Sensitivity analysis for HJB equations with an application to a coupled backward-forward system

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Outline

1. Sensitivity analysis for Hamilton-Jacobi-Bellman (HJB) equations

2. An application to mean field games

- $\mathbf{C}$: bounded continuous $f : \mathbf{R}^d \to \mathbf{R}$ with $\lim_{x\to\infty} f(x) = 0$ with sup norm.
- $\mathbf{C}^1$: continuously differentiable and bounded $f : \mathbf{R}^d \to \mathbf{R}$ such that $f' \in \mathbf{C}$, $\|f\|_{\mathbf{C}^1} := \sup_x |f(x)| + \sup_x |f'(x)|$.
- $\mathbf{C}^2$: twice continuously differentiable and bounded $f : \mathbf{R}^d \to \mathbf{R}$ such that $f' \in \mathbf{C}$ and $f'' \in \mathbf{C}$, $\|f\|_{\mathbf{C}^2} := \sup_x |f(x)| + \sup_x |f'(x)| + \sup_x |f''(x)|$.
- $\mathbf{C}_{Lip}$: Lipschitz continuous $f : \mathbf{R}^d \to \mathbf{R}$ with the norm $\|f\|_{\mathbf{C}_{Lip}} := \sup_x |f(x)| + \sup_{x,y} \frac{|f(x)-f(y)|}{|x-y|}$.
- $\mathcal{L}(\mathbf{C}^2, \mathbf{C})$: linear bounded operators from $\mathbf{C}^2$ to $\mathbf{C}$ equipped with $\|\cdot\|_{\mathbf{C}^2\to\mathbf{C}}$ and the strong operator topology.

Consider an optimal control problem

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An important example is

$$\mathbf{S} = (\mathbf{C}^2)^*, \text{the dual space of } \mathbf{C}^2$$
$$\mathcal{M} = \mathcal{P}(\mathbf{R}^d), \text{the set of probabilities on } \mathbf{R}^d$$
$$\|\mu\|_{(\mathbf{C}^2)^*} = \sup_{\|g\|_{\mathbf{C}^2}\le 1} \left| \int_{\mathbf{R}^d} g(x)\mu(dx) \right|.$$

$$\left\{ A[t, \mu, u] \in \mathcal{L}(\mathbf{C}^2, \mathbf{C}) : t \in [0, T], \mu \in \mathcal{M}, u \in \mathcal{U} \right\}$$

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For each $(t,\mu,u)\in[0,T]\times\mathcal{M}\times\mathcal{U}$, $A[t,\mu,u]:\mathbf{C}^2\mapsto\mathbf{C}$ is assumed to generate a Feller process with values in $\mathbf{R}^d$ and to be of the form

$$A[t,\mu,u]f(z)=(h(t,z,\mu,u),\nabla f(z))+L[t,\mu]f(z)$$

- $h:[0,T]\times\mathbf{R}^d\times\mathcal{M}\times\mathcal{U}\to\mathbf{R}^d$
- $L[t,\mu]\in\mathcal{L}(\mathbf{C}^2,\mathbf{C})$ is of the form:

$$L[t,\mu]f(z)=\frac{1}{2}(G(t,z,\mu)\nabla,\nabla)f(z)+(b(t,z,\mu),\nabla f(z))$$
$$+\int_{\mathbf{R}^d}(f(z+y)-f(z)-(\nabla f(z),y)\mathbf{1}_{B_1}(y))\nu(t,z,\mu,dy).$$

For given

$$\{u_\cdot\} = \{u_t \in \mathcal{U}, t \in [0,T]\}, \quad \{\mu_\cdot\} = \{\mu_t \in \mathcal{M}, t \in [0,T]\},$$

$\left(X_t^{\{\mu_\cdot\},\{u_\cdot\}} : t \in [0,T]\right)$ denotes the Markov process generated by $\{A[t,\mu_t,u_t]\}$ with

$$A[t,\mu_t,u_t]f(z) = (h(t,z,\mu_t,u_t), \nabla f(z)) + L[t,\mu_t]f(z).$$

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Example: If $L[t,\mu] = \frac{1}{2}\sigma^2\Delta$ with a constant σ,

$$dX_t^{\{\mu.\},\{u.\}} = h(t, X_t^{\{\mu.\},\{u.\}}, \mu_t, u_t)\, dt + \sigma dW_t.$$

The value function V is defined by

$$V(t, x; \{\mu.\}) := \sup_{\{u.\}} \mathbb{E}_x \left[\int_t^T J(s, X_s^{\{\mu.\},\{u.\}}, \mu_s, u_s)\, ds + V^T(X_T^{\{\mu.\},\{u.\}}, \mu_T) \right]$$

- $J : [0, T] \times \mathbf{R}^d \times \mathcal{M} \times \mathcal{U} \to \mathbf{R}$
- $V^T : \mathbf{R}^d \times \mathcal{M} \to \mathbf{R}$

Hamilton-Jacobi-Bellman (HJB) equation

$$-\frac{\partial V}{\partial t}(t,x;\{\mu.\}) = H(t,x,\nabla V(t,x;\{\mu.\}),\mu_t) + L[t,\mu_t]V(t,x;\{\mu.\})$$
$$V(T,x;\{\mu.\}) = V^T(x;\mu_T)$$

where $H : [0,T] \times \mathbf{R}^d \times \mathbf{R}^d \times \mathcal{M} \to \mathbf{R}$ is defined by

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We aim to investigate the sensitivity of the solution $V(t,x;\{\mu.\})$ with respect to the functional parameter $\{\mu.\}$.

Assumptions:

- (A_H) on $H(t, x, p, \mu)$
- (A_{V^T}) on $V^T(x; \mu)$
- (A_L) on $\{L[t, \mu] \in \mathcal{L}(\mathbf{C}^2, \mathbf{C}) : t \in [0, T], \mu \in \mathcal{M}\}$

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- (A_U) on backward propagator $\left\{U^{t,s}_{\{\mu.\}} \in \mathcal{L}(\mathbf{C},\mathbf{C}), t \le s\right\}$

For any $\{\mu.\}$, $f \in \mathbf{C}^2$ and all $t \le s \le r$, we have

$$\frac{d}{ds} U^{t,s}_{\{\mu.\}} f = U^{t,s}_{\{\mu.\}} L[s,\mu_s] f, \quad \frac{d}{ds} U^{s,r}_{\{\mu.\}} f = -L[s,\mu_s] U^{s,r}_{\{\mu.\}} f.$$

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- (A_U) on backward propagator $\left\{U^{t,s}_{\{\mu.\}} \in \mathcal{L}(\mathbf{C}, \mathbf{C}), t \leq s\right\}$

Smoothing property

For each $0 \leq t < s \leq T$,

$$U^{t,s}_{\{\mu.\}} : \mathbf{C} \to \mathbf{C}^1, \quad U^{t,s}_{\{\mu.\}} : \mathbf{C}_{Lip} \to \mathbf{C}^2,$$

and there exists a $\beta \in (0, 1)$ and constants $c_1, c_2 > 0$ such that

$$\|U^{t,s}_{\{\mu.\}}\phi\|_{\mathbf{C}^1} \leq c_1(s-t)^{-\beta}\|\phi\|_{\mathbf{C}}, \quad \|U^{t,s}_{\{\mu.\}}\psi\|_{\mathbf{C}^2} \leq c_2(s-t)^{-\beta}\|\psi\|_{\mathbf{C}_{Lip}}$$

for all $\phi \in \mathbf{C}$ and $\psi \in \mathbf{C}_{Lip}$.

Theorem 1 (KY2013)

Suppose $(A_H), (A_L), (A_U)$. For each fixed $\{\mu_\cdot\}$, if $V^T(\cdot; \mu_T) \in \mathbf{C}^1$, then there exists a unique mild solution V satisfying $V(t, \cdot; \{\mu_\cdot\}) \in \mathbf{C}^1$ for all $t \in [0, T]$.

For any given $\{\mu^1_\cdot\}, \{\mu^2_\cdot\}$, define $\bar{V} : [0, T] \times \mathbf{R}^d \times [0, 1] \to \mathbf{R}$ by

$$\bar{V}(t, x; \alpha) := V\left(t, x; \{\mu^1_\cdot\} + \alpha\{(\mu^2 - \mu^1)_\cdot\}\right)$$

where $\{\mu^1_\cdot\} + \alpha\left\{(\mu^2 - \mu^1)_\cdot\right\} := \left\{\mu^1_t + \alpha(\mu^2_t - \mu^1_t) \in \mathcal{M}, t \in [0, T]\right\}$.

For any given $\{\mu^1_\cdot\}, \{\mu^2_\cdot\}$, define $\bar{V} : [0, T] \times \mathbf{R}^d \times [0, 1] \to \mathbf{R}$ by

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$$\bar{V}(t, x; 1) - \bar{V}(t, x; 0) = V\left(t, x; \{\mu^2_\cdot\}\right) - V\left(t, x; \{\mu^1_\cdot\}\right)$$

For any given $\{\mu_\cdot^1\}, \{\mu_\cdot^2\}$, define $\bar{V} : [0,T] \times \mathbf{R}^d \times [0,1] \to \mathbf{R}$ by

$$\bar{V}(t,x;\alpha) := V\left(t,x;\{\mu_\cdot^1\} + \alpha\{(\mu^2 - \mu^1)_\cdot\}\right)$$

where $\{\mu_\cdot^1\} + \alpha\left\{(\mu^2 - \mu^1)_\cdot\right\} := \left\{\mu_t^1 + \alpha(\mu_t^2 - \mu_t^1) \in \mathcal{M}, t \in [0,T]\right\}$.

$$\bar{V}(t,x;1) - \bar{V}(t,x;0) = V\left(t,x;\{\mu_\cdot^2\}\right) - V\left(t,x;\{\mu_\cdot^1\}\right)$$

Define
$$\begin{aligned}\bar{H}(t,x,p,\alpha) &:= H(t,x,p,\mu_t^1 + \alpha(\mu_t^2 - \mu_t^1))\\ \bar{L}[t,\alpha] &:= L(t,\mu_t^1 + \alpha(\mu_t^2 - \mu_t^1))\\ \bar{V}^T(x;\alpha) &:= V^T(x;\mu_T^1 + \alpha(\mu_T^2 - \mu_T^1))\end{aligned}$$

with $\alpha \in [0,1]$, $t \in [0,T]$ and $(x,p) \in \mathbf{R}^{2d}$.

The sensitivity analysis of the solution $V(t,x;\{\mu.\})$ of

$$-\frac{\partial V}{\partial t}(t,x;\{\mu.\}) = H(t,x,\nabla V(t,x;\{\mu.\}),\mu_t) + L[t,\mu_t]V(t,x;\{\mu.\})$$
$$V(T,x;\{\mu.\}) = V^T(x;\mu_T)$$

w.r.t. a functional parameter$\{\mu.\}$ is reduced to that of the solution $\bar{V}(t,x;\alpha)$ of

$$-\frac{\partial \bar{V}}{\partial t}(t,x;\alpha) = \bar{H}(t,x,\nabla \bar{V}(t,x;\alpha),\alpha) + \bar{L}[t,\alpha]\bar{V}(t,x;\alpha)$$
$$\bar{V}(T,x;\alpha) = \bar{V}^T(x;\alpha).$$

w.r.t. a real parameter $\alpha \in [0,1]$.

Theorem 2 (KY2013)

Assume $(A_{V^T}), (A_L), (A_U)$. Define

$$W : [0,T] \times \mathbf{R}^d \times [0,1] \to \mathbf{R}^d, \qquad W(t,x;\alpha) = \bar{U}_{\alpha}^{t,T} \bar{V}^T(x;\alpha).$$

Then for $\alpha_1, \alpha_2 \in [0,1]$ with $\alpha_1 \neq \alpha_2$, there exists $c > 0$ s. t.

$$\frac{\|W(t,\cdot;\alpha_1) - W(t,\cdot;\alpha)\|_{\mathbf{C}^1}}{|\alpha_1 - \alpha_2|} \le c\Bigg((T-t)^{1-\beta} \|\bar{V}^T(\cdot;\alpha_2)\|_{\mathbf{C}^2} \sup_{\substack{s\in[t,T] \\ \gamma\in[\alpha_1,\alpha_2]}} \left\| \frac{\partial \bar{L}[s,\gamma]}{\partial \alpha} \right\|_{\mathbf{C}^2 \to \mathbf{C}} + \sup_{\gamma\in[\alpha_1,\alpha_2]} \left\| \frac{\partial \bar{V}^T(\cdot;\gamma)}{\partial \alpha} \right\|_{\mathbf{C}^1} \Bigg)$$

for every $t \in [0,T]$.

Theorem 3 (KY2013)

Assume $(A_H), (A_{V^T}), (A_L), (A_U)$. Then,

(a) For any $T > 0$, there exists a constant $c = c(T) > 0$ such that for each $\alpha_1, \alpha_2 \in [0,1]$ with $\alpha_1 \neq \alpha_2$,

$$\sup_{t\in[0,T]} \frac{\|\bar{V}(t,\cdot;\alpha_1) - \bar{V}(t,\cdot;\alpha_2)\|_{\mathbf{C}^1}}{|\alpha_1 - \alpha_2|}$$

$$\leq c\Bigg(\sup_{\gamma\in[\alpha_1,\alpha_2]} \left\| \frac{\partial \bar{V}^T(\cdot;\gamma)}{\partial\alpha} \right\|_{\mathbf{C}^1} + \sup_{\substack{(t,p)\in\mathcal{O}\\ \gamma\in[\alpha_1,\alpha_2]}} \left\| \frac{\partial \bar{H}(t,\cdot,p,\gamma)}{\partial\alpha} \right\|_{\mathbf{C}}$$

$$+ \sup_{\substack{t\in[0,T]\\ \gamma\in[\alpha_1,\alpha_2]}} \left\| \frac{\partial \bar{L}}{\partial\alpha}[t,\gamma] \right\|_{\mathbf{C}^2\to\mathbf{C}} \left(\left\| \bar{V}^T(\cdot;\alpha_2) \right\|_{\mathbf{C}^2} + 1 \right) \Bigg),$$

where $\mathcal{O} = \{(t,p) : t \in [0,T], |p| \leq \sup_{t\in[0,T]} \|\bar{V}(t,\cdot;\alpha)\|_{\mathbf{C}^1}\}$.

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(a) For any $T > 0$, there exists a constant $c = c(T) > 0$ such that for each $\alpha_1, \alpha_2 \in [0, 1]$ with $\alpha_1 \neq \alpha_2$,

$$\sup_{t\in[0,T]} \frac{\|\bar{V}(t,\cdot;\alpha_1) - \bar{V}(t,\cdot;\alpha_2)\|_{\mathbf{C}^1}}{|\alpha_1 - \alpha_2|}$$

$$\leq c\left(\sup_{\gamma\in[\alpha_1,\alpha_2]} \left\| \frac{\partial \bar{V}^T(\cdot;\gamma)}{\partial\alpha} \right\|_{\mathbf{C}^1} + \sup_{\substack{(t,p)\in\mathcal{O} \\ \gamma\in[\alpha_1,\alpha_2]}} \left\| \frac{\partial \bar{H}(t,\cdot,p,\gamma)}{\partial\alpha} \right\|_{\mathbf{C}} \right.$$

$$\left. + \sup_{\substack{t\in[0,T] \\ \gamma\in[\alpha_1,\alpha_2]}} \left\| \frac{\partial \bar{L}}{\partial\alpha}[t,\gamma] \right\|_{\mathbf{C}^2\to\mathbf{C}} \left(\left\| \bar{V}^T(\cdot;\alpha_2) \right\|_{\mathbf{C}^2} + 1 \right) \right),$$

where $\mathcal{O} = \{(t,p) : t \in [0,T], |p| \leq \sup_{t\in[0,T]} \|\bar{V}(t,\cdot;\alpha)\|_{\mathbf{C}^1}\}$.

$$\bar{V}(t,x;1) - \bar{V}(t,x;0) = V\left(t,x;\{\mu_\cdot^2\}\right) - V\left(t,x;\{\mu_\cdot^1\}\right)$$

Theorem 3 (KY2013)-continued

(b) For $\{\mu_\cdot^1\}, \{\mu_\cdot^2\} \in C([0,T], \mathcal{M})$, there exists $k > 0$ such that

$$\sup_{t\in[0,T]} \|V(t,\cdot;\{\mu_\cdot^1\}) - V(t,\cdot;\{\mu_\cdot^2\})\|_{\mathbf{C}^1} \le k \sup_{t\in[0,T]} \|\mu_t^1 - \mu_t^2\|_{\mathbf{S}}$$

and

$$\sup_{t\in[0,T]} \|\nabla V(t,\cdot;\{\mu_\cdot^1\}) - \nabla V(t,\cdot;\{\mu_\cdot^2\})\|_{\mathbf{C}} \le k \sup_{t\in[0,T]} \|\mu_t^1 - \mu_t^2\|_{\mathbf{S}}.$$

Outline

Backward HJB equation

$$-\frac{\partial V}{\partial t}(t,x;\{\mu.\}) = H(t,x,\nabla V(t,x;\{\mu.\}),\mu_t) + L[t,\mu_t]V(t,x;\{\mu.\})$$
$$V(T,x;\{\mu.\}) = V^T(x;\mu_T)$$
$$H(t,x,p,\mu) = \max_{u\in\mathcal{U}}(h(t,x,\mu,u)p + J(t,x,\mu,u))$$

Let $\hat{u}(t,x,\{\mu.\})$ denote the unique optimal control function.

Backward HJB equation

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Regularity condition of feedback control

For every $\{\eta.\}, \{\xi.\} \in C_{\mu_0}([0,T],\mathcal{P}(\mathbf{R}^d))$,

$$\sup_{(t,x)\in[0,T]\times\mathbf{R}^d} |\hat{u}(t,x;\{\eta.\}) - \hat{u}(t,x;\{\xi.\})| \le k_1 \sup_{s\in[0,T]} ||\eta_s - \xi_s||_{(\mathbf{C}^2)^*}$$

with some constant $k_1 > 0$.

Backward HJB equation

$$-\frac{\partial V}{\partial t}(t,x;\{\mu.\}) = H(t,x,\nabla V(t,x;\{\mu.\}),\mu_t) + L[t,\mu_t]V(t,x;\{\mu.\})$$

$$V(T,x;\{\mu.\}) = V^T(x;\mu_T)$$

$$H(t,x,p,\mu) = \max_{u\in\mathcal{U}}(h(t,x,\mu,u)p + J(t,x,\mu,u))$$

Let $\hat{u}(t,x,\{\mu.\})$ denote the unique optimal control function.

Forward kinetic equation

$$\frac{d}{dt}(g,\mu_t) = (A[t,\mu_t,\hat{u}(t,.,\{\mu.\})]g,\mu_t), \quad \mu_0 = \mu$$

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