# A second order model for macroscopic crowd movements with congestion

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(ongoing joint work F. Santambrogio)

Mean Field Games and Related Topics - 2, Padova, Sept. 4-6, 2013

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1. Deterministic crowd movements with congestion 1 - Microscopic model (briefly)

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4. Final remarks, conclusions, open questions

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- This introduces the presence of a projection operator $P_{adm(q)}$ acting on the velocities onto the set of admissible velocities:

$$q \in K := \{q = (q_i)_i \in \Omega^N : |q_i - q_j| \geq 2R\},$$

$$adm(q) = \{v = (v_i)_i : (v_i - v_j) \cdot (q_i - q_j) \geq 0 \forall (i,j) : |q_i - q_j| = 2R\}.$$

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- Finally we solve $q'(t) = P_{adm(q(t))} u(t)$ (with $q(0)$ given).

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# The continuous model - part 1

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- For every time $t$, we consider $u_t : \Omega \to \mathbb{R}^d$ a vector field, possibly depending on $\rho$;
- For every density $\rho$ we have a set of admissible velocities, characterized by the sign of the divergence on the saturated region $\{\rho = 1\}$, so the set is: $adm(\rho) := \left\{ v : \Omega \to \mathbb{R}^d : \nabla \cdot v \geq 0 \text{ on } \{\rho = 1\} \right\};$

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- In this sense $v = P_{adm(\rho)}[u]$ and $u = v + \nabla p$, $v \in adm(\rho)$ and

$$p \in press(\rho) := \left\{ p \in H^1(\Omega) : p \geq 0, p(1-\rho) = 0 \right\}.$$

# Wasserstein distances and Kantorovich potentials

- For two (absolutely cont.) probability measures $\mu, \nu \in \mathcal{P}(\Omega)$ de define

$$W_2^2(\mu,\nu) := \inf \left\{ \int \frac{1}{2}|x - T(x)|^2 d\mu : T : \Omega \to \Omega,\, T_{\#}\mu = \nu \right\}$$
$$= sup_{\phi,\psi \in Lip(\Omega)} \left\{ \int \phi d\mu + \int \psi d\nu : \phi(x) + \psi(y) \leq \frac{|x-y|^2}{2} \right\}$$

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- $W_2$ metrizes the weak-* topology on $\mathcal{P}(\Omega)$ for compact domains $\Omega$.

# Solution of the continuity equation for gradient fields

- If the vector field of the particles is given by $u_t := -\nabla V_t, \forall t$, then the solution of (1) can be obtained by the gradient flow of the functional

$$\mathcal{F}(\rho) := \int_\Omega V_t(x) d\rho(x) + I_K(\rho),$$

where $I_K$ is the indicator function of $K := \{\rho \in \mathcal{P}(\Omega) : \rho \leq 1\}$, which is 0 in $K$ and $+\infty$ outside of $K$;

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- Use the JKO scheme to get a solution, which is for $\tau > 0$, $\rho_0^\tau := \rho_0$ and for $k \in \mathbb{N}, k \geq 0$ we consider the scheme

$$\rho_{k+1}^\tau \in \mathrm{argmin}_{\rho \in \mathcal{P}(\Omega)} \left( \mathcal{F}(\rho) + \frac{1}{2\tau} W_2^2(\rho, \rho_k^\tau) \right) \tag{2}$$

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- Construct piecewise constant and geodesic interpolations;
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- Pass to the limit $\tau \to 0$.

# Deriving the pressure via the projection

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$$\rho = \begin{cases} 1, & \text{on } \phi < l, \\ \in [0,1], & \text{on } \phi = l, \\ 0, & \text{on } \phi > l. \end{cases}$$

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It follows that $p := (t - \phi)_+ \geq 0$ satisfies $p(1 - \rho) = 0$, hence it is an admissible pressure, and we have that

$$T(x) = x - \nabla\phi(x) = x + \nabla p, \ \rho - \text{a.e.}$$

is the optimal transport map from the projected field to the original one.

# In our case

Remark that

$$||\nabla p||_{L^2(\rho^\tau_{n+1})} = W_2(\rho^\tau_{n+1}, \tilde{\rho}^\tau_{n+1}) \leq W_2(\rho^\tau_n, \tilde{\rho}^\tau_{n+1}) \leq \tau ||u_{n\tau}||_{L^2(\rho^\tau_n)}.$$

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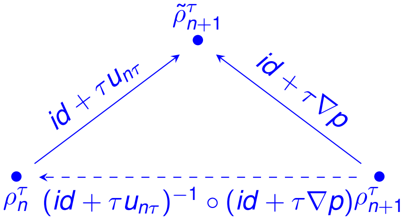
This indicates us to rescale the pressure with $\tau$, hence we have



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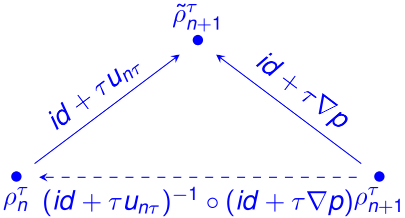


Remark: $(id + \tau u_{n\tau})^{-1} \circ (id + \tau\nabla p) = id - \tau(u_{n\tau} - \nabla p) + o(\tau)$, provided $u$ is regular enough. This will allow us to take the limit as $\tau \to 0$ and get a solution of the continuity equation.

# Adding a diffusion term

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- The Fokker-Planck type equation, we get is

$$\partial_t \rho_t - \Delta \rho_t + \nabla \cdot (P_{adm(\rho_t)}[u_t]\rho_t) = 0, \tag{3}$$

which is exactly

$$\partial_t \rho_t - \Delta \rho_t + \nabla \cdot \left( P_{adm(\rho_t)} \left[ u_t - \frac{\nabla \rho_t}{\rho_t} \right] \rho_t \right) = 0,$$

because $\frac{\nabla \rho}{\rho} = 0$ on $\{\rho = 1\}$.

# How to show the existence of a solution of (3)?

If the velocity field is again a gradient ($u_t = -\nabla V_t$), then we can argue similarly as in the deterministic case by the JKO scheme[3] using the gradient flow[4] of the perturbed entropy functional

$$\mathcal{F}(\rho) := \int_\Omega V_t d\rho + \int_\Omega \rho \ln \rho + I_K(\rho).$$

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# The splitting algorithms

## First approach

- Take a random variable with $X \sim \rho_n^\tau$.
- Construct a new r.v. $Y = (id + \tau u_{n\tau}) \circ X + W_\tau$, where $W$ is a Brownian motion independent of $X$.
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Set $\rho_{n+1}^\tau = P_K(\tilde{\rho}_{n+1}^\tau)$, where $\tilde{\rho}_{n+1}^\tau = \rho_\tau$.

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- These are available under higher regularity assumptions (BV for the initial data).

Third approach which is working, but not so natural:

- Construct $\tilde{\rho}^\tau_{n+1} := (id + \tau u_{n\tau})_\# \rho^\tau_n$.
- Define

$$\rho^\tau_{n+1} := \mathrm{argmin}_{\rho \in K} \int_\Omega \rho \ln \rho + \frac{1}{2\tau} W_2^2(\rho, \tilde{\rho}^\tau_{n+1}).$$

# Deriving the Fokker-Planck equation

As in the deterministic case the optimizer in the above problem for the optimal $\rho$ we have: that $\exists l$ s.t.

$$\rho = \begin{cases} 1, & \text{on } \left(\ln \rho + \frac{\phi}{\tau}\right) < l, \\ \in [0,1], & \text{on } \left(\ln \rho + \frac{\phi}{\tau}\right) = l, \\ 0, & \text{on } \left(\ln \rho + \frac{\phi}{\tau}\right) > l, \end{cases}$$

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from where we define the admissible pressure $p := \left(t - \ln\rho - \frac{\phi}{\tau}\right)_+$. This will imply that the optimal transport map from $\rho^\tau_{n+1}$ to $\tilde{\rho}^\tau_{n+1}$ is $id + \tau\left(\nabla p + \frac{\nabla\rho}{\rho}\right)$.

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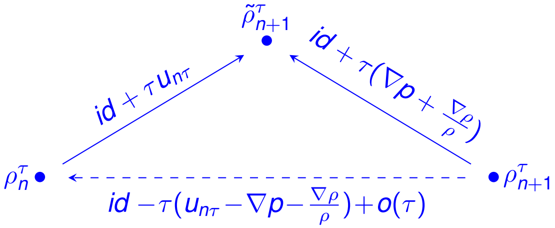
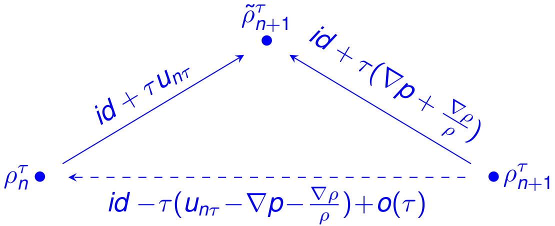
The situation is described on the picture below:



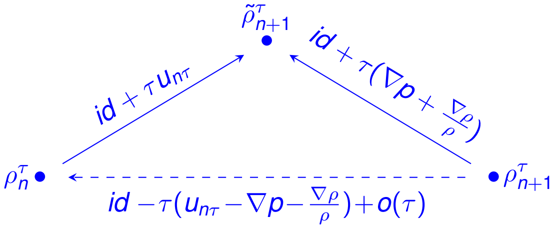
$\tilde{\rho}^{\tau}_{n+1}$
$id + \tau u_{n\tau}$
$id + \tau(\nabla p + \frac{\nabla \rho}{\rho})$
$\rho^{\tau}_{n}$
$\rho^{\tau}_{n+1}$
$id - \tau(u_{n\tau} - \nabla p - \frac{\nabla \rho}{\rho}) + o(\tau)$




Notice again, that

$$(id+\tau u_{n\tau})^{-1} \circ \left(id + \tau\left(\nabla p + \frac{\nabla\rho}{\rho}\right)\right) = id - \tau\left(u_{n\tau} - \nabla p - \frac{\nabla\rho}{\rho}\right) + o(\tau),$$

provided $u$ has enough regularity.

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provided $u$ has enough regularity.
Hence letting $\tau \to 0$, we derive

$$\partial_t\rho_t - \Delta\rho_t + \nabla\cdot(\rho_t(u_t-\nabla p_t)) = 0.$$

# Final conclusions, remarks and perspectives

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- Need more work on the first two approaches to get uniform estimates.
- Open question: Invariance of the BV densities under the projection $P_K$.
- Perspective: insert this model into second order MFG systems with density constraints.

Thank you for your attention!