Non-asymptotic mean-field games

Tembine Hamidou

KAUST and Supelec

Mean Field Games and Related Topics - 2,
Padova, September 4-6, 2013

1. Introduction

2. Non-Asymptotic Mean-Field Games
    - How to exploit efficiently indistinguishability property?
    - Explicit Error Bound for arbitrary number of players

3. Auction with Asymmetric Bidders
    - Equilibrium structure: open problem
    - Long-term revenue of the auctionner

- Recently there has been renewed interest in large-scale optimatization, control, and games in several research disciplines, with its uses in
  - crowd safety, wireless networks, financial markets,
  - biology, power grid, big data analytics and cloud social networks.

  Classical works provide rich mathematical foundations and equilibrium concepts in the asymptotic regime, but relatively little in the way of computational and representational insights that would allow a non-asymptotic mean-field approach.
- This talk: non-asymptotic regime.

Indistinguishable Games.

- Indistinguishability: $\mathcal{A}_j = \mathcal{A}_i, \ \forall i,$
  $r_j(a_1, \ldots, a_n) = r_{\pi(j)}(a_{\pi(1)}, \ldots, a_{\pi(n)}), \ \forall \ \pi \in S_n$ (Permutation Group)

Generically, $r_j(a_j, a_{-j}) = \bar{r}\left(a_j, \overbrace{\frac{1}{n-1}\sum_{j' \neq j} \delta_{a_{j'}}}^{empirical \ \ measure \ \ of \ \ others' \ \ action}\right)$

Equilibrium: $a^*, \ \ \bar{r}(a_j^*, \frac{1}{n-1}\sum_{j'\neq j} \delta_{a^*_{j'}}) = \max_{a_j \in \mathcal{A}} \bar{r}(a_j, \frac{1}{n-1}\sum_{j'\neq j}\delta_{a^*_{j'}})$

- players change their behavior in relation to the aggregate.
- the aggregate should be consistent with the optimal decisions.

Refs.: Borel 1921, Volterra 1926, von Neumann'44, Nash'51, Wardrop'52, Aumann'64, Selten'70, Schmeidler'73, Dubey et al. 1980

For any $\epsilon > 0$, $\exists\ n_\epsilon |\ \ \forall\ n \geq n_\epsilon,\ |\bar{r}_n - \bar{r}_\infty| \leq \epsilon$. Thus, for $n < n_\epsilon$ the current theory does not give a meaningful approximation.

How about games with few players?

Is the mean-field approach extendable to games with few players?

Non-Asymptotic Mean-Field

suitable not only for large systems but also for small network with few number of players.

Key Idea

Efficient use of the indistinguishability property to derive approximation results.

Lemma

- $\partial_{a_j} r(\bar{m}^{\bigotimes n}) = \partial_{a_1} r(\bar{m}^{\bigotimes n})$ *where* $\bar{m}^{\bigotimes n} := (\bar{m}, \ldots, \bar{m})$
- *The structure of the payoff function implies that the first order term in the Taylor expansion is cancelled out.*
- $\partial^2_{a_i a_j} r(\bar{m}^{\bigotimes n}) = \partial^2_{a_1 a_2} r(\bar{m}^{\bigotimes n})$

Result

$$\| (a_1, \ldots, a_n) - \bar{m}^{\bigotimes n} \| \leq c_{\bar{m},r} \Longrightarrow \| r(a) - \bar{r}(\bar{m}) \| \leq \delta c^2_{\bar{m},r}$$

where $\bar{r}(\bar{m}) := r(\bar{m}, \ldots, \bar{m}) = r(\bar{m}^{\bigotimes n}), \delta > 0.$

Idea of proof: Thanks to indistinguishability, $\forall n \geq 2$,

$$\sum_{j=1}^{n} (a_j - \bar{m}) \underbrace{\partial_{a_j} r(\bar{m}^{\bigotimes n})}_{Independent\ \ of\ \ j} = 0$$

Works for all n. There is NO need for n to be large.

Explicit Error Bound for arbitrary number of players

Result

The gap $r(a) - \bar{r}(\bar{m})$ *is in order of* $O_2 := \delta_{\bar{m},\bar{r}} \sum_{j=1}^{n} (a_j - \bar{m})^2$, *where*

$$\delta_{\bar{m},\bar{r}} = \left| \frac{n}{2(n-1)} \left(-\frac{1}{n^2} \bar{r}''(\bar{m}) + \partial^2_{a_1 a_1} r(\bar{m}^{\bigotimes n}) \right) \right|$$

In order to compute the error bound, one needs only $\bar{r}$, and $\partial^2_{a_1 a_1} r(\bar{m}^{\bigotimes n})$.

Works for all $n \geq 2$. There is NO need for n to be large.

Display Ads and Online Auction

Examples: textual advertisements; commercial messages sent to mobile device users, or email; web display ads.

Problem

- $n \geq 2$ bidders, v_j :valuation of bidder j
- $\tilde{v}_j$ has a C^1-cumulative distribution function with support $[\underline{v}, \bar{v}]$
- a strategy of bidder j is a mapping : $v_j \longmapsto b_j(v_j)$
- payoff: $(v_j - b_j)\mathbb{P}\left(\max_{j' \neq j} b_{j'}(\tilde{v}_{j'})) < b_j)\right)$
- Revenue of the seller: $\mathbb{E}\max_j b_j(\tilde{v}_j)$.

Pure equilibrium: $r_{j,\mu_j}(b_j^*(v_j), b_{-j}^* | \ v_j) = \sup_{b_j} r_{j,\mu_j}(b_j, b_{-j}^* | \ v_j)$

Equilibrium: **Intractability** (even with small number of bidders).

Solve n ODEs with $2n$ boundary conditions... (G. Fibich, N. Gavish et al.'03)

$$\eta_j'(b) = \frac{F_j(\eta_j(b))}{F_j'(\eta_j(b))}\left[\left(\frac{1}{n-1}\sum_{j'=1}^{n}\frac{1}{\eta_{j'}(b)-b}\right) - \frac{1}{\eta_j(b)-b}\right],$$

$$\eta_j(\underline{v}) = \underline{v}, \ \eta_j(\bar{b}) = \bar{v}, j \in \{1, 2, \ldots, n\}$$

Equilibrium Strategies and Payoffs

From Vickrey[1] 1961, we know that in symmetric case,
$b(s) = s - \frac{\int_{\underline{v}}^{s} F^{n-1}(x)dx}{F^{n-1}(s)}$
For asymmetric distribution $F_j(v) = \bar{m}(v) + \epsilon\gamma_j(v)$,
$\bar{m}(v) = \frac{1}{n}\sum_{j'=1}^{n} F_{j'}(v)$,
$\epsilon = \max_j \max_{[\underline{v},\bar{v}]} \ |F_j(v) - \bar{m}(v)|$,
$\gamma_j(v) = \frac{F_j(v)-\bar{m}(v)}{\epsilon}$.
Under non-asymptotic mean-field framework :

- Good approximate of the asymmetric equilibrium strategies,
- Equilibrium payoff with deviation order of $O(\epsilon^2)$.
- This is true in first-price auction as well as in LUBA and second-price auction !

[1] Vickrey W. 1961. Counterspeculation, auctions, and competitive sealed tenders. J. Finance 16 8-37.

- FOC:

$$H_j(\bar{b},\eta) = 1 + (b - \eta_j(b)) \sum_{j' \neq j} \frac{F_j'(\eta_j(b))}{F_j(\eta_j(b))} \eta_j'(b).$$

- Polynomial truncature expansion of inverse-bid functions

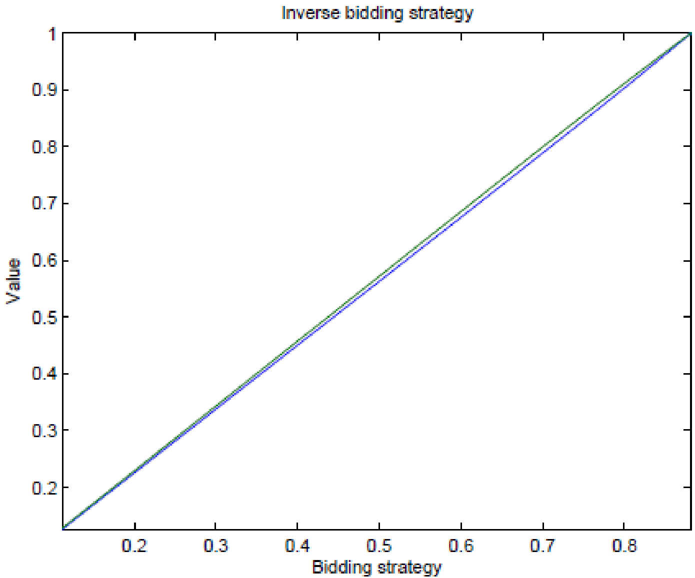
$$\hat{\eta}_{j,K}(b) = \bar{b} - \sum_{k=0}^{K} \xi_{j,k} (\bar{b} - b)^k.$$

- Adding $2n$ boundary conditions:

$$L(\bar{b},\eta) = \sum_{j=1}^{n} H_j(\bar{b},\eta)^2 + \sum_{j=1}^{n} (\eta_j(\bar{b}) - \bar{v})^2 + \sum_{j=1}^{n} (\eta_j(\underline{b}) - \underline{v})^2 \geq 0$$

- PROBLEM: $\inf_{(\hat{\bar{b}}_t)_t, (\xi_{j,k})_{j,k}} \sum_{t=1}^{T} L(\hat{\bar{b}}_t, \hat{\eta}_K)$.

Illustration: Inverse optimal strategy for $F_1(v) = v^7$, $F_2(v) = v^8$.


Inverse bidding strategy
Value
Bidding strategy
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.2
0.3
0.4
0.5
0.6
0.7
0.8

$$\mathcal{T} = \{0, 1, \ldots, T-1\},\ s_{j,t+1} = s_{j,t} - b_{j,t} \mathbb{1}_{\{b_{j,t} > \max_{j' \neq j} b_{j',t}\}}$$

The dynamic game is played as follows. At opportunity $t \in \mathcal{T}$, every player $j,\ j \in \mathcal{N}$,

- realizes his current value $v_{j,t} \in [\underline{v}, \bar{v}]$ distributed according to $F_{j,t}$
- submits a bid $b_{j,t} = \tau_{j,t}(v_{j,t}, h_{j,t})$, where $\tau_{j,t}: \ [\underline{v}, \bar{v}] \times \mathcal{H}_{j,t} \longrightarrow [\underline{b}, \bar{b}]$ denotes his bidding strategy at auction t;
- updates $h_{j,t+1} \in \mathcal{H}_{j,t+1}$, set of beliefs $b_{-j,t+1} \propto (F_{j',t+1},\ j' \neq j)$.

$$\hat{r}_{j,\mu_j} := \begin{cases} (v_j - b_j) & \text{if } j \text{ wins} \\ -\mu_j(v_k - b_k) & \text{if } k \neq j \text{ wins} \end{cases}$$

Each bidder maximizes her long-term payoff:

$$R_{j,T} = \mathbb{E}\left[g(s_{j,T}) + \sum_{t \in \mathcal{T}} \hat{r}_{j,\mu_j}(s_{j,t}, b_{j,t}, b_{-j,t}) \middle|\ v_j \right]$$

Result (Symmetric beliefs)

$$b_t^*(s_t, v) \quad \in \quad \arg\max_{b \le s_t} \mathbb{E}_{\bar{b}_t} \left[r_\mu + \mathbb{1}_{\{b > \bar{b}_t\}} V_{t+1}(s_t - b) + \mathbb{1}_{\{b < \bar{b}_t\}} V_{t+1}(s_t) \mid v \right],$$

$$b_t^*(s_t, v) = \begin{cases} x_1^* & \text{if } W_t > V_{t+1}(s_t) \\ x_2^* < \min\left(\bar{b}_t, s_t\right) & \text{otherwise} \end{cases}$$

$$\begin{aligned} V_t(s_t) \quad = \quad & \mathbb{E}_v \sup_b \mathbb{E}_{\bar{b}_t} \left[(v - b)\mathbb{1}_{\{b > \bar{b}_t\}} + \mathbb{1}_{\{b > \bar{b}_t\}} V_{t+1}(s_t - b) \right. \\ & \left. + \mathbb{1}_{\{b < \bar{b}_t\}} V_{t+1}(s_t) \mid v \right] \end{aligned} \tag{1}$$

Result (Non-asymptotic mean-field)

Let $\mu_j = \mu, F_{j,t}(v) = \bar{m}_t(v) + \epsilon\gamma_{j,t}(v)$, *where*

$$\gamma_{j,t}(v) = \frac{F_{j,t}(v) - \bar{m}_t(v)}{\epsilon}, \ \bar{m}_t(v) = \frac{1}{n}\sum_{j'=1}^{n} F_{j',t}(v),$$

$$\epsilon = \max_j \sup_{t\in\mathcal{T}} \sup_{[\underline{v},\bar{v}]} |F_{j,t}(v) - \bar{m}_t(v)|.$$

- *The long-term revenue of the auctionner* $R_\mu(s_0; F_1, \ldots, F_n)$ *satisfies Indistinguishability property.*
- $R_\mu(s_0; F_1, \ldots, F_n)$ *is in order of* $R_\mu(s_0; \bar{m}, \ldots, \bar{m}) + O(\epsilon^2)$ *for any* $n \geq 2$.

We have revisited mean field games in a very basic setup

Main assumption:

- Indistinguishability (and smoothness)

Observations:

- limitation of the asymptotic mean field approach,
- basic but interesting results from non-asymptotic mean field approach

Thank you !