A representation formula for the best constant in the Sobolev immersion $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$

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Abstract

Let $\lambda_q := \inf\left\{ \|\nabla u\|_p^p \,/\, \|u\|_q^p : u \in W_0^{1,p}(\Omega) \setminus \{0\} \right\}$, where Ω is a bounded and smooth domain of $\mathbb{R}^N$, $1 < p < N$ and $1 \le q \le p^\star := \frac{Np}{N-p}$. ($\sqrt[p]{\lambda_q}$ is the best constant in the Sobolev immersion $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$.)

For each $q \in [1, p^\star)$ let

$$E_q := \left\{ u \in W_0^{1,p}(\Omega) : \|u\|_q = 1 \text{ and } \|\nabla u\|_p = \sqrt[p]{\lambda_q} \right\}$$

denote the set of the L^q-normalized extremal functions corresponding to λ_q.

We prove that the following representation formula

$$\lambda_q = \lambda_1 \exp\left(-p \int_1^q \frac{1}{s^2} \int_\Omega |u_s|^s \log |u_s|^s \, dx ds \right)$$

is valid for all $q \in [1, p^\star)$, where $u_s \in E_s$.

For this, after presenting some properties of the function $q \in [1, p^\star) \mapsto \lambda_q$ (among them the absolute continuity) we verify that

$$\lambda_q' + \lambda_q \left(\frac{p}{q^2} \int_\Omega |u_q|^q \log |u_q|^q \, dx \right) = 0$$

at each point q where the derivative λ_q' of λ_q exists.

It follows from our results that λ_q is differentiable at any $q \in [1, p]$ and, moreover, that λ_q is differentiable at $q \in (p, p^\star)$ if, and only if, the functional $I_q : W_0^{1,p}(\Omega) \to \mathbb{R}$, defined by

$$I_q(u) := \int_\Omega |u_q|^q \log |u_q|^q \, dx,$$

is constant on E_q. Thus, I_q is constant on E_q for almost all $q \in (p, p^\star)$ and, in the particular case where Ω is a ball, λ_q is also differentiable at any point of this interval.

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