Regularity for mean-field games in the subquadratic case

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Outline

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A model MFG problem

1. We consider the following MFG problem:

$$\begin{cases} -u_t + H(x, Du) = \Delta u + g(m) \\ m_t - \operatorname{div}(D_p H m) = \Delta m, \end{cases} \tag{1}$$

on $\mathbb{T}^d \times [0, T]$, where $T > 0$ is a fixed terminal instant.

2. Initial-terminal boundary conditions

$$\begin{cases} u(x, T) = u_0(x) \\ m(x, 0) = m_0(x) \end{cases} \tag{2}$$

3. Spatially-periodic solutions

General assumptions on H

1. $H \in \mathcal{C}^2\left(\mathbb{T}^d \times \mathbb{R}^d\right)$ is coercive and, for fixed x, $p \mapsto H(x,p)$ is strictly convex
2. There are $c,\ C > 0$ s.t.:

$$D_pH(x,p)\cdot p - H(x,p) \geq cH(x,p) - C$$

3. H satisfies

$$|D_xH|\,,\ |D^2_{xx}H|, \leq CH + C$$

4. For any symmetric matrix M and $\delta > 0$, exists C_δ s.t.:

$$\mathrm{Tr}(D^2_{px}HM) \leq \delta\,\mathrm{Tr}(D^2_{pp}HMM) + C_\delta H$$

Subquadratic Hamiltonians

Consider $\gamma \in (1, 2)$

1. There exists $C > 0$ s.t.

$$H(x, p) \leq C\,|p|^{\gamma} + C$$

2. There exists $C > 0$ s.t.

$$|D_p H| \leq C\,|p|^{\gamma-1} + C$$

Subquadratic Hamiltonians

Consider $\gamma \in (1,2)$

1. There exists $C > 0$ s.t.

$$\left|D^2_{pp}H\right|^2 \leq CH$$

2. There exists $C > 0$ s.t.

$$\left|D^2_{pp}HM\right|^2 \leq C\,\mathrm{Tr}(D^2_{pp}HMM),$$

where M is any symmetric matrix

A model Hamiltonian

Let $\gamma \in (1,2)$ and consider

$$H(x,p) \;=\; a(x)\left(1+|p|^2\right)^{\frac{\gamma}{2}} + V(x)$$

Existence of solutions

1. Existence of weak solutions to (1)-(2)
   - J-M. Lasry and P-L. Lions, 2006
2. Existence of weak solutions to the planning problem
   - A. Porreta, 2013
3. Smooth solutions for quadratic Hamiltonians
   - P. Cardaliaguet, J-M. Lasry, P-L. Lions and A. Porreta, 2012

Existence of solutions

- Hamiltonians with quadratic or subquadratic growth and $g(m) = m^{\alpha}$:
  - Existence of smooth solutions for $\alpha > 0$ provided that $\gamma \in \left(1, 1 + \frac{1}{d+1}\right)$
  - Existence of smooth solutions for $\alpha < \frac{2}{d-2}$ provided that $\gamma \in \left(1 + \frac{1}{d+1}, 2\right)$
- P-L. Lions, 2012

Existence of classical solutions

Theorem (D. Gomes, E.P., H. Sanchez-Morgado)

Suppose that H is subquadratic and assume that $g(m) = m^{\alpha}$. If $\alpha < \alpha_{\gamma,d}$, then there exists a classical solution (u, m) to (1).

Existence of classical solutions

$$\alpha_{\gamma,d} = C\frac{2}{d-2}$$

where

$$C = \frac{2(\gamma-1)(4-\gamma)\gamma^2 + d(4+2(2-\gamma)\gamma)((4-\gamma)(2-\gamma)\gamma-4)}{(\gamma-1)(4-\gamma)\gamma(2(4-\gamma)\gamma-d(4+(2-\gamma)\gamma))}$$

Notice that

$$\lim_{\gamma\to 2}\alpha_{\gamma,d} = \frac{2}{d-2}$$

and

$$\alpha_{\gamma,d} > \frac{2}{d-2}$$

Outline

Regularity for the HJ equation

Lemma (Upper bounds)

Let (u, m) *be a solution to* (1)-(2) *and assume that* H *is subquadratic. Then,*

$$\|u\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))} \leq C + C\,\|g(m)\|_{L^{\frac{2(\gamma-1)r}{\gamma}}(0,T;L^{\frac{2(\gamma-1)p}{\gamma}}(\mathbb{T}^d))}$$

Regularity for the HJ equation

Lemma ($L^r L^p$-estimates)

Let (u, m) *be a solution to* (1)-(2) *and assume that* H *is subquadratic. Then,*

$$\|Du\|_{L^{2(\gamma-1)r}(0,T;L^{2(\gamma-1)p}(\mathbb{T}^d))} \leq C + C\,\|g(m)\|^{\frac{4-\gamma}{2(2-\gamma)}}_{L^{\frac{2(\gamma-1)r}{\gamma}}(0,T;L^{\frac{2(\gamma-1)p}{\gamma}}(\mathbb{T}^d))}$$

Polynomial estimates for the FP equation

Lemma (Polynomial estimates)

Let (u, m) be a solution to (1)-(2) and assume that H is subquadratic. Then,

$$\|g(m)\|_{L^{\frac{2(\gamma-1)r}{\gamma}}(0,T;L^{\frac{2(\gamma-1)p}{\gamma}}(\mathbb{T}^d))} \leq C + C\left\| |D_pH|^2 \right\|^{\theta}_{L^r(0,T;L^p(\mathbb{T}^d))}$$

$L^r L^p$-regularity

Lemma

Let (u, m) be a solution to (1)-(2) and assume that H is subquadratic. Then,

$$\|Du\|_{L^{2(\gamma-1)r}(0,T;L^{2(\gamma-1)p}(\mathbb{T}^d))} \leq C + C\,\|Du\|^{\theta}_{L^{2(\gamma-1)r}(0,T;L^{2(\gamma-1)p}(\mathbb{T}^d))}.$$

Moreover, if $\alpha < \alpha_{\gamma,d}$, we have that $\theta < 1$

Integrability for the FP equation

Corollary

Let (u, m) be a solution to (1)-(2) and assume that H is subquadratic. Then, if

$$\alpha < \alpha_{\gamma,d}$$

we have

$$m \in L^{\infty}(0, T; L^{\beta}(\mathbb{T}^d))$$

for every $\beta > 1$.

Outline

1. Introduction
   - Time dependent mean-field games
   - Subquadratic Hamiltonians
   - Previous results
   - Existence of classical solutions
2. Critical estimates
   - Gagliardo-Niremberg estimates
   - Polynomial dependence
   - Regularity for the HJ equation
   - Regularity for the FP equation
3. Further regularity
   - Further regularity for the HJ equation
   - Further regularity for the FP equation
4. Work in progress

$L^\infty L^\infty$-estimates

Corollary

Let (u, m) be a solution to (1)-(2) and assume that H is subquadratic. Then, if

$$\alpha < \alpha_{\gamma,d}$$

we have

$$Du \in L^\infty(\mathbb{T}^d \times [0, T]).$$

Lipschitz regularity for $\ln m$

Corollary

Let (u, m) be a solution to (1)-(2) and assume that H is subquadratic. Then, if $\alpha < \alpha_{\gamma,d}$, we have that $\ln m$ is Lipschitz and, therefore, m is bounded by above and below.

Hölder regularity for the FP equation

Corollary

Let (u, m) *be a solution to* (1)-(2) *and assume that H is. Then, if* $\alpha < \alpha_{\gamma,d}$, *we have that*

$$m \in \mathcal{C}^{0,\lambda}(\mathbb{T}^d \times [0, T]).$$

Outline

Improved regularity

- Improve the parameter $\alpha_{\gamma,d}$ by combining previous estimates with the fact that $Du \in L^\gamma(\mathbb{T}^d \times [0, T])$