

SPECTRAL CONVERGENCE AND NONLINEAR DYNAMICS

JOSE M. ARRIETA[†]

These notes are divided in five sections, each of them constitute approximately the contents of a lecture in the Minicourse.

1. INTRODUCTION AND A FAST OVERVIEW ON ATTRACTORS.

The purpose of these notes is to address the behavior of the asymptotic dynamics of a reaction-diffusion equation when the domain is perturbed. The equations are given by

$$\begin{cases} u_t - \Delta u + u = f(u) & \text{in } \Omega_\epsilon \\ \frac{\partial u}{\partial n} = 0 & \text{in } \partial\Omega_\epsilon. \end{cases} \quad (1.1)$$

where Ω_ϵ , $0 \leq \epsilon \leq \epsilon_0$, are bounded Lipschitz domains in \mathbb{R}^N , $N \geq 2$. We analyze how the asymptotic dynamics of the evolutionary problem (1.1) changes when we vary the domain. In particular, we are interested in studying how the behavior of the spectral properties of the linear operator $-\Delta$ under variations of the domain, determines the behavior of the nonlinear dynamics of (1.1).

The nonlinearity f is assumed to be smooth enough, say C^2 and to simplify, we will assume that the following condition holds:

$$|f(s)| + |f'(s)| + |f''(s)| \leq L_f \quad \forall s \in \mathbb{R} \quad (1.2)$$

for some positive constant L_f .

We will regard Ω_ϵ as a perturbation of the fixed domain Ω_0 and to simplify the exposition, we will assume throughout these notes that the domains are uniformly bounded, that the perturbation is an “exterior” perturbation of the domain, and that they converge in measure. We summarize this conditions in the following hypothesis:

$$\left\{ \begin{array}{l} \text{For each } 0 \leq \epsilon \leq \epsilon_0, \quad \Omega_\epsilon \text{ is a Lipschitz domain, there exists } R > 0 \\ \text{such that } \Omega_0 \subset \Omega_\epsilon \subset B(0, R), \text{ and } |\Omega_\epsilon \setminus \Omega_0| \rightarrow 0, \text{ as } \epsilon \rightarrow 0 \end{array} \right\} \quad (1.3)$$

One of the main difficulties when treating domain perturbation problems is that our functions live in different spaces (say $u_\epsilon \in H^1(\Omega_\epsilon)$ and $u_0 \in H^1(\Omega_0)$) and therefore statements of the type $u_\epsilon - u_0$ should be interpreted clearly. In these notes we will consider the space

$$H_\epsilon^1 = H^1(\Omega_0) \oplus H^1(\Omega_\epsilon \setminus \bar{\Omega}_0) \quad (1.4)$$

that is $H_\epsilon^1 = \{\phi \in L^2(\Omega_\epsilon), \text{ such that } \phi|_{\Omega_0} \in H^1(\Omega_0), \phi|_{\Omega_\epsilon \setminus \bar{\Omega}_0} \in H^1(\Omega_\epsilon \setminus \bar{\Omega}_0)\}$ with the norm $\|u\|_{H_\epsilon^1}^2 = \|u\|_{H^1(\Omega_0)}^2 + \|u\|_{H^1(\Omega_\epsilon \setminus \bar{\Omega}_0)}^2$. Notice that extending by zero outside Ω_0 we have $H^1(\Omega_0) \hookrightarrow H_\epsilon^1$, with embedding constant 1 and in a natural way we have $H^1(\Omega_\epsilon) \hookrightarrow H_\epsilon^1$, with embedding constant also 1. Hence if $u_\epsilon \in H^1(\Omega_\epsilon)$, $u_0 \in H^1(\Omega_0)$ we can write $\|u_\epsilon - u_0\|_{H_\epsilon^1}$. Moreover with certain abuse of notation we will say that $u_\epsilon \rightarrow u_0$ in H_ϵ^1 if $\|u_\epsilon - u_0\|_{H_\epsilon^1} \rightarrow 0$ as $\epsilon \rightarrow 0$.

Also, with an extension by zero outside Ω_ϵ or Ω_0 , $L^2(\Omega_\epsilon) \hookrightarrow L^2(\mathbb{R}^N)$ and $L^2(\Omega_0) \hookrightarrow L^2(\mathbb{R}^N)$. Hence, for functions $V_\epsilon \in L^2(\Omega_\epsilon)$, $V_0 \in L^2(\Omega_0)$, statements of the type $V_\epsilon \rightarrow V_0$ in $L^2(\mathbb{R}^N)$ or $w - L^2(\mathbb{R}^N)$ make perfect sense. Moreover, if we have an operator T acting on $L^2(\Omega_\epsilon)$ we may also

[†] Universidad Complutense de Madrid.

regard this operator as acting on $L^2(\Omega_0)$ by just viewing any element $u_0 \in L^2(\Omega_0)$ as an element of $L^2(\Omega_\epsilon)$ by extending u_0 outside Ω_0 by zero and then making the restriction to Ω_ϵ . Similarly we can do with operators defined in $L^2(\Omega_0)$ with the restriction operator.

Notice also, that since the domain Ω_0 is Lipschitz, we have a bounded extension operator, $E : H^1(\Omega_0) \rightarrow H^1(\mathbb{R}^N)$, which is also a extension operator from $L^2(\Omega_0) \rightarrow L^2(\mathbb{R}^N)$.

We want to analyze equation (1.1) for a fixed value of ϵ , establishing the existence of solutions and obtaining basic properties. We regard the differential operator $-\Delta + I$ as an unbounded operator

$$\begin{aligned} A_\epsilon : D(A_\epsilon) \subset L^2(\Omega_\epsilon) &\rightarrow L^2(\Omega_\epsilon) \\ u &\rightarrow -\Delta u + u \end{aligned}$$

where $D(A_\epsilon) = \{u \in H^2(\Omega_\epsilon) : \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega_\epsilon\}$. This operator is selfadjoint, that is, $(Au, v)_{L^2} = (u, Av)_{L^2}$ and since Ω_ϵ is bounded and smooth, the spectrum is discrete. That is, $\sigma(A_\epsilon) = \{\lambda_i^\epsilon\}_{i=1}^\infty$ with $1 = \lambda_1^\epsilon \leq \lambda_2^\epsilon \leq \dots$ where $\lambda_m^\epsilon \rightarrow +\infty$ as $m \rightarrow +\infty$. Notice that the first eigenvalue is $\lambda_1^\epsilon = 1$, due to the Neumann boundary condition and the fact that $A_\epsilon = -\Delta + I$. The eigenfunctions are denoted by $\{\phi_i^\epsilon\}_{i=1}^\infty$ which we will assume they form a complete orthonormal set in $L^2(\Omega_\epsilon)$, that is,

$$(\phi_i^\epsilon, \phi_j^\epsilon)_{L^2(\Omega_\epsilon)} = \int_{\Omega_\epsilon} \phi_i^\epsilon(x) \phi_j^\epsilon(x) dx = \delta_{ij}$$

and they form a basis in $L^2(\Omega_\epsilon)$ (and also in $H^1(\Omega_\epsilon)$). We will always denote by (\cdot, \cdot) the inner product in $L^2(\Omega_\epsilon)$. Notice that if $z_\epsilon \in L^2(\Omega_\epsilon)$ we can write $z_\epsilon = \sum_{i=1}^\infty (z_\epsilon, \phi_i^\epsilon) \phi_i^\epsilon$. In particular,

$$\|z_\epsilon\|_{L^2(\Omega_\epsilon)}^2 = \sum_{i=1}^\infty (z_\epsilon, \phi_i^\epsilon)^2.$$

In case $z_\epsilon \in H^1(\Omega_\epsilon)$ we also have

$$\|z_\epsilon\|_{H^1(\Omega_\epsilon)}^2 = \sum_{i=1}^\infty (z_\epsilon, \phi_i^\epsilon)^2 \lambda_i^\epsilon$$

and if $z_\epsilon \in D(A_\epsilon)$ we have

$$\|z_\epsilon\|_{H^2(\Omega_\epsilon)}^2 \leq C_\epsilon \sum_{i=1}^\infty (z_\epsilon, \phi_i^\epsilon)^2 (\lambda_i^\epsilon)^2,$$

where the constant C_ϵ comes from the embedding $D(A_\epsilon) \hookrightarrow H^2(\Omega_\epsilon)$ and $D(A_\epsilon)$ is endowed with the graph norm $\|A_\epsilon \cdot\|_{L^2(\Omega_\epsilon)}$.

Moreover, the operator A_ϵ generates a family of operators $S_\epsilon(t) \equiv e^{-A_\epsilon t}$,

$$\begin{aligned} e^{-A_\epsilon t} : L^2(\Omega_\epsilon) &\rightarrow L^2(\Omega_\epsilon) \\ z_\epsilon &\rightarrow e^{-A_\epsilon t} z_\epsilon \end{aligned}$$

which is defined as the unique solution of the linear evolution problem

$$\begin{cases} u_t - \Delta u + u = 0 & \text{in } \Omega_\epsilon \\ \frac{\partial u}{\partial n} = 0 & \text{in } \partial\Omega_\epsilon \\ u(0) = z_\epsilon \in L^2(\Omega_\epsilon) \end{cases} \quad (1.5)$$

This operators admit a very nice expression in terms of the eigenvalues and eigenfunctions of the operators A_ϵ . As a matter of fact, we may write

$$e^{-A_\epsilon t} z_\epsilon = \sum_{i=1}^{\infty} (z_\epsilon, \phi_i^\epsilon) e^{-\lambda_i^\epsilon t} \phi_i^\epsilon.$$

This operator, which we will also denote as $S_\epsilon(t)$ has several nice properties:

- $e^{-A_\epsilon t}$ is a bounded operator from $L^2(\Omega_\epsilon)$ into itself, with bound $\|e^{-A_\epsilon t} z_\epsilon\|_{L^2(\Omega_\epsilon)} \leq e^{-\lambda_1^\epsilon t} \|z_\epsilon\|_{L^2(\Omega_\epsilon)} = e^{-t} \|z_\epsilon\|_{L^2(\Omega_\epsilon)}$, where we have used that $\lambda_1^\epsilon = 1$. We also have $\|e^{-A_\epsilon t} z_\epsilon\|_{H^1(\Omega_\epsilon)} \leq e^{-t} \|z_\epsilon\|_{H^1(\Omega_\epsilon)}$
- For fixed $t > 0$ the operator $e^{-A_\epsilon t} : L^2(\Omega_\epsilon) \rightarrow L^2(\Omega_\epsilon)$ is a compact operator. As a matter of fact, it is not difficult to see that $S_\epsilon(t)$ transforms $L^2(\Omega_\epsilon)$ in $H^1(\Omega_\epsilon)$ and it is a bounded operator between these two spaces. Actually,

$$\|e^{-A_\epsilon t} z_\epsilon\|_{H^1(\Omega_\epsilon)}^2 = \sum_{i=1}^{\infty} (z_\epsilon, \phi_i^\epsilon)^2 \lambda_i^\epsilon e^{-2\lambda_i^\epsilon t} = t^{-1} e^{-\lambda_1^\epsilon t} \sum_{i=1}^{\infty} (z_\epsilon, \phi_i^\epsilon)^2 \lambda_i^\epsilon t e^{-\lambda_i^\epsilon t}$$

but

$$\lambda_i^\epsilon t e^{-\lambda_i^\epsilon t} \leq \sup_{x \geq 0} x e^{-x} = e^{-1} \leq 1$$

which implies that

$$\|e^{-A_\epsilon t} z_\epsilon\|_{H^1(\Omega_\epsilon)}^2 \leq e^{-t} t^{-1} \|z_\epsilon\|_{L^2(\Omega_\epsilon)}^2$$

where we have used that $\lambda_1^\epsilon \geq 1$. This implies,

$$\|e^{-A_\epsilon t}\|_{\mathcal{L}(L^2(\Omega_\epsilon), H^1(\Omega_\epsilon))} \leq e^{-t/2} t^{-1/2} \quad (1.6)$$

A very similar argument will show that $e^{-A_\epsilon t} : L^2(\Omega_\epsilon) \rightarrow H^2(\Omega_\epsilon)$ and it is a bounded operator. This time the estimate obtained is of the type

$$\|e^{-A_\epsilon t}\|_{\mathcal{L}(L^2(\Omega_\epsilon), H^2(\Omega_\epsilon))} \leq C_\epsilon e^{-t/2} t^{-1} \quad (1.7)$$

where the constant C_ϵ comes from the embedding of $D(A_\epsilon) \hookrightarrow H^2(\Omega_\epsilon)$.

- $S_\epsilon(0) = I$
- $S_\epsilon(t+s) = S_\epsilon(t) \circ S_\epsilon(s)$, for each $t, s \geq 0$ (this is the semigroup property).

With respect to the nonlinear evolution problem (1.1), we first notice that if the nonlinearity f satisfies (1.2), then considering the composition operator generated by f acting in $L^2(\Omega_\epsilon)$ we have, that $f \in C^{0,1}(L^2(\Omega_\epsilon), L^2(\Omega_\epsilon))$ with Lipschitz constant L_f , see (1.2) (which is uniformly bounded in ϵ). This is not difficult to see:

$$\|f(u) - f(v)\|_{L^2(\Omega_\epsilon)}^2 = \int_{\Omega_\epsilon} |f(u_\epsilon(x)) - f(v_\epsilon(x))|^2 dx \leq L_f^2 \int_{\Omega_\epsilon} |u_\epsilon(x) - v_\epsilon(x)|^2 dx = L_f^2 \|u_\epsilon - v_\epsilon\|_{L^2(\Omega_\epsilon)}^2$$

Moreover $f \in C^{1,\theta}(H^1(\Omega_\epsilon), L^2(\Omega_\epsilon))$ where the value $\theta = 1$ in dimension ≤ 4 and $\theta = 2/(N-2)$ in dimension $N \geq 5$. As a matter of fact the differentiability is better in dimensions $N = 1, 2, 3$ but for the purposes of these notes, we will not need more than just the $C^{1,\theta}$ regularity. To show this, notice that the Frechet derivative of f at u_ϵ as function from $H^1(\Omega_\epsilon) \rightarrow L^2(\Omega_\epsilon)$ is given by

$$\begin{aligned} Df(u_\epsilon) : H^1(\Omega_\epsilon) &\rightarrow L^2(\Omega_\epsilon) \\ \varphi_\epsilon &\rightarrow f'(u_\epsilon) \varphi_\epsilon \end{aligned}$$

which is well defined because of (1.2). Moreover,

$$\begin{aligned} \|Df(u_\epsilon) \varphi_\epsilon - Df(v_\epsilon) \varphi_\epsilon\|_{L^2(\Omega_\epsilon)} &= \|(f'(u_\epsilon) - f'(v_\epsilon)) \varphi_\epsilon\|_{L^2(\Omega_\epsilon)} \\ &\leq \|f'(u_\epsilon) - f'(v_\epsilon)\|_{L^N(\Omega_\epsilon)} \|\varphi_\epsilon\|_{L^{2N/(N-2)}(\Omega_\epsilon)} \leq C_\epsilon \|f'(u_\epsilon) - f'(v_\epsilon)\|_{L^N(\Omega_\epsilon)} \|\varphi_\epsilon\|_{H^1(\Omega_\epsilon)} \end{aligned}$$

where C_ϵ comes from the embedding $H^1(\Omega_\epsilon) \hookrightarrow L^{2N/(N-2)}(\Omega_\epsilon)$. Hence,

$$\|Df(u_\epsilon) - Df(v_\epsilon)\|_{\mathcal{L}(H^1(\Omega_\epsilon), L^2(\Omega_\epsilon))} \leq C_\epsilon \|f'(u_\epsilon) - f'(v_\epsilon)\|_{L^N(\Omega_\epsilon)}$$

But, from the fact that f' is Lipschitz and also bounded, (see (1.2)), we easily get

$$|f'(u_\epsilon(x)) - f'(v_\epsilon(x))| \leq 2L_f \max\{1, |u_\epsilon(x) - v_\epsilon(x)|\}$$

and this implies that for any $0 < \alpha \leq 1$, we have

$$|f'(u_\epsilon(x)) - f'(v_\epsilon(x))| \leq 2L_f |u_\epsilon(x) - v_\epsilon(x)|^\alpha$$

Hence, choosing $\alpha = 2/(N-2)$ for $N \geq 4$, we get

$$\|f'(u_\epsilon) - f'(v_\epsilon)\|_{L^N(\Omega_\epsilon)} \leq C_\epsilon \|u_\epsilon - v_\epsilon\|_{L^{2N/(N-2)}(\Omega_\epsilon)}^{2/(N-2)}$$

which shows the above result.

The solutions of (1.1) with initial condition $u(0) = z_\epsilon$ are obtained through the Variation of Constant Formula:

$$u(t, z_\epsilon) = e^{-A_\epsilon t} z_\epsilon + \int_0^t e^{-A_\epsilon(t-s)} f(u(s, z_\epsilon)) ds, \quad (1.8)$$

which is proved to have solutions through an appropriate fixed point argument.

Since the nonlinearity f is globally Lipschitz, we have that solutions of (1.1) or more exactly of (1.8) are globally defined (that is, they exist for $t \geq 0$). This allows us to define a family of nonlinear operators

$$\begin{aligned} T_\epsilon(t) : H^1(\Omega_\epsilon) &\rightarrow H^1(\Omega_\epsilon) \\ z_\epsilon &\rightarrow u(t, z_\epsilon) \end{aligned}$$

where $u(t, z_\epsilon)$ is given by (1.8). This family has the following properties

- $T_\epsilon(0) = I$
- $T_\epsilon(t+s) = T_\epsilon(t) \circ T_\epsilon(s)$
- $T_\epsilon(t) \in C^1(H^1(\Omega_\epsilon), H^1(\Omega_\epsilon))$
- $t \rightarrow T_\epsilon(t)z_\epsilon$ is in $C([0, \infty), H^1(\Omega_\epsilon))$

and we will refer to it as the “nonlinear semigroup” (or nonlinear semiflow) generated by (1.1). This nonlinear semigroup contains all the information of the solutions of (1.1). It actually enjoys several nice properties in terms of the asymptotic behavior of the solutions of (1.1).

Dissipativity. This is a key property and it has to do with the “ultimate boundedness” of the orbits of T_ϵ . We will show that there exists $R > 0$ such that for any $z_\epsilon \in H^1(\Omega_\epsilon)$ there exists a positive time $\tau = \tau(\|z_\epsilon\|_{H^1(\Omega_\epsilon)})$ with the property that

$$\|T_\epsilon(t)z_\epsilon\|_{H^1(\Omega_\epsilon)} \leq R, \quad \forall t \geq \tau$$

Moreover, this R can be chosen independent to ϵ . To see this, let us consider an initial condition $z_\epsilon \in H^1(\Omega_\epsilon)$ and estimate in (1.8):

$$\begin{aligned} \|u(t, z_\epsilon)\|_{H^1(\Omega_\epsilon)} &\leq e^{-t} \|z_\epsilon\|_{H^1(\Omega_\epsilon)} + \int_0^t \|e^{-A_\epsilon(t-s)}\|_{\mathcal{L}(L^2, H^1)} \|f(u(s, z_\epsilon))\|_{L^2} ds \\ &\leq e^{-t} \|z_\epsilon\|_{H^1} + L_f |\Omega_\epsilon|^{1/2} \int_0^t e^{-(t-s)/2} (t-s)^{-1/2} ds \\ &\leq e^{-t} \|z_\epsilon\|_{H^1} + C \int_0^t e^{-s/2} s^{-1/2} ds \end{aligned}$$

where C is an upper bound of $L_f|\Omega_\epsilon|^{1/2}$. Now, if $\tau = \log(1 + \|z_\epsilon\|_{H^1})$ we have that for $t \geq \tau$,

$$\|u(t)\|_{H^1(\Omega_\epsilon)} \leq 1 + C \int_0^\infty e^{-s/2} s^{-1/2} ds \equiv R$$

Orbits of bounded sets are bounded. We can see that if we have $\rho > 0$, then the positive orbit of the bounded set $\{z_\epsilon \in H^1(\Omega_\epsilon) : \|z_\epsilon\|_{H^1(\Omega_\epsilon)} \leq \rho\}$, that is

$$\bigcup_{t>0} \bigcup_{\|z_\epsilon\| \leq \rho} T_\epsilon(t)z_\epsilon$$

is bounded in $H^1(\Omega_\epsilon)$. Actually, from the dissipativity proof, we get that

$$\|T_\epsilon(t)z_\epsilon\|_{H^1(\Omega_\epsilon)} \leq \rho + C \int_0^\infty e^{-s/2} s^{-1/2} ds$$

Compactness. For each $t > 0$ fixed, the nonlinear operator $T_\epsilon(t) : H^1(\Omega_\epsilon) \rightarrow H^1(\Omega_\epsilon)$ is a compact map. To see this, we just observe from (1.8) that T_ϵ is the sum of two maps: a linear one, given by $e^{-A_\epsilon t}$ and a nonlinear one given by the integral part. The linear is definitely compact as we have shown above. The integral can be written as follows (where $\eta > 0$ is a small number):

$$\int_0^t e^{-A_\epsilon s} f(T(t-s)z_\epsilon) ds = \int_0^\eta e^{-A_\epsilon s} f(T(t-s)z_\epsilon) ds + e^{-A_\epsilon \eta} \int_\eta^t e^{-A_\epsilon(s-\eta)} f(T(t-s)z_\epsilon) ds$$

But the first integral is as small as we want by making $\eta \rightarrow 0$. The second integral lies in a compact set of $H^1(\Omega_\epsilon)$ for each $\eta > 0$. This decomposition implies the compactness, see [5, Theorem 4.2.2, page 73].

With these three properties we have

Theorem 1.1. *The dynamical system $T_\epsilon(t)$, generated by (1.1) has an attractor $\mathcal{A}_\epsilon \subset H^1(\Omega_\epsilon)$*

Definition 1.2. *The attractor of the dynamical system $T_\epsilon(t)$ is the set $\mathcal{A}_\epsilon \subset H^1(\Omega_\epsilon)$ which is:*

- **Invariant.** *That is, $T_\epsilon(t)\mathcal{A}_\epsilon = \mathcal{A}_\epsilon$.*
- **Compact.**
- **Attracts bounded sets:** *$\forall B \subset H^1(\Omega_\epsilon)$ bounded set, we have*

$$\text{dist}_{H^1(\Omega_\epsilon)}(T_\epsilon(t)B, \mathcal{A}_\epsilon) \rightarrow 0, \quad \text{as } t \rightarrow +\infty$$

In the definition above dist_X is the (non-symmetric) Hausdorff distance in the metric space X that is,

$$\text{dist}_X(A, B) = \sup_{a \in A} \inf_{b \in B} d_X(a, b).$$

Because of the properties above, the attractor has the following properties:

- (1) All bounded global orbits are contained in the attractor \mathcal{A}_ϵ . In particular all stationary solutions (equilibria), all periodic orbits, all connecting orbits between orbits are contained in the attractor \mathcal{A}_ϵ .
- (2) For all $\varphi_\epsilon \in \mathcal{A}_\epsilon$ there exists a global orbit contained in the attractor passing through this point.

For the equation we are dealing with, we have some extra important tool:

Liapunov functions. Consider the functional $E : H^1(\Omega_\epsilon) \rightarrow \mathbb{R}$ defined as

$$E(u) = \frac{1}{2} \int_{\Omega} (|\nabla u(x)|^2 + |u(x)|^2) dx - \int_{\Omega} F(u(x)) dx$$

where F is a primitive of f , that is $F'(s) = f(s)$ for all $s \in \mathbb{R}$. It is not difficult to see that $E \in C^1(H^1(\Omega_\epsilon), \mathbb{R})$ and moreover

$$\frac{d}{dt} E(T_\epsilon(t)z_\epsilon) = DE(T_\epsilon(t)z_\epsilon) \circ \frac{d}{dt} T_\epsilon(t) = - \int_{\Omega_\epsilon} u_t^2 \leq 0.$$

So the function $t \rightarrow E(T_\epsilon(t)z_\epsilon)$ is non increasing along solutions (actually it is strictly decreasing along solutions except at stationary solutions, where $u_t = 0$). In particular this implies:

- No periodic orbits
- No homoclinic connections or even loops.

A system with a Lyapunov function is called a Gradient System and its attractor has the simplest possible structure: it is formed by equilibria and connections among them.

In particular, if we have only a finite number of stationary solutions, $\{e_1^\epsilon, \dots, e_n^\epsilon\} \subset H^1(\Omega_\epsilon)$ then the attractor is characterized as follows: $z_\epsilon \in \mathcal{A}_\epsilon$ then we have only two possibilities

- z_ϵ is an equilibrium point, or
- z_ϵ has a global orbit $\gamma_\epsilon(t)$ with $\gamma_\epsilon(0) = z_\epsilon$ and $\gamma_\epsilon(t) \rightarrow e_j^\epsilon$ as $t \rightarrow +\infty$ and $\gamma_\epsilon(t) \rightarrow e_i^\epsilon$ as $t \rightarrow -\infty$ for some $i \neq j$.

So we are in a situation where for each value of the parameter ϵ we have an attractor $\mathcal{A}_\epsilon \subset H^1(\Omega_\epsilon)$ and we want to understand its behavior when we perturb the domain. As a matter of fact, in these notes we give conditions on the behavior of Ω_ϵ as $\epsilon \rightarrow 0$ and on the unperturbed problem, (1.1) with $\epsilon = 0$, that guarantee the continuity (upper and lower semicontinuity) of the attractors \mathcal{A}_ϵ in H_ϵ^1 as $\epsilon \rightarrow 0$. More precisely, we show the following two results:

i) **The upper semicontinuity of the attractors \mathcal{A}_ϵ in H_ϵ^1** , which is obtained just requiring the spectral convergence in H_ϵ^1 of the Neumann Laplacian as $\epsilon \rightarrow 0$; that is, requiring that the eigenvalues and eigenfunctions of the Laplace operator with homogeneous Neumann boundary conditions behave continuously in H_ϵ^1 as $\epsilon \rightarrow 0$.

ii) **The lower semicontinuity of the attractors \mathcal{A}_ϵ in H_ϵ^1** . Once upper semicontinuity is attained, lower semicontinuity in H_ϵ^1 is obtained by requiring that every equilibrium of the unperturbed problem is hyperbolic. To obtain the lower semicontinuity we will use the gradient structure of the flow.

By upper semicontinuity of the attractors in H_ϵ^1 we mean that

$$\sup_{u_\epsilon \in \mathcal{A}_\epsilon} \inf_{u_0 \in \mathcal{A}_0} \|u_\epsilon - u_0\|_{H_\epsilon^1} \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0$$

By lower semicontinuity of the attractors in H_ϵ^1 we mean that

$$\sup_{u_0 \in \mathcal{A}_0} \inf_{u_\epsilon \in \mathcal{A}_\epsilon} \|u_\epsilon - u_0\|_{H_\epsilon^1} \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0$$

2. SPECTRAL BEHAVIOR.

It will become very clear that understanding the spectral behavior of the Laplace operator is extremely important when analysing the continuity properties of nonlinear dynamics. Therefore, in this section we are interested in obtaining necessary and sufficient conditions that guarantee that the eigenvalues and eigenfunctions behave continuously when the domain undergoes a perturbation satisfying (1.3).

As a matter of fact we will be a little more general and consider operators of the form $-\Delta + V_\epsilon$ where the potentials may depend also on ϵ . We specify their behavior as $\epsilon \rightarrow 0$ in the following definition.

Definition 2.1. *A family $\{V_\epsilon : 0 \leq \epsilon \leq \epsilon_0\}$ of potentials is said admissible if $V_\epsilon \in L^\infty(\Omega_\epsilon)$, $\sup_{0 \leq \epsilon \leq \epsilon_0} \|V_\epsilon\|_{L^\infty(\Omega_\epsilon)} \leq C < \infty$ and $V_\epsilon \rightarrow V_0$ weakly in $L^2(\mathbb{R}^N)$.*

To fix the notations we consider the eigenvalue problems

$$\begin{cases} -\Delta u + V_\epsilon u = \lambda u, & \Omega_\epsilon \\ \frac{\partial u}{\partial n} = 0, & \partial\Omega_\epsilon \end{cases}$$

where $\{V_\epsilon : 0 \leq \epsilon \leq \epsilon_0\}$ is admissible. We denote by $\{\lambda_n^\epsilon\}_{n=1}^\infty$, for $\epsilon \in [0, \epsilon_0]$, the set of eigenvalues, ordered and counting multiplicity, of the operator $-\Delta + V_\epsilon$ with Neumann boundary conditions in Ω_ϵ and by $\{\phi_n^\epsilon\}_{n=1}^\infty$ a corresponding complete family of orthonormalized eigenfunctions.

Definition 2.2. *We will say that the spectra behaves continuously at $\epsilon = 0$, if for fixed $n \in \mathbb{N}$ we have that $\lambda_n^\epsilon \rightarrow \lambda_n^0$ as $\epsilon \rightarrow 0$ and the spectral projections converge in H_ϵ^1 , that is, if a $\notin \{\lambda_n^0\}_{n=0}^\infty$, and $\lambda_n^0 < a < \lambda_{n+1}^0$, then if we define the projections $P_a^\epsilon : L^2(\mathbb{R}^N) \rightarrow H^1(\Omega_\epsilon)$, $P_a^\epsilon(\psi) = \sum_{i=1}^n (\phi_i^\epsilon, \psi)_{L^2(\Omega_\epsilon)} \phi_i^\epsilon$ then*

$$\sup\{\|P_a^\epsilon(\psi) - P_a^0(\psi)\|_{H_\epsilon^1}, \psi \in L^2(\mathbb{R}^N), \|\psi\|_{L^2(\mathbb{R}^N)} = 1\} \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0.$$

Remark 2.3. *The convergence of the spectral projections is equivalent to the following: for each sequence $\epsilon_k \rightarrow 0$ there exists a subsequence, that we denote again by ϵ_k and a complete system of orthonormal eigenfunctions of the limiting problem $\{\phi_n^0\}_{n=1}^\infty$ such that $\|\phi_n^{\epsilon_k} - \phi_n^0\|_{H_{\epsilon_k}^1} \rightarrow 0$ as $k \rightarrow \infty$.*

In order to characterize when the spectra behaves continuously we define

$$\tau_\epsilon = \inf_{\substack{\phi \in H^1(\Omega_\epsilon) \\ \phi=0, \text{ in } \Omega_0}} \frac{\int_{\Omega_\epsilon} |\nabla \phi|^2}{\int_{\Omega_\epsilon} |\phi|^2} \quad (2.1)$$

Observe that, in case $\Omega_\epsilon \setminus \bar{\Omega}_0$ is smooth, τ_ϵ is the first eigenvalue of the following problem,

$$\begin{cases} -\Delta u = \tau u, & \Omega_\epsilon \setminus \bar{\Omega}_0 \\ u = 0, & \partial\Omega_0 \\ \frac{\partial u}{\partial n} = 0, & \partial\Omega_\epsilon \end{cases}$$

We have the following useful characterization

Proposition 2.4. *Assume the family of domains $\{\Omega_\epsilon\}_{0 \leq \epsilon \leq \epsilon_0}$ satisfy (1.3) and that $|\Omega_\epsilon \setminus \Omega_0| \rightarrow 0$ as $\epsilon \rightarrow 0$. Then, the following four statements are equivalent:*

i) The spectra of $-\Delta + V_\epsilon$ behaves continuously as $\epsilon \rightarrow 0$ for any admissible family of potentials $\{V_\epsilon, 0 \leq \epsilon \leq \epsilon_0\}$

ii) $\tau_\epsilon \rightarrow \infty$ as $\epsilon \rightarrow 0$.

iii) For any family of functions ψ_ϵ with $\|\psi_\epsilon\|_{H^1(\Omega_\epsilon)} \leq C$ then $\|\psi_\epsilon\|_{L^2(\Omega_\epsilon \setminus \bar{\Omega}_0)} \rightarrow 0$ as $\epsilon \rightarrow 0$.

iv) For any family of functions ψ_ϵ with $\|\psi_\epsilon\|_{H^1(\Omega_\epsilon)} \leq C$ then there exists a sequence ψ_{ϵ_k} and a function $\psi_0 \in H^1(\Omega_0)$ such that $\psi_{\epsilon_k} \rightarrow \psi_0$, in $L^2(\mathbb{R}^N)$ and for any $\chi \in H^1(\mathbb{R}^N)$ we have that

$$\int_{\Omega_{\epsilon_k}} \nabla \psi_{\epsilon_k} \nabla \chi \rightarrow \int_{\Omega_0} \nabla \psi_0 \nabla \chi$$

Proof.

ii) \Rightarrow iii). If it is not true then there will exist a sequence of functions ϕ_{ϵ_k} with $\|\phi_{\epsilon_k}\|_{H^1(\Omega_{\epsilon_k})} \leq C_1$ and $\|\phi_{\epsilon_k}\|_{L^2(\Omega_{\epsilon_k} \setminus \Omega_0)} \geq C_2 > 0$, for some constants C_1 and C_2 independent of ϵ_k . If we consider the functions $\psi_{\epsilon_k} = E(\phi_{\epsilon_k}|_{\Omega_0})$ then $\psi_{\epsilon_k} \in H^1(\mathbb{R}^N)$ with $\|\psi_{\epsilon_k}\|_{H^1(\mathbb{R}^N)} \leq C$ independent of ϵ_k . Moreover, by Hölder's inequality and Sobolev embeddings, we have

$$\|\psi_{\epsilon_k}\|_{L^2(\Omega_{\epsilon_k} \setminus \Omega_0)} \leq \|\psi_{\epsilon_k}\|_{L^{2N/(N-2)}(\Omega_{\epsilon_k} \setminus \Omega_0)} |\Omega_{\epsilon_k} \setminus \Omega_0|^{\frac{1}{N}} \leq C \|\psi_{\epsilon_k}\|_{H^1(\mathbb{R}^N)} |\Omega_{\epsilon_k} \setminus \Omega_0|^{\frac{1}{N}}, \quad (N \geq 3)$$

$$\|\psi_{\epsilon_k}\|_{L^2(\Omega_{\epsilon_k} \setminus \Omega_0)} \leq \|\psi_{\epsilon_k}\|_{L^{2p}(\Omega_{\epsilon_k} \setminus \Omega_0)} |\Omega_{\epsilon_k} \setminus \Omega_0|^{\frac{1}{2} - \frac{1}{p}} \leq C \|\psi_{\epsilon_k}\|_{H^1(\mathbb{R}^N)} |\Omega_{\epsilon_k} \setminus \Omega_0|^{\frac{1}{2} - \frac{1}{2p}}, \quad (N = 1, 2)$$

where p can be chosen arbitrarily large in the last inequality. These two last inequalities imply that there exists a $\theta > 0$, such that

$$\|\psi_{\epsilon_k}\|_{L^2(\Omega_{\epsilon_k} \setminus \Omega_0)} \leq C |\Omega_{\epsilon_k} \setminus \Omega_0|^\theta \rightarrow 0, \text{ as } \epsilon_k \rightarrow 0$$

We consider now $\chi_{\epsilon_k} = \phi_{\epsilon_k} - \psi_{\epsilon_k}$. By construction $\chi_{\epsilon_k} = 0$ in Ω_0 and $\|\chi_{\epsilon_k}\|_{H^1(\Omega_{\epsilon_k})} \leq C$. Moreover, $\|\chi_{\epsilon_k}\|_{L^2(\Omega_{\epsilon_k} \setminus \Omega_0)} \geq \|\phi_{\epsilon_k}\|_{L^2(\Omega_{\epsilon_k} \setminus \Omega_0)} - \|\psi_{\epsilon_k}\|_{L^2(\Omega_{\epsilon_k} \setminus \Omega_0)} \geq C_2/2$ as long as ϵ_k is small enough. This contradicts ii).

iii) \Rightarrow iv). If ψ_{ϵ_k} is a sequence with $\|\psi_{\epsilon_k}\|_{H^1(\Omega_{\epsilon_k})} \leq C$, then we can extract a subsequence of ψ_{ϵ_k} , that we denote again by ψ_{ϵ_k} and we obtain a function $\psi_0 \in H^1(\Omega_0)$ such that $\psi_{\epsilon_k} \rightarrow \psi_0$, w- $H^1(\Omega_0)$, s- $L^2(\Omega_0)$. Moreover, from iii) we have that $\|\psi_\epsilon\|_{L^2(\Omega_\epsilon \setminus \Omega_0)} \rightarrow 0$ as $\epsilon \rightarrow 0$. From this, we easily get that $\psi_{\epsilon_k} \rightarrow \psi_0$ in $L^2(\mathbb{R}^N)$.

Now if $\chi \in H^1(\mathbb{R}^N)$ we have

$$\left| \int_{\Omega_{\epsilon_k}} \nabla \psi_{\epsilon_k} \nabla \chi - \int_{\Omega_0} \nabla \psi_0 \nabla \chi \right| \leq \left| \int_{\Omega_0} (\nabla \psi_{\epsilon_k} - \nabla \psi_0) \nabla \chi \right| + \int_{\Omega_{\epsilon_k} \setminus \Omega_0} |\nabla \psi_{\epsilon_k}| |\nabla \chi| \rightarrow 0$$

since $\nabla \psi_{\epsilon_k} \rightarrow \nabla \psi_0$ w- $H^1(\Omega_0)$ and $\|\psi_\epsilon\|_{L^2(\Omega_\epsilon \setminus \Omega_0)} \rightarrow 0$.

iv) \Rightarrow i). Fix n with the property that $\lambda_n^0 < \lambda_{n+1}^0$ and consider the family of eigenfunctions $\{\phi_1^0, \dots, \phi_n^0\}$. If we denote by E a extension operator from $H^1(\Omega_0)$ to $H^1(\mathbb{R}^N)$, and by \mathcal{T}_ϵ the restriction operator to Ω_ϵ , we construct the functions $\xi_i^\epsilon = \mathcal{T}_\epsilon E \phi_i^0$, $i = 1, \dots, n$. Since iv) implies v) we easily see that $\|\xi_i^\epsilon\|_{H^1(\Omega_\epsilon \setminus \Omega_0)} \rightarrow 0$ as $\epsilon \rightarrow 0$ for $i = 1, \dots, n$. By the min-max characterization of eigenvalues, we easily obtain that $\lambda_i^\epsilon \leq \lambda_i^0 + o(1)$ as $\epsilon \rightarrow 0$.

We can choose a sequence $\epsilon_k \rightarrow 0$ and numbers $\kappa_i \leq \lambda_i^0$, $i = 1, \dots, n$, such that $\lambda_i^{\epsilon_k} \rightarrow \kappa_i$, for $i = 1, \dots, n$. Since $\phi_i^{\epsilon_k}$, for $i = 1, \dots, n$ is a bounded sequence in $H^1(\Omega_{\epsilon_k})$, then by iv) we can extract another subsequence, that we still denote by $\phi_i^{\epsilon_k}$, and get functions $\xi_i^0 \in H^1(\Omega_0)$, $i = 1, \dots, n$, such that $\phi_i^{\epsilon_k} \rightarrow \xi_i^0$ in $L^2(\mathbb{R}^N)$ and

$$\int_{\Omega_{\epsilon_k}} \nabla \phi_i^{\epsilon_k} \nabla \chi \rightarrow \int_{\Omega_0} \nabla \xi_i^0 \nabla \chi, \quad i = 1, \dots, n$$

for any $\chi \in H^1(\mathbb{R}^N)$.

In particular $\int_{\Omega_0} \xi_i^0 \xi_j^0 = \delta_{ij}$ and passing to the weak limit in the equation, we get

$$\int_{\Omega_0} \nabla \xi_i^0 \nabla \chi + \int_{\Omega_0} V_0 \xi_i^0 \chi = \kappa_i \int_{\Omega_0} \xi_i^0 \chi, \quad i = 1, \dots, n.$$

This implies that necessarily κ_i and ξ_i^0 are eigenvalues and eigenfunctions of the limiting problem. Since we already know that $\kappa_i \leq \lambda_i^0$ we necessarily have that $\kappa_i = \lambda_i^0$ for $i = 1, \dots, n$ and $\{\xi_1^0, \dots, \xi_n^0\}$ is a system of orthonormal eigenfunctions associated to $\lambda_1^0, \dots, \lambda_n^0$.

In order to prove the convergence in $H_{\epsilon_k}^1$ we notice that $\phi_i^0, i = 1, \dots, n$, satisfy

$$\int_{\Omega_{\epsilon_k}} |\nabla \phi_i^{\epsilon_k}|^2 = \lambda_i^{\epsilon_k} \int_{\Omega_{\epsilon_k}} |\phi_i^{\epsilon_k}|^2 - \int_{\Omega_{\epsilon_k}} V_{\epsilon_k} |\phi_i^{\epsilon_k}|^2 \rightarrow \lambda_i^0 \int_{\Omega_0} |\xi_i^0|^2 - \int_{\Omega_0} V_0 |\xi_i^0|^2 = \int_{\Omega_0} |\nabla \xi_i^0|^2$$

where we have used that $\phi_i^{\epsilon_k} \rightarrow \xi_i^0$ in $L^2(\mathbb{R}^N)$, the weak convergence of V_{ϵ_k} to V_0 and the uniform bound of $\|V_{\epsilon_k}\|_{L^\infty(\Omega_{\epsilon_k})}$. Hence,

$$\int_{\mathbb{R}^N} |\nabla \phi_i^{\epsilon_k} - \nabla \xi_i^0|^2 = \int_{\Omega_{\epsilon_k}} |\nabla \phi_i^{\epsilon_k}|^2 + \int_{\Omega_0} |\nabla \xi_i^0|^2 - 2 \int_{\Omega_{\epsilon_k}} \nabla \phi_i^{\epsilon_k} \nabla \xi_i^0$$

But,

$$\int_{\Omega_{\epsilon_k}} |\nabla \phi_i^{\epsilon_k}|^2 \rightarrow \int_{\Omega_0} |\nabla \xi_i^0|^2, \text{ as } \epsilon_k \rightarrow 0$$

and if we define $\tilde{\xi}_i^0 \in H^1(\mathbb{R}^N)$ an extension of ξ_i^0 we get that

$$\int_{\Omega_{\epsilon_k}} \nabla \phi_i^{\epsilon_k} \nabla \xi_i^0 = \int_{\Omega_{\epsilon_k}} \nabla \phi_i^{\epsilon_k} \nabla \tilde{\xi}_i^0 + \int_{\Omega_{\epsilon_k}} \nabla \phi_i^{\epsilon_k} (\nabla \xi_i^0 - \nabla \tilde{\xi}_i^0) \rightarrow \int_{\Omega_0} |\nabla \xi_i^0|^2, \text{ as } \epsilon_k \rightarrow 0$$

because

$$\left| \int_{\Omega_{\epsilon_k}} \nabla \phi_i^{\epsilon_k} (\nabla \xi_i^0 - \nabla \tilde{\xi}_i^0) \right| \leq \|\nabla \phi_i^{\epsilon_k}\|_{L^2(\Omega_{\epsilon_k})} \|\nabla \xi_i^0 - \nabla \tilde{\xi}_i^0\|_{L^2(\Omega_{\epsilon_k})} \rightarrow 0, \text{ as } \epsilon_k \rightarrow 0.$$

This implies that

$$\int_{\mathbb{R}^N} |\nabla \phi_i^{\epsilon_k} - \nabla \xi_i^0|^2 \rightarrow 0, \text{ as } \epsilon_k \rightarrow 0.$$

i) \Rightarrow ii). If this is not the case then we will have again a sequence ϵ_k approaching zero and a positive number a with $\tau_{\epsilon_k} < a$, for all k . From the definition of τ_{ϵ_k} we can get functions ϕ_{ϵ_k} with $\phi_{\epsilon_k} = 0$ in Ω_{ϵ_k} , $\|\phi_{\epsilon_k}\|_{L^2(\Omega_{\epsilon_k})} = 1$ and $\|\nabla \phi_{\epsilon_k}\|_{L^2(\Omega_{\epsilon_k})}^2 \leq a$.

Observe that

$$\int_{\Omega_{\epsilon_k}} |\nabla \phi_{\epsilon_k}|^2 + \int_{\Omega_{\epsilon_k}} V_{\epsilon_k} |\phi_{\epsilon_k}|^2 \leq a + \|V_{\epsilon_k}\|_{L^\infty(\Omega_{\epsilon_k})} \leq \tilde{a},$$

for some constant \tilde{a} independent of ϵ_k . Choose $n \in \mathbb{N}$ with the property that $\tilde{a} < \lambda_n^0 < \lambda_{n+1}^0$, denote by $\phi_1^{\epsilon_k}, \dots, \phi_n^{\epsilon_k}$ the first n eigenfunctions and consider the linear subspace $[\phi_1^{\epsilon_k}, \dots, \phi_n^{\epsilon_k}, \phi_{\epsilon_k}] \subset H^1(\Omega_{\epsilon_k})$. By the spectral convergence we can get a subsequence, that we denote by ϵ_k again and eigenfunctions of the limiting problem $\phi_1^0, \dots, \phi_n^0$ such that $\|\phi_i^{\epsilon_k} - \phi_i^0\|_{H_{\epsilon_k}^1} \rightarrow 0$ as $\epsilon_k \rightarrow 0$. This implies that $\|\phi_i^{\epsilon_k}\|_{L^2(\Omega_{\epsilon_k} \setminus K_{\epsilon_k})} \rightarrow 0$ as $\epsilon_k \rightarrow 0$. From here we get that

$$\int_{\Omega_{\epsilon_k}} \phi_i^{\epsilon_k} \phi_{\epsilon_k} \rightarrow 0, \text{ as } \epsilon_k \rightarrow 0, \text{ for } i = 1, \dots, n,$$

which means that $[\phi_1^{\epsilon_k}, \dots, \phi_n^{\epsilon_k}, \phi_{\epsilon_k}]$ is almost an orthonormal system in $L^2(\Omega_{\epsilon_k})$. By the min-max characterization of the eigenvalues, we have that

$$\lambda_{n+1}^{\epsilon_k} \leq \max_{\phi \in [\phi_1^{\epsilon_k}, \dots, \phi_n^{\epsilon_k}, \phi_{\epsilon_k}]} \frac{\int_{\Omega_{\epsilon_k}} |\nabla \phi|^2 + \int_{\Omega_{\epsilon_k}} V_{\epsilon_k} |\phi|^2}{\int_{\Omega_{\epsilon_k}} |\phi|^2}.$$

But if $\phi \in [\phi_1^{\epsilon_k}, \dots, \phi_n^{\epsilon_k}, \phi_{\epsilon_k}]$ we can write $\phi = \sum_{i=1}^n \alpha_i \phi_i^{\epsilon_k} + \beta \phi_{\epsilon_k}$. Using that $\phi_i^{\epsilon_k}$ is an eigenfunction corresponding to the eigenvalue $\lambda_i^{\epsilon_k}$ and that the family $\{\phi_1^{\epsilon_k}, \dots, \phi_n^{\epsilon_k}, \phi_{\epsilon_k}\}$ is almost orthonormal, by direct calculation of the above quotient we get that

$$\lambda_{n+1}^{\epsilon_k} \leq \frac{\sum_{i=1}^n \alpha_i^2 \lambda_i^{\epsilon_k} + \tilde{a} \beta^2 + o(1)}{\sum_{i=1}^n \alpha_i^2 + \tilde{a} + o(1)} \leq \lambda_n^0 + o(1).$$

This contradicts the continuity of the eigenvalues given by i).

And the proposition is proved. ■

We analyse now the convergence properties of the resolvent operators associated to the operators.

Definition 2.5. *We say that a family $\{\Omega_\epsilon : 0 \leq \epsilon \leq \epsilon_0\}$ is admissible if it satisfies (1.3) and one of the conditions i) to iv) of Proposition 2.4.*

We have the following result.

Proposition 2.6. *Assume that the family of potentials $\{V_\epsilon, 0 \leq \epsilon \leq \epsilon_0\}$ and the family of domains $\{\Omega_\epsilon : 0 \leq \epsilon \leq \epsilon_0\}$ are admissible. Assume also that $0 \notin \sigma(-\Delta + V_0)$. Then, for ϵ small enough $0 \notin \sigma(-\Delta + V_\epsilon)$ and there exists a constant C independent of ϵ such that*

$$\|(-\Delta + V_\epsilon)^{-1} g_\epsilon\|_{H^1(\Omega_\epsilon)} \leq C \|g_\epsilon\|_{L^2(\Omega_\epsilon)}, \quad g_\epsilon \in L^2(\Omega_\epsilon) \quad (2.2)$$

Moreover, if $g_\epsilon \rightarrow g_0$ weakly in $L^2(\mathbb{R}^N)$, then

$$\|(-\Delta + V_\epsilon)^{-1} g_\epsilon - (-\Delta + V_0)^{-1} g_0\|_{H^1_\epsilon} \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0. \quad (2.3)$$

Proof. Let us show first (2.2). By the continuity of the spectra given by Proposition 2.4 we have that for ϵ small enough $0 \notin \sigma(-\Delta + V_\epsilon)$. In particular, for $g_\epsilon \in L^2(\Omega_\epsilon)$ given, we have a unique solution $w_\epsilon \in H^1(\Omega_\epsilon)$ of

$$\begin{cases} -\Delta w_\epsilon + V_\epsilon w_\epsilon = g_\epsilon, & \Omega_\epsilon \\ \frac{\partial w_\epsilon}{\partial n} = 0, & \partial \Omega_\epsilon \end{cases} \quad (2.4)$$

We show first that if $\|g_\epsilon\|_{L^2(\Omega_\epsilon)} \leq C$, with C independent of ϵ , then $\|w_\epsilon\|_{L^2(\Omega_\epsilon)}$ is bounded. Suppose not. Then there is a subsequence, which we again denote by $\{w_\epsilon\}$, such that $\|w_\epsilon\|_{L^2(\Omega_\epsilon)} \rightarrow \infty$. Consider $\tilde{w}_\epsilon = \frac{w_\epsilon}{\|w_\epsilon\|_{L^2(\Omega_\epsilon)}}$, so that $\|\tilde{w}_\epsilon\|_{L^2(\Omega_\epsilon)} = 1$. Then

$$\begin{cases} -\Delta \tilde{w}_\epsilon + V_\epsilon \tilde{w}_\epsilon = \frac{g_\epsilon}{\|w_\epsilon\|_{L^2(\Omega_\epsilon)}}, & \Omega_\epsilon \\ \frac{\partial \tilde{w}_\epsilon}{\partial n} = 0, & \partial \Omega_\epsilon. \end{cases} \quad (2.5)$$

Multiplying this equation by \tilde{w}_ϵ and integrating by parts we obtain that

$$\int_{\Omega_\epsilon} |\nabla \tilde{w}_\epsilon|^2 + \int_{\Omega_\epsilon} V_\epsilon |\tilde{w}_\epsilon|^2 = \int_{\Omega_\epsilon} \frac{\tilde{g}_\epsilon}{\|w_\epsilon\|_{L^2(\Omega_\epsilon)}} \tilde{w}_\epsilon$$

from where it follows that

$$\int_{\Omega_\epsilon} |\nabla \tilde{w}_\epsilon|^2 \leq C,$$

with C independent of ϵ . Applying Proposition 2.4 iv) we can extract a sequence, denoted still by \tilde{w}_ϵ , so that $\tilde{w}_\epsilon \rightarrow \tilde{w}_0$ in $L^2(\mathbb{R}^N)$ and for any $\chi \in H^1(\mathbb{R}^N)$ we have

$$\int_{\Omega_\epsilon} \nabla \tilde{w}_\epsilon \nabla \chi \rightarrow \int_{\Omega_0} \nabla \tilde{w}_0 \nabla \chi.$$

Notice in particular that $\|\tilde{w}_0\|_{L^2(\Omega_0)} = 1$.

Let $\xi \in H^1(\Omega_0)$ and consider $\tilde{\xi} \in H^1(\mathbb{R}^N)$ an extension of ξ to \mathbb{R}^N . If we multiply the equation (2.5) by $\tilde{\xi} \in H^1(\Omega_\epsilon)$ and integrating by parts we have that

$$\int_{\Omega_\epsilon} \nabla \tilde{w}_\epsilon \nabla \tilde{\xi} + \int_{\Omega_\epsilon} V_\epsilon \tilde{w}_\epsilon \tilde{\xi} = \int_{\Omega_\epsilon} \frac{g_\epsilon}{\|w_\epsilon\|_{L^2(\Omega_\epsilon)}} \tilde{\xi}.$$

Taking the limit, we get that

$$\int_{\Omega_0} \nabla \tilde{w}_0 \nabla \xi + \int_{\Omega_0} V_0 \tilde{w}_0 \xi = 0,$$

where we have used that $\|V_\epsilon\|_{L^\infty(\Omega_\epsilon)} \leq C$, $V_\epsilon \rightarrow V_0$, $w\text{-}L^2(\mathbb{R}^N)$ and $\tilde{w}_\epsilon \rightarrow \tilde{w}_0$ in $L^2(\mathbb{R}^N)$. Thus

$$\begin{cases} -\Delta \tilde{w}_0 + V_0 \tilde{w}_0 = 0, & \Omega_0 \\ \frac{\partial \tilde{w}_0}{\partial n} = 0, & \partial\Omega_0, \end{cases} \quad (2.6)$$

and since $0 \notin \sigma(-\Delta + V_0)$, we get $\tilde{w}_0 = 0$. This contradicts the fact that $\|\tilde{w}_0\|_{L^2(\Omega_0)} = 1$. Hence, we obtain that $\|w_\epsilon\|_{L^2(\Omega_\epsilon)}$ is uniformly bounded in ϵ .

To show that $\|\nabla w_\epsilon\|_{L^2(\Omega_\epsilon)}$ is uniformly bounded in ϵ we note that V_ϵ are uniformly bounded in $L^\infty(\Omega_\epsilon)$ and that

$$\int_{\Omega_\epsilon} |\nabla w_\epsilon|^2 = - \int_{\Omega_\epsilon} V_\epsilon |w_\epsilon|^2 + \int_{\Omega_\epsilon} g_\epsilon w_\epsilon.$$

To show (2.3), notice that by the weak convergence of g_ϵ , we have that g_ϵ is uniformly bounded in $L^2(\mathbb{R}^N)$. Applying (2.2) we obtain that $\|(-\Delta + V_\epsilon)^{-1} g_\epsilon\|_{H^1(\Omega_\epsilon)}$ is uniformly bounded in ϵ . Using iv) in Proposition 2.4 and taking the limit in the equation we obtain that if $u_\epsilon = (-\Delta + V_\epsilon)^{-1} g_\epsilon$ and $u_0 = (-\Delta + V_0)^{-1} g_0$, then $u_\epsilon \rightarrow u_0$ in $L^2(\mathbb{R}^N)$ and $\nabla u_\epsilon \rightarrow \nabla u_0$ $w\text{-}L^2(\mathbb{R}^N)$. . Now with a similar argument as in the proof that iv) implies i) in Proposition 2.4 we obtain that $u_\epsilon \rightarrow u_0$ in H^1_ϵ . This concludes the proof of the lemma. \blacksquare

3. CONVERGENCE OF THE LINEAR AND NONLINEAR SEMIGROUPS. UPPER-SEMICONITINUITY OF ATTRACTORS.

With the continuity of the spectra of the operators $A_\epsilon = -\Delta + I$ we will be able to obtain estimates on the behavior of the linear semigroups $e^{-A_\epsilon t}$. Notice that the semigroup $e^{-A_\epsilon t}$ acts on functions defined in Ω_ϵ . We will need to estimate expressions of the type $e^{-A_\epsilon t}u_0$ where, for instance $u_0 \in L^2(\Omega_0)$. As we said in the introduction, by this we mean that we extend the function u_0 by zero outside Ω_0 and restrict to Ω_ϵ . In this way we can also regard $u_0 \in L^2(\Omega_\epsilon)$ and evaluate $e^{-A_\epsilon t}u_0$. Similarly we can give a meaning to $e^{-A_0 t}u_\epsilon$.

We have the following result

Proposition 3.1. *Assume that the family of domains $\{\Omega_\epsilon : 0 \leq \epsilon \leq \epsilon_0\}$ is admissible and Proposition 2.4 holds true. Then, there exists a number $\gamma < 1$ and a function $\theta(\epsilon)$ with $\theta(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$ such that*

$$\|e^{-A_\epsilon t}u_\epsilon - e^{-A_0 t}u_\epsilon\|_{H_\epsilon^1} \leq \theta(\epsilon)t^{-\gamma}e^{-t/4}\|u_\epsilon\|_{L^2(\Omega_\epsilon)}, \quad u_\epsilon \in L^2(\Omega_\epsilon), \quad t > 0$$

Proof. Notice first that from (1.6) we have

$$\|e^{-A_\epsilon t}u_\epsilon\|_{H^1(\Omega_\epsilon)} \leq t^{-\frac{1}{2}}e^{-t/2}\|u_\epsilon\|_{L^2(\Omega_\epsilon)}, \quad u_\epsilon \in L^2(\Omega_\epsilon), \quad t > 0, \quad \epsilon \in [0, \epsilon_0]$$

Now, we separate the estimate for t small and t large. Choose $\gamma \in (\frac{1}{2}, 1)$ fixed. Let $\delta > 0$ be a small parameter and let us consider two different cases according to $t \in (0, \delta]$ or $t > \delta$.

i) If $t \in (0, \delta]$ we easily check that

$$\begin{aligned} \|e^{-A_\epsilon t}u_\epsilon - e^{-A_0 t}u_\epsilon\|_{H_\epsilon^1} &\leq \|e^{-A_\epsilon t}u_\epsilon\|_{H_\epsilon^1} + \|e^{-A_0 t}u_\epsilon\|_{H_\epsilon^1} \\ &\leq 2t^{-\frac{1}{2}}e^{-t/2}\|u_\epsilon\|_{L^2(\Omega_\epsilon)} \leq 2\delta^{\gamma-\frac{1}{2}}t^{-\gamma}e^{-t/2}\|u_\epsilon\|_{L^2(\Omega_\epsilon)} \end{aligned} \quad (3.1)$$

ii) If $t > \delta$ we proceed as follows. We know that the function $xe^{-x} \rightarrow 0$ as $x \rightarrow \infty$. Hence, for $\delta > 0$ given we may choose a number $x_0(\delta)$ large enough so that $xe^{-x} \leq \delta^2$ for $x \geq x_0(\delta)$. If we consider now the function ze^{-2zt} , then if $l(\delta) = x_0(\delta)/\delta$ that without loss of generality we may assume ≥ 1 , then for $z \geq l(\delta)$ we have

$$ze^{-2zt} = t^{-1}tze^{-zt}e^{-zt} \leq t^{-1}\delta^2e^{-t}$$

Since we have $\lambda_k^\epsilon \xrightarrow{\epsilon \rightarrow 0} \lambda_k^0$ and $\lambda_k^0 \xrightarrow{k \rightarrow \infty} +\infty$, there exists $N = N(\delta)$ large enough, such that $\lambda_k^\epsilon \geq l(\delta)$, $\epsilon \in [0, \epsilon_0)$ with $k \geq N(\delta)$. Without loss of generality we can assume that we have $\lambda_{N(\delta)}^0 < \lambda_{N(\delta)+1}^0$. Hence, from the spectral decomposition of the linear semigroups, we obtain

$$\begin{aligned} \|e^{-A_\epsilon t}u_\epsilon - e^{-A_0 t}u_\epsilon\|_{H_\epsilon^1} &\leq \left\| \sum_{k=1}^{N(\delta)} e^{-\lambda_k^\epsilon t}(u_\epsilon, \phi_k^\epsilon)\phi_k^\epsilon - \sum_{k=1}^{N(\delta)} e^{-\lambda_k^0 t}(u_\epsilon, \phi_k^0)\phi_k^0 \right\|_{H_\epsilon^1} + \\ &+ \left\| \sum_{N(\delta)+1}^{\infty} e^{-\lambda_k^\epsilon t}(u_\epsilon, \phi_k^\epsilon)\phi_k^\epsilon \right\|_{H^1(\Omega_\epsilon)} + \left\| \sum_{N(\delta)+1}^{\infty} e^{-\lambda_k^0 t}(u_\epsilon, \phi_k^0)\phi_k^0 \right\|_{H^1(\Omega_0)} = I_1 + I_2 + I_3 \end{aligned} \quad (3.2)$$

Analyzing I_2 , I_3 and I_1 respectively, we get

$$\begin{aligned} I_2^2 &\leq \sum_{N(\delta)+1}^{\infty} \lambda_k^\epsilon e^{-2\lambda_k^\epsilon t} |(u_\epsilon, \phi_n^\epsilon)|^2 \leq \delta^2 t^{-1} e^{-t} \|u_\epsilon\|_{L^2(\Omega_\epsilon)}^2 \\ I_3^2 &\leq \sum_{N(\delta)+1}^{\infty} \lambda_k^0 e^{-2\lambda_k^0 t} |(u_0, \phi_n^0)|^2 \leq \delta^2 t^{-1} e^{-t} \|u_\epsilon\|_{L^2(\Omega_\epsilon)}^2 \end{aligned}$$

$$\begin{aligned}
I_1 &= \left\| \sum_{k=1}^{N(\delta)} e^{-\lambda_k^\epsilon t} (u_\epsilon, \phi_k^\epsilon) \phi_k^\epsilon - \sum_{k=1}^{N(\delta)} e^{-\lambda_k^0 t} (u_\epsilon, \phi_k^0) \phi_k^0 \right\|_{H_\epsilon^1} \\
&\leq \left\| \sum_{k=1}^{N(\delta)} (e^{-\lambda_k^\epsilon t} - e^{-\lambda_k^0 t}) (u_\epsilon, \phi_k^\epsilon) \phi_k^\epsilon \right\|_{H_\epsilon^1} + \left\| \sum_{k=1}^{N(\delta)} e^{-\lambda_k^0 t} ((u_\epsilon, \phi_k^\epsilon) \phi_k^\epsilon - (u_\epsilon, \phi_k^0) \phi_k^0) \right\|_{H_\epsilon^1} \\
&\leq \sum_{k=1}^{N(\delta)} (\lambda_k^\epsilon)^{1/2} |e^{-\lambda_k^\epsilon t} - e^{-\lambda_k^0 t}| \|u_\epsilon\|_{L^2(\Omega_\epsilon)} + \sum_{i=1}^{n_i} e^{-\mu_i t} \left\| \sum_{k=n_{i-1}+1}^{n_i} ((u_\epsilon, \phi_k^\epsilon) \phi_k^\epsilon - (u_\epsilon, \phi_k^0) \phi_k^0) \right\|_{H_\epsilon^1}
\end{aligned}$$

where we have used $\|\phi_i^\epsilon\|_{H^1(\Omega_\epsilon)}^2 = \lambda_i^\epsilon$, $|(u_\epsilon, \phi_i^\epsilon)| \leq \|u_\epsilon\|_{L^2(\Omega_\epsilon)}$ and triangular inequality in the first sum. For the second sum we have denoted by μ_i the points of the spectrum of A_0 , each with multiplicity $n_{i+1} - n_i$, that is, $\mu_1 < \mu_2 < \dots < \mu_{k(\delta)}$, with

$$\begin{aligned}
\mu_1 &= \lambda_1^0 = \dots = \lambda_{n_1}^0 \\
\mu_2 &= \lambda_{n_1+1}^0 = \dots = \lambda_{n_2}^0 \\
&\dots\dots \\
\mu_i &= \lambda_{n_{i-1}+1} = \dots = \lambda_{n_i} \\
&\dots\dots
\end{aligned}$$

Moreover, since $\delta > 0$ is fixed and therefore the number of terms in the sums above is finite and fixed and $\lambda_k^\epsilon \geq 1$, then

$$\begin{aligned}
&\sum_{k=1}^{N(\delta)} (\lambda_k^\epsilon)^{1/2} |e^{-\lambda_k^\epsilon t} - e^{-\lambda_k^0 t}| \leq C e^{-t}, \quad \forall t > 0 \\
&\sum_{k=1}^{N(\delta)} (\lambda_k^\epsilon)^{1/2} |e^{-\lambda_k^\epsilon t} - e^{-\lambda_k^0 t}| \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0, \quad \text{uniformly } t > 0
\end{aligned}$$

Both statements above imply that we can choose $\epsilon_1(\delta) > 0$ such that for $\epsilon < \epsilon_1(\delta)$ we have

$$\sum_{k=1}^{N(\delta)} (\lambda_k^\epsilon)^{1/2} |e^{-\lambda_k^\epsilon t} - e^{-\lambda_k^0 t}| \leq \delta e^{-t/2}, \quad \forall t > 0$$

Moreover, with a very similar argument, using this time the convergence of the spectral projections, we get that there exists another $\epsilon_2(\delta) > 0$ such that for $0 < \epsilon < \epsilon_2(\delta)$ we have

$$\sum_{i=1}^{k(\delta)} e^{-\mu_i t} \left\| \sum_{k=n_{i-1}+1}^{n_i} ((u_\epsilon, \phi_k^\epsilon) \phi_k^\epsilon - (u_\epsilon, \phi_k^0) \phi_k^0) \right\|_{H_\epsilon^1} \leq \delta e^{-t/2} \|u_\epsilon\|_{L^2(\Omega_\epsilon)}$$

Putting everything together, we get that for $t > \delta$ and for $0 < \epsilon < \epsilon_0(\delta) = \min\{\epsilon_1(\delta), \epsilon_2(\delta)\}$,

$$\|e^{-A_\epsilon t} u_\epsilon - e^{-A_0 t} u_\epsilon\|_{H_\epsilon^1} \leq (2\delta e^{-t/2} + \delta t^{-1/2} e^{-t/2}) \|u_\epsilon\|_{L^2(\Omega_\epsilon)} \leq C \delta t^{-\gamma} e^{-t/4} \|u_\epsilon\|_{L^2(\Omega_\epsilon)} \quad (3.3)$$

where $1/2 < \gamma < 1$. Putting together (3.1), which is valid for $0 < t < \delta$ and all ϵ and (3.3), which is valid for $t > \delta$ and $0 < \epsilon \leq \epsilon_0(\delta)$ and taking into account that δ is an arbitrary small number, we get the result. \blacksquare

Once we have proved a result on the continuity of the linear semigroups (Proposition 3.1), we will analyze the convergence of the nonlinear semigroups with the aid of the Variation of Constants

Formula and will see that the attractors and the stationary states (solutions of the nonlinear elliptic problem) are upper semicontinuous with respect to these perturbations.

We will show the following result

Proposition 3.2. *Assume that the family of domains $\{\Omega_\epsilon : 0 \leq \epsilon \leq \epsilon_0\}$ is admissible. Then, there exist $0 \leq \gamma < 1$, a function $c(\epsilon)$ with $c(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$ and a constant M such that*

$$\|T_\epsilon(t, u_\epsilon) - T_0(t, u_\epsilon)\|_{H_\epsilon^1} \leq M c(\epsilon) t^{-\gamma}, \quad t \in (0, \tau], \quad \|u_\epsilon\|_{L^2(\Omega_\epsilon)} \leq R, \quad \epsilon \in (0, \epsilon_0) \quad (3.4)$$

where $M = M(\tau, R)$.

Moreover the attractors are upper semicontinuous at $\epsilon = 0$ in H_ϵ^1 , in the sense that

$$\sup_{u_\epsilon \in \mathcal{A}_\epsilon} d_{H_\epsilon^1}(u_\epsilon, \mathcal{A}_0) = \sup_{u_\epsilon \in \mathcal{A}_\epsilon} \left[\inf_{u_0 \in \mathcal{A}_0} \{\|u_\epsilon - u_0\|_{H_\epsilon^1}\} \right] \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0 \quad (3.5)$$

Also, if we denote by \mathcal{E}_ϵ , $\epsilon \in [0, \epsilon_0]$ the set of stationary states of (1.1), then

$$\sup_{u_\epsilon \in \mathcal{E}_\epsilon} d_{H_\epsilon^1}(u_\epsilon, \mathcal{E}_0) = \sup_{u_\epsilon \in \mathcal{E}_\epsilon} \left[\inf_{u_0 \in \mathcal{E}_0} \{\|u_\epsilon - u_0\|_{H_\epsilon^1}\} \right] \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0 \quad (3.6)$$

Proof. Notice that the nonlinear semigroups $T_\epsilon(t)$ are given by the variation of constants formula.

$$T_\epsilon(t, u_\epsilon) = e^{-A_\epsilon t} u_\epsilon + \int_0^t e^{-A_\epsilon(t-s)} f(T_\epsilon(s, u_\epsilon)) ds, \quad \epsilon \in [0, \epsilon_0) \quad (3.7)$$

Hence, calculating $T_\epsilon(t, u_\epsilon) - T_0(t, u_\epsilon)$ and with some elementary computations we obtain

$$\begin{aligned} \|T_\epsilon(t, u_\epsilon) - T_0(t, u_\epsilon)\|_{H_\epsilon^1} &\leq \|e^{-A_\epsilon t} u_\epsilon - e^{-A_0 t} u_\epsilon\|_{H_\epsilon^1} + \\ &\int_0^t \|e^{-A_\epsilon t} f(T_\epsilon(s, u_\epsilon)) - e^{-A_0 t} f(T_\epsilon(s, u_\epsilon))\|_{H_\epsilon^1} ds \\ &+ \int_0^t \|e^{-A_0 t} (f(T_\epsilon(s, u_\epsilon)) - f(T_0(s, u_\epsilon)))\|_{H_\epsilon^1} ds \quad \epsilon \in [0, \epsilon_0) \end{aligned}$$

Applying now Proposition 3.1 and (1.6), we get

$$\begin{aligned} \|T_\epsilon(t, u_\epsilon) - T_0(t, u_\epsilon)\|_{H_\epsilon^1} &\leq \theta(\epsilon) t^{-\gamma} e^{-t/4} \|u_\epsilon\|_{L^2(\Omega_\epsilon)} \\ &+ \theta(\epsilon) \int_0^t (t-s)^{-\gamma} e^{-(t-s)/4} \|f(T_\epsilon(s, u_\epsilon))\|_{L^2(\Omega_\epsilon)} \\ &+ \int_0^t (t-s)^{-1/2} e^{-(t-s)/2} \|f(T_\epsilon(t, u_\epsilon)) - f(T_0(t, u_\epsilon))\|_{L^2(\Omega_0)} \end{aligned}$$

and using that f is bounded and Lipschitz, we have

$$\begin{aligned} &\leq \theta(\epsilon) t^{-\gamma} e^{-t/4} \|u_\epsilon\|_{L^2(\Omega_\epsilon)} + \theta(\epsilon) L |\Omega_\epsilon|^{1/2} \int_0^t s^{-\gamma} e^{-s/4} ds \\ &+ L \int_0^t (t-s)^{-1/2} e^{-(t-s)/2} \|T_\epsilon(t, u_\epsilon) - T_0(t, u_\epsilon)\|_{H_\epsilon^1} \end{aligned}$$

and since $\|u_\epsilon\|_{L^2} \leq R$ and considering $0 \leq t \leq \tau$, we finally have, with $M = M(\tau, R)$, that

$$\|T_\epsilon(t, u_\epsilon) - T_0(t, u_\epsilon)\|_{H_\epsilon^1} \leq M \theta(\epsilon) t^{-\gamma} + L \int_0^t (t-s)^{-1/2} e^{-(t-s)/2} \|T_\epsilon(t, u_\epsilon) - T_0(t, u_\epsilon)\|_{H_\epsilon^1}$$

Applying now Gronwall's lemma, see [6], we obtain statement (3.4). ■

Now, the upper semicontinuity of the attractors in H_ϵ^1 , statement (3.5) follows directly from (3.4) and the fact that the attractor \mathcal{A}_0 attracts bounded sets. In particular, $B = \bigcup_{0 < \epsilon \leq \epsilon_0} \mathcal{A}_\epsilon|_{\Omega_0}$ is

a bounded set in $H^1(\Omega_0)$. Hence, if we fix $\eta > 0$ small, we have a time $\tau > 0$ such that the orbit of B under the semi flow T_0 , enters an η -neighborhood of \mathcal{A}_0 at time τ and it will not abandon this neighborhood ever. That is, for each $w_\epsilon \in \mathcal{A}_\epsilon$, $d_{H_\epsilon^1}(T_0(\tau)w_\epsilon, \mathcal{A}_0) \leq \eta$. Moreover, if $u_\epsilon \in \mathcal{A}_\epsilon$ and if $\tau > 0$ is the above, then from the invariance of the attractors, there exists $w_\epsilon \in \mathcal{A}_\epsilon$ such that $T_\epsilon(\tau)w_\epsilon = u_\epsilon$. Hence,

$$d_{H_\epsilon^1}(u_\epsilon, \mathcal{A}_0) = d_{H_\epsilon^1}(T_\epsilon(\tau)w_\epsilon, \mathcal{A}_0) \leq \|T_\epsilon(\tau)w_\epsilon - T_0(\tau)w_\epsilon\|_{H_\epsilon^1} + d_{H_\epsilon^1}(T_0(\tau)w_\epsilon, \mathcal{A}_0) \leq Mc(\epsilon)\tau^{-\gamma} + \eta$$

and this implies that choosing ϵ_1 small enough so that $Mc(\epsilon)\tau^{-\gamma} \leq \eta$ then

$$\sup_{u_\epsilon \in \mathcal{A}_\epsilon} d_{H_\epsilon^1}(u_\epsilon, \mathcal{A}_0) \leq 2\eta, \quad \forall 0 < \epsilon \leq \epsilon_1.$$

Since $\eta > 0$ is arbitrarily small, we have shown the result on uppersemicontinuity of attractors.

To show the upper semicontinuity in H_ϵ^1 of the stationary states we will prove that for any sequence of $\epsilon \rightarrow 0$ and for any $u_\epsilon \in \mathcal{E}_\epsilon$ we can extract a subsequence, that we still denote by ϵ , and obtain a $u_0 \in \mathcal{E}_0$ such that $\|u_\epsilon - u_0\|_{H_\epsilon^1} \rightarrow 0$ as $\epsilon \rightarrow 0$. From the upper semicontinuity of the attractors given by (3.5), we obtain the existence of a $u_0 \in \mathcal{A}_0$ such that $\|u_\epsilon - u_0\|_{H_\epsilon^1} \rightarrow 0$ as $\epsilon \rightarrow 0$. To show that $u_0 \in \mathcal{E}_0$ we first observe that for any $t > 0$, $\|u_\epsilon - T_0(t, u_0)\|_{H_\epsilon^1} \rightarrow \|u_0 - T_0(t, u_0)\|_{H^1(\Omega_0)}$. Moreover, for a fixed $\tau > 0$ and for any $t \in (0, \tau)$ we have that,

$$\|u_\epsilon - T_0(t, u_0)\|_{H_\epsilon^1} = \|T_\epsilon(t, u_\epsilon) - T_0(t, u_0)\|_{H_\epsilon^1} \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0$$

where we have used that u_ϵ is a stationary state and (3.4). In particular we have that for each $t > 0$, $u_0 = T_0(t, u_0)$, which implies that u_0 is a stationary state. This concludes the proof of the Proposition. ■

4. CONTINUITY OF EQUILIBRIA, UNSTABLE MANIFOLDS AND ATTRACTORS

In order to obtain lower semicontinuity of attractors in H_ϵ^1 we must ensure that the set of equilibria \mathcal{E}_ϵ behaves lower-semicontinuously. In this section we prove that, for the sort of domain perturbations considered here and assuming that the equilibria of the limiting problem are all hyperbolic, \mathcal{E}_ϵ is a finite set with constant cardinality; that is, $\mathcal{E}_\epsilon = \{u_1^\epsilon, \dots, u_n^\epsilon\}$, $0 \leq \epsilon \leq \epsilon_0$. This set behaves continuously with respect to ϵ in H_ϵ^1 , that is,

$$\max_{1 \leq k \leq n} \{\|u_k^\epsilon - u_k^0\|_{H_\epsilon^1}\} \xrightarrow{\epsilon \rightarrow 0} 0.$$

We also indicate in this section, that the local unstable manifolds of equilibrium solutions are continuous as $\epsilon \rightarrow 0$. For that we use the convergence of equilibria to obtain the continuity of the spectrum of the linearization around such equilibria and consequently the continuity of the local unstable manifolds.

With all these ingredients and using that the system is gradient, we will show the continuity of the attractors.

4.1. Continuity of hyperbolic equilibrium. Consider the following family of elliptic problems

$$(P)_\epsilon \quad \begin{cases} \Delta u - u + f(u) = 0 & \text{in } \Omega_\epsilon \\ \frac{\partial u}{\partial n} = 0 & \text{in } \partial\Omega_\epsilon. \end{cases}$$

for each $0 \leq \epsilon \leq \epsilon_0$ ($\epsilon_0 > 0$). We can show the following

Proposition 4.1. *Assume that the family of domains $\{\Omega_\epsilon : 0 \leq \epsilon \leq \epsilon_0\}$ is admissible. Assume also that problem $(P)_0$ has a solution $u^0 \in H^1(\Omega_0)$ and that zero is not in the spectrum of the operator $\Delta - I + f'(u^0) : H_n^2(\Omega_0) \subset L^2(\Omega_0) \rightarrow L^2(\Omega_0)$. Consider the extension operator $E : H^1(\Omega_0) \rightarrow H^1(\mathbb{R}^N)$ and let $u^{0,\epsilon} = E(u^0)|_{\Omega_\epsilon} \in H^1(\Omega_\epsilon)$. Then, there exists $\epsilon_0 > 0$ and $\delta > 0$ so that problem*

$(P)_\epsilon$ has exactly one solution, u^ϵ , in $\{w_\epsilon, \|w_\epsilon - u^{0,\epsilon}\|_{H^1(\Omega_\epsilon)} \leq \delta\}$ for $0 < \epsilon \leq \epsilon_0$. Furthermore,

$$\|u^\epsilon - u^0\|_{H_\epsilon^1} \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0.$$

Proof: Define the operators

$$\begin{aligned} \Theta_\epsilon &: H^1(\Omega_\epsilon) \rightarrow H^1(\Omega_\epsilon) \\ \Theta_\epsilon(z_\epsilon) &= (-\Delta + I - f'(u^{0,\epsilon})I)^{-1} (f(z_\epsilon) - f'(u^{0,\epsilon})z_\epsilon). \end{aligned} \quad (4.1)$$

The operators Θ_ϵ are well defined by applying Proposition 2.6, since $f'(u^{0,\epsilon}) \rightarrow f'(u^0)$ in $L^2(\mathbb{R}^N)$ and $0 \notin \sigma(-\Delta + I + f'(u^0)I)$. Notice also that v_ϵ is a fixed point of Θ_ϵ if and only if v_ϵ is a solution of $(P)_\epsilon$.

We will show that there exists $\delta > 0$ and $\epsilon_0 > 0$, such that the operator Θ_ϵ , for $0 < \epsilon < \epsilon_0$, is a strict contraction from $B_\delta(u^{0,\epsilon}) = \{v_\epsilon \in H^1(\Omega_\epsilon) : \|v_\epsilon - u^{0,\epsilon}\|_{H^1(\Omega_\epsilon)} \leq \delta\}$ into itself.

To prove this, let us start by showing that $\Theta_\epsilon : B_\delta(u^{0,\epsilon}) \rightarrow H^1(\Omega_\epsilon)$ is a strict contraction, that is, there exists a $\rho < 1$ such that $\|\Theta_\epsilon v_\epsilon - \Theta_\epsilon w_\epsilon\|_{H^1(\Omega_\epsilon)} \leq \rho \|v_\epsilon - w_\epsilon\|_{H^1(\Omega_\epsilon)}$ for any $v_\epsilon, w_\epsilon \in B_\delta(u^{0,\epsilon})$. We have,

$$\begin{aligned} &\|\Theta_\epsilon(v_\epsilon) - \Theta_\epsilon(w_\epsilon)\|_{H^1(\Omega_\epsilon)} \\ &\leq \|(-\Delta + I - f'(u^{0,\epsilon})I)^{-1}\|_{\mathcal{L}(L^2(\Omega_\epsilon), H^1(\Omega_\epsilon))} \|f(v_\epsilon) - f(w_\epsilon) - f'(u^{0,\epsilon})(v_\epsilon - w_\epsilon)\|_{L^2(\Omega_\epsilon)} \\ &\leq C \|f(v_\epsilon) - f(w_\epsilon) - f'(u^{0,\epsilon})(v_\epsilon - w_\epsilon)\|_{L^2(\Omega_\epsilon)}. \end{aligned} \quad (4.2)$$

Where we have used Lemma 2.6 to obtain that $\|(-\Delta + I - f'(u^{0,\epsilon})I)^{-1}\|_{\mathcal{L}(L^2(\Omega_\epsilon), H^1(\Omega_\epsilon))} \leq C$ for some constant C independent of ϵ .

Next we study $\|f(v_\epsilon) - f(w_\epsilon) - f'(u^{0,\epsilon})(v_\epsilon - w_\epsilon)\|_{L^2(\Omega_\epsilon)}$. We prove

Lemma 4.2. *There exists a constant C such that for all v_ϵ, w_ϵ with $\|v_\epsilon - u^{0,\epsilon}\|_{H^1(\Omega_\epsilon)} < \delta$, $\|v_\epsilon - u^{0,\epsilon}\|_{H^1(\Omega_\epsilon)} < \delta$, we have*

$$\|f(v_\epsilon) - f(w_\epsilon) - f'(u^{0,\epsilon})(v_\epsilon - w_\epsilon)\|_{L^2(\Omega_\epsilon)} \leq C\left(\frac{1}{\tau_\epsilon} + \delta^{2/N}\right)\|v_\epsilon - w_\epsilon\|_{H^1(\Omega_\epsilon)}$$

where τ_ϵ is given by (2.1).

If we assume the lemma proved, then we have

$$\|\Theta_\epsilon(v_\epsilon) - \Theta_\epsilon(w_\epsilon)\|_{H^1(\Omega_\epsilon)} \leq C\left(\frac{1}{\tau_\epsilon} + \delta^{2/N}\right)\|v_\epsilon - w_\epsilon\|_{H^1(\Omega_\epsilon)}$$

Now, given $\rho < 1$ choose ϵ small enough such that $C\frac{1}{\tau_\epsilon} \leq \frac{\rho}{2}$ and δ small enough so that $C\delta^{2/N} < \frac{\rho}{2}$. This shows Θ_ϵ is a strict contraction from $B_\delta(u^{0,\epsilon})$ into $H^1(\Omega_\epsilon)$.

In order to prove that Θ_ϵ maps $B_\delta(u^{0,\epsilon})$ into itself we show first that $\|\Theta_\epsilon u^{0,\epsilon} - u^{0,\epsilon}\|_{H^1(\Omega_\epsilon)} \rightarrow 0$ as $\epsilon \rightarrow 0$, for all $k = 1, \dots, m$. Notice that

$$\|\Theta_\epsilon u^{0,\epsilon} - u^{0,\epsilon}\|_{H^1(\Omega_\epsilon)} \leq \|\Theta_\epsilon u^{0,\epsilon} - u^0\|_{H^1_\epsilon} + \|u^{0,\epsilon} - u^0\|_{H^1_\epsilon} = \|\Theta_\epsilon u^{0,\epsilon} - u^0\|_{H^1_\epsilon} + \|u^{0,\epsilon}\|_{H^1(\Omega_\epsilon \setminus \Omega_0)}$$

But $\|u^{0,\epsilon}\|_{H^1(\Omega_\epsilon \setminus \Omega_0)} \rightarrow 0$ as $\epsilon \rightarrow 0$. Hence we just need to show that $\|\Theta_\epsilon u^{0,\epsilon} - u^0\|_{H^1_\epsilon} \rightarrow 0$ as $\epsilon \rightarrow 0$. If we denote by $v_\epsilon = \Theta_\epsilon u^{0,\epsilon}$, then $v_\epsilon \in H^1(\Omega_\epsilon)$ is the solution of

$$\begin{cases} -\Delta v_\epsilon + v_\epsilon - f'(u^{0,\epsilon})v_\epsilon = f(u^{0,\epsilon}) - f'(u^{0,\epsilon})u^{0,\epsilon}, & \Omega_\epsilon \\ \frac{\partial v_\epsilon}{\partial n} = 0, & \partial\Omega_\epsilon \end{cases}$$

and u^0 is the solution of

$$\begin{cases} -\Delta u^0 + u^0 - f'(u^0)u^0 = f(u^0) - f'(u^0)u^0, & \Omega_0 \\ \frac{\partial u^0}{\partial n} = 0, & \partial\Omega_0 \end{cases}$$

But by the resolvent convergence estimates (2.3) we get that $\|v_\epsilon - u^0\|_{H^1_\epsilon} \rightarrow 0$ as $\epsilon \rightarrow 0$.

To show that Θ_ϵ maps $B_\delta(u^{0,\epsilon})$ into itself we just observe that if $v_\epsilon \in B_\delta(u^{0,\epsilon})$

$$\|\Theta_\epsilon v_\epsilon - u^{0,\epsilon}\|_{H^1(\Omega_\epsilon)} \leq \|\Theta_\epsilon v_\epsilon - \Theta_\epsilon u^{0,\epsilon}\|_{H^1(\Omega_\epsilon)} + \|\Theta_\epsilon u^{0,\epsilon} - u^{0,\epsilon}\|_{H^1(\Omega_\epsilon)} \leq \rho\delta + \|\Theta_\epsilon u^{0,\epsilon} - u^{0,\epsilon}\|_{H^1(\Omega_\epsilon)}$$

Choosing ϵ small enough again we can guarantee that $\|\Theta_\epsilon u^{0,\epsilon} - u^{0,\epsilon}\|_{H^1(\Omega_\epsilon)} < (1 - \rho)\delta$ and therefore $\|\Theta_\epsilon v_\epsilon - u^{0,\epsilon}\|_{H^1(\Omega_\epsilon)} < \delta$. This concludes the proof of the Proposition. \blacksquare

Proof of Lemma 4.2: Note that

$$|f(v_\epsilon(x)) - f(w_\epsilon(x)) - f'(u^{0,\epsilon}(x))(v_\epsilon(x) - w_\epsilon(x))| \leq \bar{C}\gamma_{\epsilon,\delta}(x)|v_\epsilon(x) - w_\epsilon(x)|$$

where

$$\gamma_{\epsilon,\delta}(x) = \min\{1, |v_\epsilon(x) - u^{0,\epsilon}(x)| + |w_\epsilon(x) - u^{0,\epsilon}(x)|\}.$$

It follows, from the definition of $\gamma_{\epsilon,\delta}$, that $\|\gamma_{\epsilon,\delta}\|_{L^\infty(\Omega_\epsilon)} \leq 1$, $0 \leq \epsilon \leq \epsilon_0$. Moreover $\|\gamma_{\epsilon,\delta}\|_{L^2(\Omega_\epsilon)} \leq \|v_\epsilon - u^{0,\epsilon}\|_{L^2(\Omega_\epsilon)} + \|w_\epsilon - u^{0,\epsilon}\|_{L^2(\Omega_\epsilon)} \leq 2\delta$, for all $v_\epsilon, w_\epsilon \in B_\delta(u^{0,\epsilon})$. Using Hölder's inequality, we get

$$\|\gamma_{\epsilon,\delta}\|_{L^p(\Omega_\epsilon)} \leq (2\delta)^{2/p} \leq 2(\delta)^{2/p}, \quad 2 \leq p < \infty, \text{ for all } v_\epsilon, w_\epsilon \in B_\delta(u^{0,\epsilon})$$

Now if $\varphi_\epsilon = v_\epsilon - w_\epsilon$ we denote by $\tilde{\varphi}_\epsilon = E(\varphi_\epsilon|_{\Omega_0})|_{\Omega_\epsilon}$. Then

$$\begin{aligned} \|\tilde{\varphi}_\epsilon - \varphi_\epsilon\|_{L^2(\Omega_\epsilon)} &= \|\tilde{\varphi}_\epsilon - \varphi_\epsilon\|_{L^2(\Omega_\epsilon \setminus \Omega_0)} \leq \frac{1}{\tau_\epsilon} \|\nabla \tilde{\varphi}_\epsilon - \nabla \varphi_\epsilon\|_{L^2(\Omega_\epsilon \setminus \Omega_0)} \\ &\leq C\frac{1}{\tau_\epsilon} (\|\varphi_\epsilon\|_{H^1(\Omega_\epsilon)} + \|\tilde{\varphi}_\epsilon\|_{H^1(\mathbb{R}^N)}) \leq C\frac{1}{\tau_\epsilon} (\|\varphi_\epsilon\|_{H^1(\Omega_\epsilon)} + \|\varphi_\epsilon\|_{H^1(\Omega_0)}) \\ &\leq C\frac{2}{\tau_\epsilon} \|\varphi_\epsilon\|_{H^1(\Omega_\epsilon)}, \end{aligned}$$

where we have used that $E : H^1(\Omega_0) \rightarrow H^1(\mathbb{R}^N)$ is bounded and τ_ϵ is the first eigenvalue of $-\Delta$ in $\Omega_\epsilon \setminus \Omega_0$ with Dirichlet boundary condition in $\partial\Omega_0$ and Neumann boundary condition in $\partial\Omega_\epsilon$. Now

$$\begin{aligned} \|\gamma_{\epsilon,\delta}\varphi_\epsilon\|_{L^2(\Omega_\epsilon)} &\leq \|\gamma_{\epsilon,\delta}(\varphi_\epsilon - \tilde{\varphi}_\epsilon)\|_{L^2(\Omega_\epsilon)} + \|\gamma_{\epsilon,\delta}\tilde{\varphi}_\epsilon\|_{L^2(\Omega_\epsilon)} \\ &\leq \|\gamma_{\epsilon,\delta}\|_{L^\infty(\Omega_\epsilon)}\|\varphi_\epsilon - \tilde{\varphi}_\epsilon\|_{L^2(\Omega_\epsilon)} + \|\gamma_{\epsilon,\delta}\|_{L^N(\Omega_\epsilon)}\|\tilde{\varphi}_\epsilon\|_{L^{2N/(N-2)}(\mathbb{R}^N)} \\ &\leq (C\frac{2}{\tau_\epsilon} + C\delta^{2/N})\|\varphi_\epsilon\|_{H^1(\Omega_\epsilon)} \end{aligned}$$

This proves the lemma. \blacksquare

As an immediate consequence of this proposition, we have

Corollary 4.3. *Assume the conditions of Proposition 4.1 hold. Assume moreover that problem $(P)_0$ has exactly m solutions u_1^0, \dots, u_m^0 and that all of them are hyperbolic in the sense that 0 is not in the spectrum of $\Delta - I + f'(u_k^0)I : H_n^2(\Omega_0) \subset L^2(\Omega_0) \rightarrow L^2(\Omega_0)$ for $k = 1, \dots, m$. Then there exists a small $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$ problem $(P)_\epsilon$ has exactly m solutions $u_1^\epsilon, \dots, u_m^\epsilon$. Moreover, we have*

$$\|u_k^\epsilon - u_k^0\|_{H_\epsilon^1} \rightarrow 0, \text{ as } \epsilon \rightarrow 0.$$

Proof: By Proposition 3.2 we have that for any solution u^ϵ of $(P)_\epsilon$ for ϵ small enough lies in a neighborhood of the set of equilibria $(P)_0$. But by Proposition 4.1, in a neighborhood of u_k^0 there is only one solution of $(P)_\epsilon$ which converges to u_k^0 in H_ϵ^1 . This proves the result. \blacksquare

4.2. Continuity of Unstable Manifolds. In this section we show that the local unstable manifolds of u^ϵ , for $k = 1, \dots, m$ fixed, are continuous in H_ϵ^1 as $\epsilon \rightarrow 0$. The existence of this manifold follows from standard invariant manifold theory, see [6], although its proof is adapted to encompass the possibility that the space changes according to a parameter and to keep track of the dependence of the invariant manifold upon the parameter. After this, we show that the unstable manifolds are close for small ϵ . For this we will use the convergence results on the linear part obtained in Section 2.

We have the following

Proposition 4.4. *Assume that the family of domains $\{\Omega_\epsilon : 0 \leq \epsilon \leq \epsilon_0\}$ is admissible. Assume also that u^0 is a solution of problem $(P)_0$ and that zero is not in the spectrum of the operator $\Delta - I + f'(u^0) : H_n^2(\Omega_0) \subset L^2(\Omega_0) \rightarrow L^2(\Omega_0)$. By Proposition 4.1, $(P)_\epsilon$ has a unique solution, u^ϵ , near u_0 . Then, there exist $\delta, \epsilon_0 > 0$ such that u^ϵ has a local unstable manifold $W_{loc}^u(u^\epsilon) \subset H^1(\Omega_\epsilon)$ for $0 \leq \epsilon \leq \epsilon_0$ and if we denote by*

$$W_\delta^u(u^\epsilon) = \{w \in W_{loc}^u(u^\epsilon), \|w - u^\epsilon\|_{H^1(\Omega_\epsilon)} < \delta\}, \quad 0 \leq \epsilon \leq \epsilon_0$$

then $W_\delta^u(u^\epsilon)$ converges in H_ϵ^1 to $W_\delta^u(u^0)$ as $\epsilon \rightarrow 0$, that is

$$\sup_{w_\epsilon \in W_\delta^u(u^\epsilon)} \inf_{w_0 \in W_\delta^u(u^0)} \|w_\epsilon - w_0\|_{H_\epsilon^1} + \sup_{w_0 \in W_\delta^u(u^0)} \inf_{w_\epsilon \in W_\delta^u(u^\epsilon)} \|w_\epsilon - w_0\|_{H_\epsilon^1} \rightarrow 0, \text{ as } \epsilon \rightarrow 0$$

Proof: See Arrieta+Carvalho

As an immediate consequence of this proposition, we have

Corollary 4.5. *Assume the conditions of Proposition 4.4 hold, that problem $(P)_0$ has exactly m solutions u_1^0, \dots, u_m^0 and that all of them are hyperbolic. Then there exist $\epsilon_0, \delta > 0$ small enough such that problem $(P)_\epsilon$ has exactly m solutions and their local unstable manifolds $W_\delta^u(u_k^\epsilon)$, $k = 1, \dots, m$ behave continuously in H_ϵ^1 as $\epsilon \rightarrow 0$.*

4.3. Continuity of Attractors. We are now in position to prove the central result of our work.

Theorem 4.6. *Assume that the family of domains $\{\Omega_\epsilon, 0 \leq \epsilon \leq \epsilon_0\}$ is admissible and that every equilibrium of the unperturbed problem $(P)_0$ is hyperbolic. Then the attractors \mathcal{A}_ϵ behave continuously in H_ϵ^1 as $\epsilon \rightarrow 0$, that is*

$$\sup_{u_\epsilon \in \mathcal{A}_\epsilon} \inf_{u_0 \in \mathcal{A}_0} \|u_\epsilon - u_0\|_{H_\epsilon^1} + \sup_{u_0 \in \mathcal{A}_0} \inf_{u_\epsilon \in \mathcal{A}_\epsilon} \|u_\epsilon - u_0\|_{H_\epsilon^1} \rightarrow 0, \text{ as } \epsilon \rightarrow 0$$

Proof: Since we have already shown in Proposition 3.2 the upper semicontinuity of attractors, we just need to show the lower semicontinuity. This will follow from the continuity of the local unstable manifolds. To see this, we argue in the following way. If $u_0 \in \mathcal{A}_0$ then u_0 belongs to the unstable manifold of u_k^0 for some $1 \leq k \leq m$. Let $\delta > 0$ be the one obtained in Proposition 4.4. If τ is such that $w_0 = T_0(-\tau, u_0) \in W_\delta^u(u_k^0)$, from the continuity of the unstable manifolds there is a sequence $w_\epsilon \in W_\delta^u(u_k^\epsilon)$ which converges to w_0 in H_ϵ^1 as $\epsilon \rightarrow 0$. Now, since the family of semigroups is continuous in H_ϵ^1 we have that $\mathcal{A}_\epsilon \ni T_\epsilon(\tau, w_\epsilon) \rightarrow T_0(\tau, w_0) = u_0$ in H_ϵ^1 as $\epsilon \rightarrow 0$. Showing the lower semicontinuity of attractors. This proves the theorem

Remark 4.7. *The dynamics of (1.1) has been compared in the space H_ϵ^1 . This means that, for instance, in the case of exterior perturbations of the domain the restriction to Ω_0 of equilibria, unstable manifolds and attractors of (1.1) in Ω_ϵ converges in $H^1(\Omega_0)$ to the equilibria, unstable manifolds and attractor of the same problem in Ω_0 .*

We may explore now the possibility of obtaining convergence in stronger norms. For this what we need is to have uniform bounds of the attractors in stronger norms. In order to accomplish this we first note that we may easily obtain uniform $L^\infty(\Omega_\epsilon)$ bounds of u_t in the attractors, that is, there exists a constant C independent of ϵ such that

$$\sup\{\|u_t(t, \phi_\epsilon)\|_{L^\infty(\Omega_\epsilon)}, \phi_\epsilon \in \mathcal{A}_\epsilon, t \in \mathbb{R}, 0 \leq \epsilon \leq \epsilon_0\} \leq C.$$

To obtain this we follow the steps given in Proposition 5.1 of [?].

Hence, we can view equation (1.1) for fixed t as an elliptic equation $-\Delta u_\epsilon + u_\epsilon = F(u_\epsilon) + u_\epsilon - u_{\epsilon,t}$ and notice that the right hand side is uniformly bounded in $L^\infty(\Omega_\epsilon)$ when $u_\epsilon(t)$ is an orbit in the attractor \mathcal{A}_ϵ . Therefore, the problem of obtaining uniform bounds in stronger norms is reduced to obtaining uniform bounds for the solution of the elliptic problem

$$\begin{cases} -\Delta u + u = g, & \Omega_\epsilon \\ \frac{\partial u}{\partial n} = 0, & \partial\Omega_\epsilon \end{cases} \quad (4.3)$$

when $g \in L^\infty(\Omega_\epsilon)$, $\|g\|_{L^\infty(\Omega_\epsilon)} \leq C$, with C independent of ϵ .

Hence if, for instance, the family of domains Ω_ϵ is uniformly Hölder then there exists a $\alpha > 0$ and a constant C such that if u is the solution of (4.3) then $\|u\|_{C^\alpha(\Omega_\epsilon)} \leq C$ (see [?]). This allows to obtain convergence in C^β for any $0 < \beta < \alpha$.

5. TWO EXAMPLES

Let us consider in this section two examples of families of admissible domains and therefore where Proposition 2.4 applies and all the results on continuity of the spectrum and of the nonlinear dynamics from these notes hold true. We refer to [2] for details on these two examples. The first one is a C^0 perturbation of the domain which admits a highly oscillatory behavior at the boundary and the second one is a “non standard” dumbbell type domain.

5.1. A C^0 perturbation of the domain. Let $\Omega_0 \subset \mathbb{R}^N$ be a $C^{0,1}$ domain and assume that for any point $\xi \in \partial\Omega_0$, up to a rigid motion we have that

$$\Omega_0 \cap \{x \in \mathbb{R}^N : |x_i - \xi_i| < \delta\} = \{x = (x', x_N) : x_N = \xi_N + f_0(x'), |x_i - \xi_i| < \delta, i = 1, \dots, N-1\}$$

for certain Lipschitz function f_0 and where, as it is done customarily, we denote by $x' = (x_1, \dots, x_{N-1})$ so that $x = (x', x_N)$.

In order to simplify the notation assume that $\xi = 0$. Hence

$$\Omega_0 \cap \{x \in \mathbb{R}^N : |x_i| < \delta\} = \{x = (x', x_N) : x_N < f_0(x'), |x_i| < \delta, i = 1, \dots, N-1\}$$

Assume that

$$\Omega_\epsilon \cap \{x \in \mathbb{R}^N : |x_i| < \delta\} = \{x = (x', x_N) : x_N < f_\epsilon(x'), |x_i| < \delta, i = 1, \dots, N-1\}$$

where $f_\epsilon \rightarrow f_0$ uniformly in $\{x' : |x'_i| < \delta\}$.

Notice also that by definition

$$\partial K_\epsilon \cap \{x \in \mathbb{R}^N : |x_i| < \delta\} = \{x = (x', x_N) : x_N = g_\epsilon(x'), |x_i| < \delta, i = 1, \dots, N-1\}$$

for certain function g_ϵ with $g_\epsilon < f_0$, $g_\epsilon < f_\epsilon$ and $g_\epsilon \rightarrow f_0$ uniformly in $\{x' : |x'_i| < \delta\}$.

If we denote by

$$R_{\epsilon, \delta} = (\Omega_\epsilon \setminus K_\epsilon) \cap \{x; |x_i| < \delta\} = \{x = (x', x_N) : |x_i| < \delta, g_\epsilon(x') < x_N < f_\epsilon(x')\}.$$

we have

$$\|\nabla u_\epsilon\|_{L^2(R_{\epsilon, \delta})}^2 = \int_{-\delta}^{\delta} \dots \int_{-\delta}^{\delta} \int_{g_\epsilon(x')}^{f_\epsilon(x')} \left| \frac{\partial u}{\partial x_n} \right|^2 dx_N dx'$$

But for x' fixed, applying Poincaré inequality in one dimension, we have

$$\int_{g_\epsilon(x')}^{f_\epsilon(x')} \left| \frac{\partial(u_\epsilon \circ \chi^{-1})}{\partial x_n} \right|^2 dx_N \geq \frac{\pi^2}{4|f_\epsilon(x') - g_\epsilon(x')|^2} \int_{g_\epsilon(x')}^{f_\epsilon(x')} |u_\epsilon|^2 dx_N$$

which implies that

$$\|\nabla u_\epsilon\|_{L^2(R_{\epsilon, \delta})}^2 \geq \frac{\pi^2}{4\|f_\epsilon - g_\epsilon\|_{L^\infty}^2} \|u_\epsilon\|_{L^2(R_{\epsilon, \delta})}^2$$

and since $f_\epsilon, g_\epsilon \rightarrow f_0$ uniformly in $\{x' : |x'_i| < \delta\}$ then there exists $\kappa_\epsilon \rightarrow \infty$ as $\epsilon \rightarrow 0$, such that

$$\|\nabla u_\epsilon\|_{L^2(R_{\epsilon, \delta})}^2 \geq \kappa_\epsilon \|u_\epsilon\|_{L^2(R_{\epsilon, \delta})}^2$$

Since this argument can be done for a finite covering of $\partial\Omega_0$ we obtain that

$$\|\nabla u_\epsilon\|_{L^2(\Omega_\epsilon \setminus K_\epsilon)}^2 \geq C\kappa_\epsilon \|u_\epsilon\|_{L^2(\Omega_\epsilon \setminus K_\epsilon)}^2$$

for certain constant C independent of ϵ . This shows that ii) holds.

Notice that the only requirement on f_ϵ is the uniform convergence to f_0 . In particular we may consider perturbations with a highly oscillating behavior. For instance

$$f_\epsilon(x') = f_0(x') + \epsilon F\left(\frac{x_1}{\epsilon^{\alpha_1}}, \dots, \frac{x_{N-1}}{\epsilon^{\alpha_{N-1}}}\right)$$

where $F : \mathbb{R}^{N-1} \rightarrow \mathbb{R}$ is a smooth bounded function.

5.2. A non standard dumbbell type perturbation. A typical dumbbell domain consists of a pair of disjoint domains Ω_L and Ω_R which are joined by a thin channel R_ϵ . Usually the shape of the channel is given by (for instance in two dimensions)

$$R_\epsilon = \{(x, y) : x \in (0, L), 0 < y < \epsilon g_\epsilon(x)\}$$

where $g_\epsilon \rightarrow g_0$ uniformly in $[0, L]$ and g_0 is some smooth strictly positive function.

The unperturbed domain is given by $\Omega_0 = \Omega_L \cup \Omega_R$. The dumbbell domain is given by $\Omega_\epsilon = \Omega_L \cup R_\epsilon \cup \Omega_R$. It represents a prototype of nonconvex perturbation and it has been extensively studied from many points of view. In terms of the spectral behavior of the Laplace operator, the results in [1] say that there is a net contribution of the spectra of the Laplace operator coming from the thin channel. That is, the eigenvalues and eigenfunctions of the dumbbell domain converge as $\epsilon \rightarrow 0$ to the eigenvalues and eigenfunctions of the unperturbed domain $\Omega_0 = \Omega_L \cup \Omega_R$ and to the eigenvalues and eigenfunctions of a problem coming from the channel:

$$\begin{cases} -\frac{1}{g_0}(g_0 u_x)_x = \mu u, & x \in (0, L) \\ u(0) = 0, & u(1) = 0 \end{cases} \quad (5.1)$$

Moreover, it is known that the eigenvalues of

$$\begin{cases} -\Delta u = \tau u, & x \in R_\epsilon \\ u = 0, & \partial R_\epsilon \cap \partial(\Omega_L \cup \Omega_R) \\ \frac{\partial u}{\partial n} = 0, & \partial R_\epsilon \setminus \partial(\Omega_L \cup \Omega_R) \end{cases} \quad (5.2)$$

converge to the eigenvalues of (5.1).

In particular, ii) of Proposition 2.4 does not hold and we cannot apply the results in this paper.

Here, we are going to construct a dumbbell domain $\Omega_\epsilon \subset \mathbb{R}^N$, $N \geq 2$, with a thin channel R_ϵ such that property ii) of Proposition 2.4 holds, that is, the first eigenvalue of (5.2) diverges to infinity as the parameter $\epsilon \rightarrow 0$. For this dumbbell domain we obtain the convergence of the spectra given by Proposition 2.4, that is, the eigenvalues and eigenfunctions in Ω_ϵ converge to the eigenvalues and eigenfunctions of Ω_0 , so that no contribution from the channel occurs. Hence, all the results of this paper will apply to this example.

The channel R_ϵ will be constructed as follows:

$$R_\epsilon = \{(x, x') : x \in (0, L), x' \in \mathbb{R}^{N-1}, |x'| < g_\epsilon(x)\}$$

where

$$g_\epsilon(x) = \begin{cases} (\frac{1}{2} - \frac{x}{2L})^\frac{1}{\epsilon}, & 0 < x < L/2 \\ (\frac{x}{2L})^\frac{1}{\epsilon}, & L/2 < x < L \end{cases}$$

We refer to [2] and [3] for details on how to show that for this channel $\tau_\epsilon \rightarrow +\infty$.

Remark 5.1. *For this kind of dumbbell domain the formation of nonconstant stable equilibrium solutions is a direct consequence of Proposition 3.2. If for instance we consider the nonlinearity $f(u) = u - u^3$, we have that for any domain the equilibria $u = 1$ and $u = -1$ are asymptotically stable. Hence if we consider u_0 an equilibria in $\Omega_0 = \Omega_L \cup \Omega_R$ given by $u_0 = 1$ in Ω_L and $u_0 = -1$ in Ω_R , we know that this equilibrium is asymptotically stable. By Proposition 3.2 there exists an equilibrium $u_\epsilon \in H^1(\Omega_\epsilon)$ which is near u_0 in H_ϵ^1 and that the linearization around u_ϵ converges to the linearization of the limit problem around u_0 . In particular u_ϵ is an asymptotically stable equilibrium (with the same index as u_0) and u_ϵ is obviously nonconstant.*

REFERENCES

- [1] J. M. Arrieta, “Neumann eigenvalue problems on exterior perturbations of the domain”, *J. Differential Equations* **118**, pp. 54-103, (1995).
- [2] J. M. Arrieta, A. N. Carvalho, Spectral Convergence and nonlinear dynamics of reaction diffusion equations under perturbation of the domain, *J. Differential Equations* **199**, 143-178, (2004)
- [3] J.M. Arrieta, D. Krejcirik, Geometric versus spectral convergence for the Neumann Laplacian under exterior perturbations of the domain, *Integral methods in science and engineering. Volume I. Analytic Methods*, pp 9-19, Birkhuser Boston, Boston, MA, (2010)
- [4] R. Courant and D. Hilbert, *Methods of Mathematical Physics, Vol I*, Wiley-Interscience, New York (1953). German Edition (1937).
- [5] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, Mathematical Surveys and Monographs **25** American Mathematical Society, Providence 1988.
- [6] D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics **840**, Springer-Verlag, Berlin, (1981).