Restriction Theorems for the Fourier Transform and Applications

Prof. Ana Vargas
Departamento de Matemáticas
Modulo17
Facultad de Ciencias, Universidad Autónoma de Madrid
ctra. de Colmenar Viejo, Km. 15
28049 Madrid
ana.vargas@uam.es

The problem of restriction of Fourier transform is about estimates for integrals of the form

$$\int_S f(\xi)e^{2\pi i x\cdot\xi}\, d\sigma(\xi),$$

where S is (for instance) a hypersurface in $\mathbb{R}^n$. These integrals arise when we write the solution of dispersive PDE's using the Fourier transform. They are also related with some classical problems in Harmonic Analysis, as the summation of Fourier series, the Kakeya problem, and to problems in Geometric Measure Theory, such as the distance problem. The aim of this course is to show the basic results for the restriction problem, its classical applications to related problems and a hint of the most recent results in this field.

1. Introduction. We will state the problem, and show with a simple example that the curvature plays some role. We will present the classical example (Knapp's example) to show what is the conjectured range and state the classical results, L^2 estimates, and the results in $\mathbb{R}^2$.

As motivation to study this problem we will see that the solutions of some equations, such as the Schödinger and wave equations, can be written in terms of the Fourier restriction operator. The L^2 restriction estimates (Strichartz's estimates) have been extensively used to deal with some non linear problems.

In order to prove some theorems, we need to review some classical tools to study the decay of the oscillatory integrals, such as the stationary phase principle. In the case of the paraboloid, this decay can also be obtained by an explicit computation of the kernel of the operator.

2. The Stein–Tomas theorem. Strichartz's estimates. We will define some auxiliary operators, the fractionary integrals, and prove the $L^p \to L^q$ estimates for them (Hardy–Littlewood–Sobolev theorem. We will use this to prove the classical L^2 restriction theorem. One classical application of this result is to prove the well possedness of the Cauchy problem for some non linear Schrödinger equations, using a fixed point argument. The restriction estimates can also be applied to obtain L^p boundedness of the operators of summation of Fourier integrals, the Bochner–Riesz multiplier operators.

3. Bilinear restriction theorems. The most recent results for the problem of restriction have been obtained via the "bilinear approach". We will present this method in $\mathbb{R}^2$, proving the full restriction conjecture in this setting.

The main achievement of this method has been the bilinear restriction theorems in higher dimensions proved by Wolff and Tao. We will show some of the ideas of their proofs. We will also discuss the trilinear theorem due to Bennett, Carbery and Tao, and its application to the most recent result in this theory, given by Bourgain and Guth.

References

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