Eigenvalue problems for elliptic operators on Riemannian Manifolds

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This serie of lectures will mainly concern the spectrum of the Laplacian on a compact Riemannian manifold (M,g) and of some other elliptic operators. The goal is to explain how to estimate the eigenvalues of the operators with respect to the geometry of the manifold (thanks to geometric invariants of the manifold like volume, diameter, curvature). In these lectures, I will mainly focuss on global geometric invariants like the Cheeger constant or the isoperimetric ratio, and I will avoid to use the specific formalism of the Riemannian geometry, like tensor calculus.

Lecture 1. Generalities on the spectrum of a manifold. Variational characterization and min-max, example of the Cheeger's dumbbell.

Lecture 2. The Cheeger's inequality. I will explain this inequality $\lambda_1 \geq \frac{h^2}{4}$, where λ_1 is the first nonzero eigenvalue of the Laplacian and h an isoperimetric constant on the manifold (M,g) and sketch its proof. Depending on the time, I will explain how to get a similar isoperimetric inequality for other differential operators.

Lecture 3. Upper bound for the first nonzero eigenvalue on an hypersurface of the Euclidean space. I will explain and prove a result du to Chavel : If Ω denote a domain of $\mathbb{R}^{n+1}$ with smooth boundary Σ, we have the relation $\lambda_1(\Sigma) \leq \frac{n}{(n+1)^2}\frac{Vol(\Sigma)^2}{Vol(\Omega)^2}$. This will be the opportunity to present a classical method to bound the first nonzero eigenvalue from above.

Lecture 4. Upper bound for the eigenvalues on an hypersurface of the Euclidean space. If Ω denote a domain of $\mathbb{R}^{n+1}$ with smooth boundary Σ, I will explain how to bound all the eigenvalues of the Laplacian and of the Steklov or Dirichlet-to-Neumann operator on Σ thanks to the isoperimetric ratio $I = \frac{Vol_n(\Sigma)}{Vol(\Omega)^{n/(n+1)}}$.

Lecture 5. The method of Korevaar and Grigor'yan-Netrusov-Yau. I will present with more details the method used in Lecture 4. This method can be used for a lot of different problems.