Approximate Solutions for Deterministic Mean Field Games: Initial Steps

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Mean Field Games and Related Topics - 2

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Outline

1. Previous work: N-player differential games
    - Problem formulation
    - Approximate solutions (2-player case)
2. Currently: Mean Field Games

Background

Differential games: several players, several objectives

$$\dot{x} = f(x, u_1(t), \dots, u_N(t))$$

$$\min_{u_1} \frac{1}{2} \int_0^T L_1(x(t), u_1(t), \dots, u_N(t)) \mathrm{d}t$$

$$\vdots$$

$$\min_{u_N} \frac{1}{2} \int_0^T L_N(x(t), u_1(t), \dots, u_N(t)) \mathrm{d}t$$

Differential Games

Solution relies on N PDE's (Hamilton-Jacobi-Isaacs equations)

$$\frac{\partial V_i}{\partial x} f(x) - \frac{1}{2} \frac{\partial V_i}{\partial x} g_i(x) g_i(x)^\top \frac{\partial V_i}{\partial x}^\top + \frac{1}{2} q_i(x)$$

$$- \frac{1}{2} \sum_{j=1, j \neq i}^{N} \frac{\partial V_j}{\partial x} g_j(x) g_j(x)^\top \frac{\partial V_j}{\partial x}^\top - \sum_{j=1, j \neq i}^{N} \frac{\partial V_i}{\partial x} g_j(x) g_j(x)^\top \frac{\partial V_j}{\partial x}^\top = 0$$

Nash strategies:

$$u_i^* = -g_i(x)^\top \frac{\partial V_i}{\partial x}^\top$$

Previous Work

Approximate Solutions for N-player games
(T. Mylvaganam, M. Sassano, A. Astolfi)

Input-affine systems

$$\dot{x} = f(x) + g_1(x)u_1 + \cdots + g_N(x)u_N\,, \quad x \in \mathbb{R}^n$$

Cost-functionals of the form

$$J_i(x(0), u_1, \ldots, u_N) \triangleq \frac{1}{2}\int_0^\infty \left(q_i(x(t)) + \|u_i(t)\|^2 - \sum_{j=1, j\neq i}^{N} \|u_j(t)\|^2 \right) \mathrm{d}t\,,$$

2-Player Problem

Nonlinear input-affine dynamics

$$\dot{x} = f(x) + g_1(x)u_1 + g_2(x)u_2$$

Assumption: $f(0) = 0 \Rightarrow f(x) = F(x)x$

Cost-functionals

$$J_1(x(0), u_1, u_2) \triangleq \frac{1}{2}\int_0^\infty q_1(x) + \|u_1(t)\|^2 - \|u_2(t)\|^2 \mathrm{d}t$$

$$J_2(x(0), u_1, u_2) \triangleq \frac{1}{2}\int_0^\infty q_2(x) + \|u_2(t)\|^2 - \|u_1(t)\|^2 \mathrm{d}t$$

Running costs s.t.

$$q_1(x) = x^\top Q_1(x)x \geq 0 \quad q_2(x) = x^\top Q_2(x)x \geq 0 \quad q_1 + q_2 > 0$$

2-Player Problem

Hamilton-Jacobi-Isaacs (HJI) partial differential equations

$$\frac{\partial V_1}{\partial x} f(x) - \frac{1}{2}\frac{\partial V_1}{\partial x} g_1(x) g_1(x)^\top \frac{\partial V_1}{\partial x}^\top + \frac{1}{2} q_1(x)$$
$$- \frac{1}{2}\frac{\partial V_2}{\partial x} g_2(x) g_2(x)^\top \frac{\partial V_2}{\partial x}^\top - \frac{\partial V_1}{\partial x} g_2(x) g_2(x)^\top \frac{\partial V_2}{\partial x}^\top = 0$$

Nash strategies

$$u_1^*(x) = -g_1(x)^\top \frac{\partial V_1}{\partial x}^\top$$

$$u_2^*(x) = -g_2(x)^\top \frac{\partial V_2}{\partial x}^\top$$

2-Player Problem

Hamilton-Jacobi-Isaacs (HJI) partial differential equations

$$\frac{\partial V_2}{\partial x} f(x) - \frac{1}{2}\frac{\partial V_2}{\partial x} g_2(x) g_2(x)^\top \frac{\partial V_2}{\partial x}^\top + \frac{1}{2} q_2(x)$$
$$- \frac{1}{2}\frac{\partial V_1}{\partial x} g_1(x) g_1(x)^\top \frac{\partial V_1}{\partial x}^\top - \frac{\partial V_2}{\partial x} g_1(x) g_1(x)^\top \frac{\partial V_1}{\partial x}^\top = 0$$

Nash strategies

$$u_1^*(x) = -g_1(x)^\top \frac{\partial V_1}{\partial x}^\top$$

$$u_2^*(x) = -g_2(x)^\top \frac{\partial V_2}{\partial x}^\top$$

2-Player Problem

Problem: HJI PDE's not generally analytically solvable

Approximate solutions:

- Linear quadratic approximation
- Dynamic feedback satisfying partial differential *inequalities*

Two methods of designing dynamic feedback strategies

- 2-player games presented today
- See previous work for *N*-player extension

Linear-Quadratic Approximation

Linearised system:

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2$$

Quadratic cost-functionals

$$J_1(x(0), u_1, u_2) \triangleq \frac{1}{2} \int_0^{\infty} x^\top Q_1 x + \|u_1(t)\|^2 - \|u_2(t)\|^2 \mathrm{d}t$$

$$J_2(x(0), u_1, u_2) \triangleq \frac{1}{2} \int_0^{\infty} x^\top Q_2 x + \|u_2(t)\|^2 - \|u_1(t)\|^2 \mathrm{d}t$$

Linear-Quadratic Approximation

Coupled algebraic Riccati equations

$$\bar{P}_1 A + A^\top \bar{P}_1 - \bar{P}_1 B_1 B_1^\top \bar{P}_1 - \bar{P}_1 B_2 B_2^\top \bar{P}_2 - \bar{P}_2 B_2 B_2^\top \bar{P}_1 - \bar{P}_2 B_2 B_2^\top \bar{P}_2 + Q_1 = 0$$

$$\bar{P}_2 A + A^\top \bar{P}_2 - \bar{P}_2 B_2 B_2^\top \bar{P}_2 - \bar{P}_2 B_1 B_1^\top \bar{P}_1 - \bar{P}_1 B_1 B_1^\top \bar{P}_2 - \bar{P}_1 B_1 B_1^\top \bar{P}_1 + Q_2 = 0$$

Nash strategies

$$u_1^*(x) = -B_1^\top \bar{P}_1 x$$

$$u_2 * (x) = -B_2^\top \bar{P}_2 x$$

Algebraic $\bar{P}$ Matrix Solutions (2-Player Case)

Both methods of designing the dynamic feedback strategies rely on *algebraic* $\bar{P}$ *solutions.*

Algebraic $\bar{P}$ Matrix Solutions (2-Player Case)

P_1 and P_2 are algebraic $\bar{P}$ matrix solutions if

$$P_1(x)F(x) + F(x)^\top P_1(x) - P_1(x)g_1(x)g_1(x)^\top P_1(x) + Q_1(x)$$
$$-\left(P_2(x) + 2P_1(x)\right)g_2(x)g_2(x)^\top P_2(x) + \Sigma_1(x) = 0$$

$$P_2(x)F(x) + F(x)^\top P_2(x) - P_2(x)g_2(x)g_2(x)^\top P_2(x) + Q_2(x)$$
$$-\left(P_1(x) + 2P_2(x)\right)g_1(x)g_1(x)^\top P_1(x) + \Sigma_2(x) = 0$$

$P_1(0)$ and $P_2(0)$ solutions of the coupled Riccati equations with Q_i replaced by $Q_i + 2\bar{\Sigma}_i$

Method I: Several Dynamic Extensions

Introduce $N = 2$ dynamic extensions: ξ_1, ξ_2

Assume existence of algebraic $\bar{P}$ matrix solutions

Extended value functions:

$$V_1(x, \xi_1) = P_1(\xi_1)x + \frac{1}{2}\|x - \xi_1\|^2_{R_1} \qquad V_2(x, \xi_2) = P_2(\xi_1)x + \frac{1}{2}\|x - \xi_2\|^2_{R_2}$$

Player strategies

$$u_1(x, \xi_1) = -g_1(x)^\top \frac{\partial V_1}{\partial x}^\top \quad u_2(x, \xi_2) = -g_2(x)^\top \frac{\partial V_2}{\partial x}^\top$$

Dynamics of ξ_1 and ξ_2

$$\dot{\xi}_1 = -k_1 \frac{\partial V_1}{\partial \xi_1}^\top \quad \dot{\xi}_2 = -k_2 \frac{\partial V_2}{\partial \xi_2}^\top$$

Method I: Several Dynamic Extensions

Structural assumption: $\text{rank}\,(g_1(x)) = n$ and $\text{rank}\,(g_2(x)) = n$

$$\exists \bar{k}_1 \geq 0 \qquad \exists \bar{k}_2 \geq 0 \qquad \exists R_1 > 0 \qquad \exists R_2 > 0 \qquad \exists \Omega \subseteq \mathbb{R}^n \times \mathbb{R}^n$$

such that

$$\forall k_1 > \bar{k}_1 \qquad \forall k_2 > \bar{k}_2 \qquad \forall (x,\, \xi) \in \Omega$$

V_1 and V_2 solve the system of PDI's

$$\frac{\partial V_1}{\partial x} f(x) + \frac{\partial V_1}{\partial \xi_1} \dot{\xi}_1 - \frac{1}{2} \frac{\partial V_1}{\partial x} g_1(x) g_1(x)^\top \frac{\partial V_1}{\partial x}^\top + \frac{1}{2} q_1(x)$$
$$- \frac{1}{2} \frac{\partial V_2}{\partial x} g_2(x) g_2(x)^\top \frac{\partial V_2}{\partial x}^\top - \frac{\partial V_1}{\partial x} g_2(x) g_2(x)^\top \frac{\partial V_2}{\partial x}^\top \leq 0$$

$(0,0)^\top$ is asymptotically stable

Method I: Several Dynamic Extensions

$$\exists \bar{k}_1 \geq 0 \qquad \exists \bar{k}_2 \geq 0 \qquad \exists R_1 > 0 \qquad \exists R_2 > 0 \qquad \exists \Omega \subseteq \mathbb{R}^n \times \mathbb{R}^n$$

such that

$$\forall k_1 > \bar{k}_1 \qquad \forall k_2 > \bar{k}_2 \qquad \forall (x,\, \xi) \in \Omega$$

V_1 and V_2 solve the system of PDI's

$$\frac{\partial V_2}{\partial x} f(x) + \frac{\partial V_2}{\partial \xi_2} \dot{\xi}_2 - \frac{1}{2} \frac{\partial V_2}{\partial x} g_2(x) g_2(x)^\top \frac{\partial V_2}{\partial x}^\top + \frac{1}{2} q_2(x)$$
$$- \frac{1}{2} \frac{\partial V_1}{\partial x} g_1(x) g_1(x)^\top \frac{\partial V_1}{\partial x}^\top - \frac{\partial V_2}{\partial x} g_1(x) g_1(x)^\top \frac{\partial V_1}{\partial x}^\top \leq 0$$

$(0,0)^\top$ is asymptotically stable

Method I: Several Dynamic Extensions

Quadratic Form:

$$-\begin{bmatrix} x^\top & (x-\xi_1)^\top \end{bmatrix} \left[M_1 - k_1 \underbrace{\frac{\partial V_1}{\partial \xi_1}\frac{\partial V_1}{\partial \xi_1}^\top}_{C_1} \right] \begin{bmatrix} x \\ (x-\xi_1) \end{bmatrix}$$

$$-\begin{bmatrix} x^\top & (x-\xi_2)^\top \end{bmatrix} \left[M_2 - k_2 \underbrace{\frac{\partial V_2}{\partial \xi_2}\frac{\partial V_2}{\partial \xi_2}^\top}_{C_2} \right] \begin{bmatrix} x \\ (x-\xi_2) \end{bmatrix}$$

Restrict to kernel of C_1 and C_2:

$$Z_1^\top M_1 Z_1 > 0 \qquad\qquad Z_2^\top M_2 Z_2 > 0$$

Method I: Several Dynamic Extensions

Candidate Lyapunov function $W = V_1 + V_2 > 0$

Partial differential inequalities imply $\dot{W} < -\frac{1}{2}(q_1 + q_2) < 0$

Zero equilibrium of $\dot{\xi}(0, \xi) = -kR_i\xi$ asymptotically stable

$(x, \xi) = (0, 0)$ asymptotically stable

Method II: Shared Dynamic Extension

Introduce *one, shared* dynamic extensions: ξ

Assume existence of algebraic $\bar{P}$ matrix solutions

Extended value functions:

$$V_1(x,\xi) = P_1(\xi)x + \frac{1}{2}\|x-\xi\|^2_{R_1} \qquad V_2(x,\xi) = P_2(\xi)x + \frac{1}{2}\|x-\xi\|^2_{R_2}$$

Player strategies

$$u_1(x,\xi) = -g_1(x)^\top \frac{\partial V_1}{\partial x}^\top \quad u_2(x,\xi) = -g_2(x)^\top \frac{\partial V_2}{\partial x}^\top$$

Dynamics of ξ

$$\dot{\xi}_1 = -k\left(\frac{\partial V_1}{\partial \xi}^\top + \frac{\partial V_2}{\partial \xi}^\top\right)$$

Method II: Shared Dynamic Extension

$$\exists \bar{k} \geq 0 \qquad \exists R_1 > 0 \qquad \exists R_2 > 0 \qquad \exists \Omega \subseteq \mathbb{R}^n \times \mathbb{R}^n$$

such that

$$\forall k > \bar{k} \qquad \forall (x, \xi) \in \Omega$$

V_1 and V_2 solve the system of PDI's

$$\frac{\partial V_1}{\partial x} f(x) + \frac{\partial V_1}{\partial \xi}\dot{\xi} - \frac{1}{2}\frac{\partial V_1}{\partial x} g_1(x) g_1(x)^\top \frac{\partial V_1}{\partial x}^\top + \frac{1}{2} q_1(x)$$
$$- \frac{1}{2}\frac{\partial V_2}{\partial x} g_2(x) g_2(x)^\top \frac{\partial V_2}{\partial x}^\top - \frac{\partial V_1}{\partial x} g_2(x) g_2(x)^\top \frac{\partial V_2}{\partial x}^\top \leq 0$$

$(0, 0)^\top$ is asymptotically stable

Method II: Shared Dynamic Extension

$$\exists \bar{k} \geq 0 \qquad \exists R_1 > 0 \qquad \exists R_2 > 0 \qquad \exists \Omega \subseteq \mathbb{R}^n \times \mathbb{R}^n$$

such that

$$\forall k > \bar{k} \qquad \forall (x,\, \xi) \in \Omega$$

V_1 and V_2 solve the system of PDI's

$$\frac{\partial V_2}{\partial x} f(x) + \frac{\partial V_2}{\partial \xi}\dot{\xi} - \frac{1}{2}\frac{\partial V_2}{\partial x} g_2(x) g_2(x)^\top \frac{\partial V_2}{\partial x}^\top + \frac{1}{2} q_2(x)$$

$$-\frac{1}{2}\frac{\partial V_1}{\partial x} g_1(x) g_1(x)^\top \frac{\partial V_1}{\partial x}^\top - \frac{\partial V_2}{\partial x} g_1(x) g_1(x)^\top \frac{\partial V_1}{\partial x}^\top \leq 0$$

$(0,0)^\top$ is asymptotically stable

Method II: Shared Dynamic Extensions

Quadratic Form:

$$-\begin{bmatrix} x^\top & (x-\xi)^\top \end{bmatrix} \left[M_1 + \underbrace{k\,\frac{\partial V_1}{\partial \xi}\left(\frac{\partial V_1}{\partial \xi} + \frac{\partial V_2}{\partial \xi}\right)}_{D_1} \right] \begin{bmatrix} x \\ (x-\xi)^\top \end{bmatrix}$$

Method II: Shared Dynamic Extensions

Quadratic Form:

$$-\begin{bmatrix} x^\top & (x-\xi)^\top \end{bmatrix} [M_1 + kD_1] \begin{bmatrix} x \\ (x-\xi) \end{bmatrix}$$

$$-\begin{bmatrix} x^\top & (x-\xi)^\top \end{bmatrix} (M_2 + kD_2) \begin{bmatrix} x \\ (x-\xi) \end{bmatrix}$$

Conditions on $P_1(x)$, $P_2(x)$, R_1 and R_2 ensure $D_1 \geq 0$ and $D_2 \geq 0$

Restrict to kernel of D_1 and D_2:

$$Z_1^\top M_1 Z_1 > 0 \qquad Z_2^\top M_2 Z_2 > 0$$

Stability shown as with several dynamic extensions

Extensions to Mean Field Games

Extend the ideas to mean field games

Still a work in progress

Consider deterministic, stationary problem :

Agent dynamics:

$$\dot{x} = u$$

Cost-functional:

$$J(x(0), m(0), u(x,t)) \triangleq \frac{1}{2}\int_0^\infty \Big(q(x, m(x)) + \|u(t)\|^2\Big)\mathrm{d}t$$

e.g. running cost:

$$q(x, m(x)) = \frac{c_0 x}{c_1 + c_2 m(x)}$$

Extensions to Mean Field Games

HJB equation

$$V_x V_x^\top = q(x, m(x))$$

Advection equation

$$div(m(x)u(x)) = 0$$

Approximate solutions using dynamic extension, ξ

Extensions to Mean Field Games

HJB equation

$$V_x V_x^\top = q(x, m(x))$$

Advection equation

$$m_x V_x^\top + m(x)\text{trace}(V_{xx}) = 0$$

Approximate solutions using dynamic extension, ξ

Extensions to Mean Field Games

Algebraic $\bar{P}$ Mean Field Game solutions:

$$-\frac{1}{2}x^\top P(x)P(x)x + \frac{1}{2}q(x, m(x)) + \sigma(x) = 0$$

$$M(x)\big(P(x) + I\text{trace}\big(P(x)\big)\big)x + \gamma(x) = 0$$

Define functions $V(x, \xi)$ and $m(x, \xi)$ in terms of $P(x)$ and $M(x)$

Design $\dot{\xi}$ s.t.

$$-\tfrac{1}{2}V_x V_x^\top + \tfrac{1}{2}q(x) + V_\xi \dot{\xi} \leq 0 \qquad div\Big(m(x,\xi)\begin{bmatrix} u \\ \dot{\xi} \end{bmatrix}\Big) \leq 0$$

Extensions to Mean Field Games

Algebraic $\bar{P}$ Mean Field Game solutions:

$$-\frac{1}{2}x^\top P(x)P(x)x + \frac{1}{2}q(x, m(x)) + \sigma(x) = 0$$

$$M(x)\big(P(x) + I\text{trace}\big(P(x)\big)\big)x + \gamma(x) = 0$$

Define functions $V(x,\xi)$ and $m(x,\xi)$ in terms of $P(x)$ and $M(x)$

Design $\dot{\xi}$ s.t.

Closed-loop stability

Conclusions and Future Work

- Approximate solutions to a class of differential games
  - Dynamic extension and algebraic $\bar{P}$ solutions
  - 2-player case
  - Several extensions
  - Shared extension

- Extension to Mean Field Games
  - Deterministic, stationary problems

Thank you for your attention.

Any questions?