LQG Mean-Field Games with ergodic cost in $\mathbb{R}^d$

Fabio S. Priuli
University of Roma Tor Vergata

joint work with: M. Bardi (Univ. Padova)

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[see also Lasry & Lions (2006, 2007), Guéant, Lasry & Lions (2011), Huang, Caines & Malhamé (2004, 2007), Bensoussan, Sung, Yam & Yung (2011)]

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We consider games with

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For simplicity we focus on nearly identical players, i.e., we assume

- all players have the same dynamics
- all players have the same cost for the control
- players are indistinguishable (*symmetry assumption*)

but analogous results hold also for general games!

Consider for $i = 1, \dots, N$

$$dX_t^i = (AX_t^i - \alpha_t^i)dt + \sigma\, dW_t^i \qquad X_0^i = x^i \in \mathbb{R}^d$$

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where $A \in \mathbb{R}^{d\times d}$, α_t^i controls, σ invertible, W_t^i Brownian, Consider for $i = 1, \ldots, N$

$$dX_t^i = (AX_t^i - \alpha_t^i)dt + \sigma\, dW_t^i \qquad X_0^i = x^i \in \mathbb{R}^d$$

$$J^i(X_0, \alpha^1, \ldots, \alpha^N) \doteq \liminf_{T\to\infty} \frac{1}{T}\,\mathbb{E}\Bigg[\int_0^T \frac{(\alpha_t^i)^T R\, \alpha_t^i}{2} + \underbrace{(X_t - \overline{X}_i)^T Q^i (X_t - \overline{X}_i)}_{F^i(X^1,\ldots,X^N)}\, dt\Bigg]$$

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where $A \in \mathbb{R}^{d\times d}$, α_t^i controls, σ invertible, W_t^i Brownian, $R \in \mathbb{R}^{d\times d}$ symm. pos. def., $X_t = (X_t^1, \dots, X_t^N) \in \mathbb{R}^{Nd}$ state var., $\overline{X}_i = (\overline{X_i^1}, \dots, \overline{X_i^N}) \in \mathbb{R}^{Nd}$ vector of favorite positions, $Q^i \in \mathbb{R}^{Nd\times Nd}$ block matrix

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$$F^i(X^1, \ldots, X^N) = \sum_{j,k=1}^{N} (X_t^j - \overline{X_i^j})^T Q^i_{jk} (X_t^k - \overline{X_i^k})$$

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- $Q^i_{ij} = Q^i_{ji} = B \quad \forall j \neq i$
- $Q^i_{jj} = C_i \quad \forall j \neq i$
- $Q^i_{jk} = Q^i_{kj} = D_i \quad \forall j \neq k \neq i \neq j$

Admissible strategies

A control α_t^i adapted to W_t^i is an *admissible strategy* if

- $\mathbb{E}[X_t^i], \mathbb{E}[X_t^i (X_t^i)^T] \leq C$ for all $t > 0$
- $\exists$ probability measure m_{α^i} s.t.

$$\lim_{T \to +\infty} \frac{1}{T} \mathbb{E}\left[\int_0^T g(X_t^i)\, dt\right] = \int_{\mathbb{R}^d} g(\xi)\, dm_{\alpha^i}(\xi)$$

for any polynomial g, with $\deg(g) \leq 2$, loc. unif. in X_0^i.

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Example. Any affine $\alpha^i(x) = Kx + c$ with "$K - A > 0$" is admissible and the corresponding diffusion process

$$dX_t^i = \big((A - K)X_t^i - c\big)dt + \sigma dW_t^i$$

is ergodic with $m_{\alpha^i} =$ multivariate Gaussian

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Nash equilibria

Any set of admissible strategies $\overline{\alpha}^1, \ldots, \overline{\alpha}^N$ such that

$$J^i(X, \overline{\alpha}^1, \ldots, \overline{\alpha}^N) = \min_{\omega} J^i(X, \overline{\alpha}^1, \ldots, \overline{\alpha}^{i-1}, \omega, \overline{\alpha}^{i+1}, \ldots, \overline{\alpha}^N)$$

for any $i = 1, \ldots, N$

For this N–players game HJB+KFP are given by

$$\begin{cases} -\mathrm{tr}(\nu D^2 v^i) + H(x, \nabla v^i) + \lambda^i = f^i(x; m^1, \dots, m^N) \\ -\mathrm{tr}(\nu D^2 m^i) + \mathrm{div}\Big(m^i \frac{\partial H}{\partial p}(x, \nabla v^i) \Big) = 0 \\ x \in \mathbb{R}^d \qquad m^i > 0 \qquad \int_{\mathbb{R}^d} m^i(x)\, dx = 1 \end{cases} \tag{1}$$

where

$$\nu = \frac{\sigma^T \sigma}{2} \qquad\qquad H(x,p) = p^T \frac{R^{-1}}{2} p - p^T A x$$

$$f^i(x; m^1, \dots, m^N) \doteq \int_{\mathbb{R}^{(N-1)d}} F^i(\xi^1, \dots, \xi^{i-1}, x, \xi^{i+1}, \dots \xi^N) \prod_{j \neq i} dm^j(\xi^j)$$

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$2N$ equations with unknowns λ^i, v^i, m^i, but always $x \in \mathbb{R}^d$!

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Search for solutions of Quadratic–Gaussian (QG) type + identically distr.

$$\lambda^i \in \mathbb{R} \qquad v^i(x) = x^T \frac{\Lambda}{2} x + \rho x \qquad m^i(x) = \gamma \exp\left\{ -\frac{1}{2}(x-\mu)^T \Sigma (x - \mu) \right\}$$

Theorem 1. For N–players LQG game

- Existence & uniquess λ^i, v^i, m^i sol. to (1) with v^i, m^i QG $\Leftrightarrow$ Algebraic conditions

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- $\overline{\alpha}^i = R^{-1}\nabla v^i(x)$ provides Nash equilibria strategies and $\lambda^i = J^i(X_0, \overline{\alpha}^1, \ldots, \overline{\alpha}^N)$ for $i = 1, \ldots, N$

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Proof.
(i) By plugging into (1)

$$v^i(x) = x^T \frac{\Lambda}{2} x + \rho x \qquad m^i(x) = \gamma \exp\left\{-\frac{1}{2}(x-\mu)^T \Sigma (x-\mu)\right\}$$

$\rightsquigarrow$ algebraic conditions on $\rho, \mu \in \mathbb{R}^d$, $\Lambda, \Sigma \in \mathbb{R}^{d\times d}$

(ii) Verification theorem, using Dynkin's formula and ergodicity

Indeed,

$$\nabla v^i(x) = \Lambda x + \rho \qquad \nabla m^i(x) = -m^i(x)\Sigma(x-\mu)$$

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HJB

$$\Sigma\,\frac{\nu R \nu}{2}\,\Sigma - \frac{A^T R A}{2} = Q$$

$$-\left(\frac{A^T R A}{2} + Q + (N-1)B\right)\mu = -Qh - (N-1)Br$$

$$(\mu)^T\,\frac{\Sigma\nu R\nu\Sigma}{2}\,\mu - \mathrm{tr}(\nu R\nu\Sigma + \nu R A) + \lambda^i = \mathfrak{f}^i(\Sigma,\mu)$$

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HJB

Σ solves ARE $\quad X\,\dfrac{\nu R\nu}{2}\,X - \left(\dfrac{A^T RA}{2} + Q\right) = 0$

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$\lambda^i =$ explicit function of Σ and μ

Algebraic conditions

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Uniqueness

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- there holds

$$\text{rank } \mathcal{B} = \text{rank } [\mathcal{B}, \mathcal{C}]$$

[iff the system $\mathcal{B}y = \mathcal{C}$ has solutions]

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- the unique $\Sigma > 0$ that solves ARE also solves Sylvester's eq.

$$X\nu R - R\nu X = RA - A^T R$$

[iff $R\big(\nu\Sigma + A\big)$ symm. matrix]

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Uniqueness

- $\mathcal{B}$ is invertible

  [iff the system $\mathcal{B}y = \mathcal{C}$ has solutions]

Mean field equations

Nearly identical players implies that the costs $F^i(X^1, \ldots, X^N)$ can be written as function of the empirical density of other players

$$F^i(X^1, \ldots, X^N) = \overline{V}_N^i \left[\frac{1}{N-1} \sum_{j \neq i} \delta_{X^j} \right] (X^i)$$

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$$\begin{aligned}
\overline{V}_N^i[m](X) \doteq (X-h)^T Q^N (X-h) \\
&+ (N-1) \int_{\mathbb{R}^d} \left((X-h)^T \frac{B^N}{2} (\xi - r) + (\xi - r)^T \frac{B^N}{2} (X-h) \right) dm(\xi) \\
&+ (N-1) \int_{\mathbb{R}^d} (\xi - r)^T (C_i^N - D_i^N)(\xi - r)\, dm(\xi) \\
&+ \left((N-1) \int_{\mathbb{R}^d} (\xi - r)\, dm(\xi) \right)^T D_i^N \left((N-1) \int_{\mathbb{R}^d} (\xi - r)\, dm(\xi) \right)
\end{aligned}$$

Assuming that the coefficients scale as follows as $N \to \infty$

$$Q^N \to \hat{Q} > 0 \qquad B^N(N-1) \to \hat{B}$$
$$C_i^N(N-1) \to \hat{C} \qquad D_i^N(N-1)^2 \to \hat{D}$$

then for any prob. measure m on $\mathbb{R}^d$ and all $i = 1, \ldots, N$

$$\overline{V}_N^i[m](X) \to \hat{V}[m](X) \qquad \text{loc. unif. in } X$$

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$$\begin{aligned}\hat{V}[m](X) \doteq\ & (X-h)^T \hat{Q}(X-h) \\ & + \int_{\mathbb{R}^d} \left((X-h)^T \frac{\hat{B}}{2} (\xi - r) + (\xi - r)^T \frac{\hat{B}}{2} (X-h) \right) dm(\xi) \\ & + \int_{\mathbb{R}^d} (\xi - r)^T \hat{C} (\xi - r)\, dm(\xi) \\ & + \left(\int_{\mathbb{R}^d} (\xi - r)\, dm(\xi) \right)^T \hat{D} \left(\int_{\mathbb{R}^d} (\xi - r)\, dm(\xi) \right)\end{aligned}$$

Thus passing formally to the limit as $N \to \infty$ in HJB+KFP

$$\begin{cases} -\mathrm{tr}(\nu D^2 u) + H(x, Du) + \lambda = \hat{V}[m](x) \\ -\mathrm{tr}(\nu D^2 m) - \mathrm{div}\left(m \, \frac{\partial H}{\partial p}(x, Du)\right) = 0 \\ x \in \mathbb{R}^d \qquad m > 0 \qquad \int_{\mathbb{R}^d} m(x)\, dx = 1 \end{cases} \qquad \text{(MFE)}$$

We look for solutions λ, u, m such that u, m is QG

$$u(x) = x^T \, \frac{\Lambda}{2} \, x + \rho x \qquad\qquad m(x) = \gamma \exp\left\{ -\frac{1}{2}(x-\mu)^T \Sigma (x-\mu) \right\}$$

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Theorem 2.

- Existence & uniquess λ, u, m sol. to MFE with u, m QG $\Leftrightarrow$ ALGEBRAIC CONDITIONS

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Theorem 2.

- Existence & uniquess λ, u, m sol. to MFE with u, m QG $\Leftrightarrow$ ALGEBRAIC CONDITIONS
- $\hat{V}$ is a monotone operator iff $\hat{B} \geq 0$ and if so QG sol. is the unique solution to MFE

Limit as $N \to \infty$

Theorem 3. Assume

(i) $Q^N \to \hat{Q}$ $\qquad B^N(N-1) \to \hat{B}$

$C_i^N(N-1) \to \hat{C}$ $\qquad D_i^N(N-1)^2 \to \hat{D}$

(ii) HJB+KFP for N–players admit QG sol. $(v_N, m_N, \lambda_N^1, \dots \lambda_N^N)$

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(iii) MFE admits unique QG solution (u, m, λ)

Then

- $v_N \to u$ in $\mathbf{C}^2_{loc}(\mathbb{R}^d)$
- $m_N \to m$ in $\mathbf{C}^k(\mathbb{R}^d)$ for all k
- $\lambda_N^i \to \lambda$ for all i

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- Algebraic conditions can be directly verified for some games

N–players game with $R = \mathfrak{r}\mathrm{I}_d$, $\nu = \mathfrak{n}\mathrm{I}_d$, A symmetric, $B \geq 0$

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- $\mathcal{B} = Q + \mathfrak{r}\dfrac{A^2}{2} + \dfrac{B}{2} > 0$

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$$\Sigma^2 = \frac{2}{\mathfrak{r}\mathfrak{n}^2}\left(\mathfrak{r}\frac{A^2}{2} + Q\right) \qquad \mathcal{B}\mu = \mathcal{C}$$

$$\Lambda = \mathfrak{r}\big(\mathfrak{n}\Sigma + A\big) \qquad \rho = -\mathfrak{r}\mathfrak{n}\Sigma\mu$$

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$$\Sigma = \frac{2}{\mathfrak{n}\sqrt{\mathfrak{r}}}\sqrt{\mathfrak{r}\frac{A^2}{2} + Q} \qquad \mu = \mathcal{B}^{-1}\mathcal{C}$$

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Thanks for Your Attention!