \sum_{i=1}^N A^i[t,\mu]f(x_1, \cdots, x_N), \tag{17}$$

with $\mu = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$. Then

$$\widehat{A}_t^N F(\mu) = \int_{\mathbf{R}^d} \left[\left(A[t,\mu]\frac{\delta F}{\delta \mu(.)} \right)(x) \right.$$
$$\left. + \frac{1}{2N} \left(G(t,x,\mu)\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \frac{\delta^2 F}{\delta \mu(x)\delta \mu(y)}\Big|_{y=x} \right] \mu(dx),$$

Differentiation with variational derivatives

To get above result we use the rule

$$\frac{\partial}{\partial x_i} F(\delta_{\mathbf{x}}/N) = \frac{1}{N} \frac{\partial}{\partial x_i} \frac{\delta F(Y)}{\delta Y(x_i)}, \quad Y = \delta_{\mathbf{x}}/N, \quad \mathbf{x} = (x_1, ..., x_N) \in \mathbf{R}^{dN},$$

which implies

$$\left(G \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i} \right) F(\delta_{\mathbf{x}}/N)$$

$$= \frac{1}{N} \left(G \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i} \right) \frac{\delta F(Y)}{\delta Y(x_i)} + \frac{1}{N^2} \left(G \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \frac{\delta^2 F(Y)}{\delta Y(y) \delta Y(z)} \Big|_{y=z=x_i},$$

and hence the above formulas.
Main conclusion: we need first and second order derivatives of the solutions to kinetic equations with respect to initial data (as in discrete case).

Interacting diffusions: Theorem on LLN

(Under some natural regularity assumptions for coefficients) suppose the initial conditions

$$\mu_0^N = \frac{1}{N}(\delta_{X_{1,0}^N} + \cdots + \delta_{X_{N,0}^N})$$

converge in $D^* = (C_\infty^2(\mathbf{R}^d))^*$, as $N \to \infty$, to a probability law $\mu_0 \in \mathcal{P}(\mathbf{R}^d)$ so that

$$\|\mu_0^N - \mu_0\|_{D^*} \leq \frac{\kappa_1}{N} \tag{18}$$

with a constant $\kappa_1 > 0$. Then, for $t \in [0, T]$ with any $T > 0$,

$$|[U_N^{0,t}F](\mu_0^N) - [\Phi^{0,t}(F)](\mu_0)| \leq \frac{C(T)}{N}(t\|F\|_{C_\infty^{2,2\times 2}(M_\Lambda(\mathbf{R}^d))} + \kappa_1) \tag{19}$$

with a constant $C(T) > 0$. Extension: Dynamic Law of Large Numbers with a Tagged Particle.

Interacting particles: fluctuations

Recall

$$B^1_\mu f(x) = \frac{1}{2}(G_\mu(x)\nabla, \nabla)f(x) + (b_\mu(x), \nabla f(x))$$

$$+\int (f(x+y) - f(x) - (\nabla f(x), y)\mathbf{1}_{B_1}(y))\nu_\mu(x, dy).$$

and μ_t solves the corresponding mean field kinetic equation

$$\frac{d}{dt}(g, \mu_t) = (B^1_{\mu_t} g, \mu_t).$$

Set

$$\Omega^h_t(Y) = h^{-1/2}(Y - \mu_t).$$

Generators, weak CLT in various models: Boltzmann, Smoluchovski, McKean-Vlasov, Landau-Fokker-Planck, evolutionary games, etc.

Part III: objectives

Recall: for $\epsilon > 0$, a strategy profile Γ in a game of N players with payoffs $V_i(\Gamma)$, $i = 1, \cdots, N$, is an *ϵ-equilibrium* (or ϵ-Nash equilibrium) if, for each player i and an eligible strategy u_i,

$$V_i(\Gamma) \geq V_i(\Gamma_{-i}, u_i) - \epsilon,$$

where (Γ_{-i}, u_i) denotes the profile obtained from Γ by substituting the strategy of player i with u_i. A profile of dynamic strategies in a dynamic game on a time interval $[0, T]$ is called a *perfect ϵ-equilibrium*, if it is an ϵ-equilibrium for any subgame started any time $t \in [0, T]$.
Main consistency equation for MFG:

$$\frac{d}{dt}(f, \mu_t) = (A[t, \mu_t, \Gamma(t, ., \{\mu_{\geq t}\})]f, \mu_t). \tag{20}$$

Basic Banach spaces $B = C_\infty(\mathbf{R}^d)$ and $D = C^2_\infty(\mathbf{R}^d)$.

Part III: theorem

Suppose $\{A[t,\mu,u] : t \geq 0, \mu \in M_\Lambda(\mathbf{R}^d), u \in U\}$ is a family of linear operators $D \to C_\infty(\mathbf{R}^d)$ of the form:

$$A[t,\mu,u]f(z) = (h(t,z,\mu,u), \nabla f(z)) + \frac{1}{2}(G(t,z,\mu)\nabla, \nabla)f(z) \tag{21}$$

with smooth enough coefficients, terminal cost $V^T(x,\mu)$ and running costs $J(t,x,\mu,u)$ are smooth enough and

$$\|\mu_0^N - \mu_0\|_{D^*} \leq \frac{\kappa_1}{N}, \quad ||X_{1,0}^N - X_{1,0}|| \leq \frac{\kappa_2}{N}.$$

Let $\{\mu.\}$ be a solution to equation (20) with a continuous strategy profile $u = \Gamma(t,x,\{\mu.\})$ which is Lipschitz continuous in x and defined via corresponding HJB.
Then this strategy profile $u = \Gamma(t,x,\{\mu.\})$ is a perfect ϵ-equilibrium in an N agents game, with

$$\epsilon = \frac{C(T)}{N}(||J||_{C([0,T]\times U, C_\infty^{2;2,2\times 2}(\mathbf{R}^d \times M_\Lambda(\mathbf{R}^d)))} + ||V^T||_{C_\infty^2(\mathbf{R}^d)} + 1).$$

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