Boundary-Domain Integral and Integro-Differential Equations for Elliptic BVPs with variable coefficients

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The boundary integral equation (BIE) method has been intensively developed over recent decades both in theory and in engineering applications. Its popularity is due to the possibility of reducing a boundary value problem (BVP) for a partial differential equation (PDE) in a domain to an integral equation on the boundary of the domain.

This approach diminishes the problem dimensionality by one which is very important for construction of various numerical algorithms. The main ingredient necessary for reduction of a BVP to a boundary integral equation is a fundamental solution to the original partial differential equation, available in an analytical and/or cheaply calculated form. After the fundamental solution is used in the corresponding Green formulae, as a first step, one can reduce the BVP to a BIE. The next essential steps are to establish the equivalence of the original BVP and the corresponding BIE, and to show the invertibility of the boundary integral operator which is a crucial moment for further numerical analysis.

However, such a fundamental solution is generally not available if the coefficients of the original partial differential equation are not constants.

To obtain a general integral representation of solutions to a PDE with variable coefficients, instead of the fundamental solution one can use a *global parametrix* (Levi function) or a *localized parametrix*, which is usually available explicitly. This allows reducing a boundary value problem not to Boundary Integral Equations but to *Boundary–Domain Integral Equations* (BDIE) or to *Localized Boundary–Domain Integral Equations* (LBDIE).

Different versions of Non-localized and Localized Boundary-Domain Integral Equation methods have been developed recently for elliptic PDEs with variable coefficients. The LBDIE method employs specially constructed localized parametrix which is represented as the product of a global parametrix and a cut-off function $\chi_\varepsilon(x-y)$ with support in the ball centered at x and radius ε. Evidently the Newtonian volume and surface potential operators constructed by means of such a localized parametrix have locally supported kernels.

The system of LBDIEs is derived with the help of the corresponding localized Green's integral representation formula. The approach is a counterpart of the so called direct boundary integral equations method, but the principal difference here is that the parametrix based LBDIE systems contain potential type domain integral (pseudodifferential) operators with unknown density being the sought for solution of the BVP under consideration and the boundary traces of these domain operators, and also layer potential type operators and their traces with unknown densities being either the Dirichlet data or the Neumann data (depending on the BVP) which are considered as independent ("segregated") unknown functions.

The most principal questions in the study of LBDIE systems are

i. The equivalence between the original BVPs and the corresponding LBDIE formulations;
ii. Investigation of the Fredholm properties of the localized operators and establishing their invertibility in appropriate function spaces.

The equivalence between the original BVP problems and the corresponding LBDIE systems, which is quite nontrivial fact, plays a crucial role in further analysis. It turned out that the localized boundary domain integral operators belong to the Boutet de Monvel algebra of operators and their invertibility in appropriate function spaces can be shown with the help of the Vishik-Eskin theory based on the factorization method (the Wiener-Hopf method). This invertibility property implies then existence and uniqueness results for the LBDIE system and the corresponding original boundary value problem.

Beside a pure mathematical interest, this approach seems to be important from the point of view of numerical analysis. The case is that the LBDIE systems can be applied in constructing convenient numerical schemes in applications, since after mesh-based or mesh-less discretization, they lead to systems of algebraic equations with sparse matrices.

We will describe the application of the localized boundary-domain integral equations method to some boundary value and transmission problems for elliptic PDEs of mathematical physics.

The titles of the lectures are:

1) Classical potential method for constant coefficient boundary value problems: Indirect and direct boundary integral equations method.

2) Scalar BVPs with one variable coefficient (isotropic case): Parametrix based BDIE approach.

3) Scalar BVPs with matrix variable coefficient (anisotropic case): Localized harmonic parametrix based BDIE approach;.

4) Applications of BDIE method to transmission problems of acoustic scattering by inhomogeneous anisotropic obstacles.

5) Method of fundamental solutions for mixed and crack type problems in the classical theory of elasticity.