Initiation into corner singularities

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Outline

Part I

Prolegomena

Outline

What it is about

It is about special combinations of

1. Domains
2. Partial Differential Equations
3. Functional spaces

General framework of corner studies

1. Domains with general and combined corner types
2. Elliptic boundary value problems
3. Weighted and standard Sobolev spaces or Hölder classes
4. Existence, regularity, expansions of solutions

A sort of maximal framework would include

1. Cones in $\mathbb{R}^n$, conical manifolds, edges, manifolds with corners, polyhedra, recursively defined corner domains,...
2. Multi-order Agmon-Douglis-Nirenberg systems with variable coefficients, possibly non-smooth with asymptotics, possibly piecewise smooth on compatible partitions
3. From L^2-based Sobolev spaces to Besov spaces, including (or not) weights involving distances to singular sets. Possibly analytic-type control of derivatives.
4. Fredholm, semi-Fredholm, regularity shift, asymptotics of various types

What it is not about

It is not about general Lipschitz domains, though relations do exist with corner domains.

Two hierarchies of domains, from smooth to rough:

1. Smooth, Lipschitz [i.e. locally Lipschitz epigraphs], weakly Lipschitz [bi-Lipschitz maps], Jones domains [extension properties], John domains [flexible cone condition of Besov].
2. Smooth, regular cones, edges, polyhedral cones, curvilinear polyhedral domains.

Note also that

- Outward cusp domains may pertain to corner domains.
- Inward cusp domains don't.
- A corner domain can be non-Lipschitz

A brief history of elliptic BVP with corners: Russian school

V. A. KONDRAT'EV
Boundary-value problems for elliptic equations in domains with conical or angular points.
Trans. Moscow Math. Soc. **16** (1967) 227–313.

1. Domains with conical points
2. Scalar elliptic BVP
3. Hilbert Sobolev spaces with or without weights
4. Fredholm, regularity, asymptotics

V. G. MAZ'YA, B. A. PLAMENEVSKII
Elliptic boundary value problems on manifolds with singularities.
Probl. Mat. Anal. **6** (1977) 85–142.

V. A. KOZLOV, V. G. MAZ'YA, J. ROSSMANN
Elliptic boundary value problems in domains with point singularities.
Mathematical Surveys and Monographs, 52. American Mathematical Society, 1997.

V. MAZ'YA AND J. ROSSMANN
Elliptic equations in polyhedral domains.
Mathematical Surveys and Monographs, 162. American Mathematical Society, 2010.

1. Hierarchy of singular sets
2. Elliptic systems
3. L^p Sobolev spaces, Schauder classes, with weights
4. Fredholm, regularity, asymptotics

A brief history of elliptic BVP with corners: Ψ-do calculus

S. Rempel, B. W. Schulze
Asymptotics for Elliptic Mixed Boundary Problems.
Akademie-Verlag, 1989.

B. W. Schulze
Pseudo-differential operators on manifolds with singularities.
Studies in Mathematics and its Applications, Vol. 24. North-Holland, 1991.

B.-W. Schulze
Boundary value problems and singular pseudo-differential operators.
Pure and Applied Mathematics (New York). John Wiley & Sons Ltd., 1998.

R. B. Melrose
Pseudodifferential operators, corners and singular limits,
Proc. International Congress of Mathematicians, Math. Soc. Japan, (1991), 217–234.

R. B. Melrose
Calculus of conormal distributions on manifolds with corners,
Internat. Math. Res. Notices (1992), no. 3, p. 51–61.

R. B. Melrose
Differential analysis on manifolds with corners ,
http://www-math.mit.edu/ rbm/book.html

A brief history of elliptic BVP with corners: In Italy

A. Avantaggiati, M. Troisi,
Spazi di Sobolev con peso e problemi ellittici in un angolo I.
Ann. Mat. Pura Appl. (4) **95** (1973) 361–408.
A. Avantaggiati, M. Troisi,
Spazi di Sobolev con peso e problemi ellittici in un angolo II.
Ann. Mat. Pura Appl. (4) **97** (1973) 153–167.
A. Avantaggiati, M. Troisi,
Spazi di Sobolev con peso e problemi ellittici in un angolo III.
Ann. Mat. Pura Appl. (4) **99** (1974) 1–51.

1. Infinite cones
2. General scalar elliptic BVP of order $2m$
3. Weighted spaces with 2 weights

A. Alvino, G. Trombetti
The Dirichlet problem in a cone of $\mathbf{R}^n$ in L^p.
Ann. Mat. Pura Appl. (4) **120** (1979) 269–291.

1. Infinite cones
2. General scalar elliptic BVP of order $2m$ with spectral parameter
3. L^p Sobolev spaces with weight

A brief history of elliptic BVP with corners: In France & Belgium

P. GRISVARD
Problèmes aux limites dans les polygones. Mode d'emploi.
Bull. Dir. Etud. Rech., Sér. C **1** (1986) 21–59.

P. GRISVARD
Singularités en élasticité.
Arch. Rational Mech. Anal. **107 (2)** (1989) 157–180.

P. GRISVARD
Boundary Value Problems in Non-Smooth Domains.
Pitman, London 1985.

1. Polygonal domains
2. Elliptic BVP (Laplace, Lamé, Δ^2)

S. NICAISE
Le laplacien sur les réseaux deux-dimensionnels polygonaux topologiques.
J. Math. Pures Appl. (9) **67**(2) (1988) 93–113.

S. NICAISE
Polygonal interface problems.
Methoden und Verfahren der Mathematischen Physik, 39. Verlag Peter D. Lang, 1993.

1. Polygonal domains
2. Elliptic transmission problems (piecewise constant on polygonal subdomains)

A brief history of elliptic BVP with corners: 'CoDa'

M. DAUGE
Elliptic Boundary Value Problems in Corner Domains.
Lecture Notes in Mathematics, Vol. 1341. Springer-Verlag, 1988.

1. Hierarchy of singular sets
2. Elliptic systems
3. Hilbert Sobolev spaces without weights (interaction with polynomials)
4. Semi-Fredholm, Fredholm, regularity, corner-edge asymptotics

M. COSTABEL, M. DAUGE
General edge asymptotics of solutions of second order elliptic boundary value problems.
Proc. Royal Soc. Edinburgh **123A** (1993) 109–155 and 157–184.

M. COSTABEL, M. DAUGE
Construction of corner singularities for Agmon-Douglis-Nirenberg elliptic systems.
Math. Nachr. **162** (1993) 209–237.

M. COSTABEL, M. DAUGE
Stable asymptotics for elliptic systems on plane domains with corners.
Comm. Partial Differential Equations n$^{\circ}$ **9 & 10** (1994) 1677–1726.

1. Edges
2. Elliptic systems
3. Hilbert Sobolev spaces
4. Structure of singularites

A brief history of elliptic BVP with corners: Maxwell

M. COSTABEL, M. DAUGE
Maxwell and Lamé eigenvalues on polyhedra.
Math. Meth. Appl. Sci. **22** (1999) 243–258.

M. COSTABEL, M. DAUGE
Singularities of electromagnetic fields in polyhedral domains.
Arch. Rational Mech. Anal. **151**(3) (2000) 221–276.

M. COSTABEL, M. DAUGE, S. NICAISE
Singularities of Maxwell interface problems.
M2AN Math. Model. Numer. Anal. **33**(3) (1999) 627–649.

1. 3D Polyhedra
2. Harmonic Maxwell
3. Hilbert Sobolev spaces
4. Regularity, structure of singularites

A brief history of elliptic BVP with corners: Analytic regularity

I. BABUŠKA, B. GUO
Regularity of the solution of elliptic problems with piecewise analytic data. I. Boundary value problems for linear elliptic equation of second order,
SIAM J. Math. Anal., 19 (1988) 172–203

I. BABUŠKA, B. GUO
Regularity of the solution of elliptic problems with piecewise analytic data. II. The trace spaces and application to the boundary value problems with nonhomogeneous boundary conditions,
SIAM J. Math. Anal., 20 (1989) 763–781

1. 2D Polygonal domains

M. COSTABEL, M. DAUGE, AND S. NICAISE
Analytic regularity for linear elliptic systems in polygons and polyhedra,
Math. Models Methods Appl. Sci., **22** (2012), 1250015, 63p.

M. COSTABEL, M. DAUGE, AND S. NICAISE
Weighted analytic regularity in polyhedra,
Comput. Math. Appl., **67** (2014) 807–817.

1. 3D Polyhedral domains

M. COSTABEL, M. DAUGE, AND S. NICAISE
GLC project (Grand Livre des Coins),
?? (20??)

Outline

1. Brief history of corner studies
2. **Standard theory for elliptic BVP and coercive forms in smooth domains**
3. Typical examples of coercive problems

Elliptic Boundary Value Problems

For scalar operators of order 2μ in a domain $\Omega \subset \mathbb{R}^n$:

$$\begin{cases} Lu = f & \text{in } \Omega \\ B_\ell u = 0 \quad (\ell = 1, \dots, \mu) & \text{on } \partial\Omega. \end{cases}$$

Smoothness and ellipticity (*cf.* Shapiro-Lopatinski condition)

1. L has smooth coefficients on $\overline{\Omega}$. In each $x \in \overline{\Omega}$, the symbol of its principal part at $L^{\mathrm{pr}}(x;\xi)$ is non-zero for any value of the dual variable $\xi \in \mathbb{R}^n \setminus \{0\}$
2. $\partial\Omega$ is smooth, B_ℓ have smooth coefficients and order $\mu_\ell < 2\mu$. In each $x' \in \partial\Omega$ and each $\xi' \in \mathbb{R}^{n-1} \setminus \{0\}$, the parametric BVP

$$\begin{cases} L^{\mathrm{pr}}(x';\xi', i\partial_{x_n})u = 0 & \text{in } \mathbb{R}_+ \\ B_\ell^{\mathrm{pr}}(x';\xi', i\partial_{x_n})u = g_\ell \quad (\ell = 1, \dots, \mu) & \text{on } x_n = 0. \end{cases}$$

has a unique exponentially decreasing solution for any $(g_1, \dots, g_\mu) \in \mathbb{R}^m$.

For $d \times d$ systems (L_{ij}) of different orders complemented by suitable boundary conditions, a general notion of ellipticity is introduced and analyzed in

S. Agmon, A. Douglis, and L. Nirenberg
Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II,
Comm. Pure Appl. Math., **17** (1964) 35–92.

Elliptic systems of order 2

In most part of these lectures, we consider elliptic $d \times d$ systems of order 2:

$$L = \big(L_{ij}\big)_{i,j=1,\ldots,d} \quad \text{with} \quad L_{ij} = \sum_{|\alpha|\le 2} L_{ij}^{\alpha}(x)\,\partial_x^{\alpha} \quad \text{and} \quad \partial_x^{\alpha} = \partial_{x_1}^{\alpha_1}\ldots\partial_{x_n}^{\alpha_n}$$

with d boundary operators $B_\ell = \big(B_{\ell j}\big)_{j=1,\ldots,d}$ of order $\mu_\ell \in \{0,1\}$, for $\ell = 1,\ldots,d$.

Examples:

1 Laplace with Dirichlet BC $(d = 1)$
$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

2 Laplace with Neumann BC $(d = 1)$
$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \partial_{\boldsymbol{n}} u = 0 & \text{on } \partial\Omega. \end{cases}$$

3 Lamé with simple support BC's $(d = n)$: Setting $\boldsymbol{u}_\tau = \boldsymbol{u} - (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n}$

$$\begin{cases} -(\mu\Delta + (\lambda+\mu)\nabla\,\mathrm{div})\boldsymbol{u} = \boldsymbol{f} & \text{in } \Omega \\ \boldsymbol{u}_\tau = 0 & \text{on } \partial\Omega \\ \mu(\partial_{\boldsymbol{n}}\boldsymbol{u})\cdot\boldsymbol{n} + (\lambda+\mu)\,\mathrm{div}\,\boldsymbol{u} = 0 & \text{on } \partial\Omega \end{cases}$$

4 Maxwell with electric perfect conductor condition $(d = n = 3)$

$$\begin{cases} (\mathbf{curl}\,\mathbf{curl} - s\nabla\,\mathrm{div})\boldsymbol{u} = \boldsymbol{f} & \text{in } \Omega \\ \boldsymbol{u}_\tau = 0 & \text{on } \partial\Omega \\ \mathrm{div}\,\boldsymbol{u} = 0 & \text{on } \partial\Omega \end{cases}$$

Transitional slide: From corso 1 to corso 2

$\{L, B\}$ is an elliptic BVP on a smooth domain $\Omega \subset \mathbb{R}^n$ (or in a smooth manifold).

Recall some elements of the definition of ellipticity, making use of the concepts of

1. model problems and 2. symbols

1. Model problems $L^{\mathrm{pr}}_{x_0}$ are attached to each $x_0 \in \overline{\Omega}$ and $B^{\mathrm{pr}}_{x_0}$ to each $x_0 \in \partial\Omega$.
They are obtained by

- Applying a local map $\phi_{x_0} : x \mapsto y$ with $x_0 \mapsto 0$ to flatten the geometry
  - $\mathbb{R}^n$ for an interior point x_0 of a manifold,
  - $\mathbb{R}^n_+ := \mathbb{R}^{n-1} \times \mathbb{R}_+$ for $x_0 \in \partial\Omega$. Denote $y =: (y', y_n)$ coord. in $\mathbb{R}^n_+$.
- Freezing coefficients at 0
- Taking the principal part.

Hence the operators $L^{\mathrm{pr}}_{x_0}(\partial_y)$ and $B^{\mathrm{pr}}_{x_0}(\partial_y)$ with homogeneous constant coefficients.

2. Symbols of the model problems are considered:

- Symbol in standard sense for interior points: $L^{\mathrm{pr}}_{x_0}(i\xi)$, $\xi \in \mathbb{R}^n$.
- Symbol in operator sense for boundary points: $\{L^{\mathrm{pr}}_{x_0}(i\xi', \partial_{y_n}), B^{\mathrm{pr}}_{x_0}(i\xi', \partial_{y_n})\}$

Ellipticity of $\{L, B\}$ means that these two symbols are invertible (in a certain sense) for all non-zero value of the dual variable.

Elliptic systems of order 2: Classical results

With $H^m(\Omega)$ the Sobolev space of exponent m, let for $m \geq 2$

$$\boldsymbol{H}^m(\Omega;B) = \{\boldsymbol{u} \in H^m(\Omega)^d, \quad B\boldsymbol{u} = 0 \ \text{ on } \ \partial\Omega\} \quad \text{and} \quad \boldsymbol{H}^{m-2}(\Omega) = H^{m-2}(\Omega)^d.$$

Example of Dirichlet conditions: If $B = \mathbb{I}_{d\times d}$, then $\boldsymbol{H}^m(\Omega;B) = (H^m \cap H^1_0)(\Omega)^d$.

Let $\{L, B\}$ be an elliptic system of order 2 with smooth coefficients on the bounded regular domain Ω.

Theorem (Fredholm property)

Let $m \geq 2$. Then L is Fredholm

$$L : \boldsymbol{H}^m(\Omega;B) \longrightarrow \boldsymbol{H}^{m-2}(\Omega)$$

i.e. the kernel of L is finite dimensional, its range has a finite codim (and is closed).

Theorem (regularity shift)

Let $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega;B)$. Let $m > 2$. If $L\boldsymbol{u}$ belongs to $\boldsymbol{H}^{m-2}(\Omega)$, then $\boldsymbol{u} \in \boldsymbol{H}^m(\Omega)$ with estimates

$$\|\boldsymbol{u}\|_{H^m(\Omega)} \leq C\Big(\|L\boldsymbol{u}\|_{H^{m-2}(\Omega)} + \|\boldsymbol{u}\|_{H^2(\Omega)}\Big)$$

with a constant C depending only on Ω, L, B, and m.

Systems of order 2 in variational form

This requires two ingredients:

1. A complex sesquilinear form a of order 1 defined on a "maximal" space $X(\Omega)$

$$a(u,v) = \sum_{i=1}^{d} \sum_{j=1}^{d} \sum_{|\alpha|\le 1} \sum_{|\beta|\le 1} \int_\Omega a_{ij}^{\alpha\beta}(x)\, \partial_x^\alpha u_i\, \partial_x^\beta \bar{v}_j \,\mathrm{d}x, \quad u,v \in X(\Omega)$$

   Examples of maximal spaces

   - For acoustics, elasticity and in most cases, $X(\Omega) = H^1(\Omega)^d$
   - For electromagnetism (Maxwell system), $X(\Omega) = H(\mathbf{curl};\Omega) \cap H(\mathrm{div};\Omega)$

2. A "variational" space $V \subset X(\Omega)$ that determines essential (or Dirichlet) boundary conditions

$$V = \{u \in X(\Omega), \quad \Pi^D u = 0 \ \text{ on } \ \partial\Omega\}$$

   where Π^D is a chosen smooth projection operator (examples will follow).

Coercivity

Definition

Let a be a sesquilinear form defined on $X(\Omega)$ and associated with $V \subset X(\Omega)$.

- a is said *continuous* if

$$(1) \qquad \exists C > 0, \quad \forall u, v \in V, \quad |a(u,v)| \le C \|u\|_{X(\Omega)} \|v\|_{X(\Omega)}$$

  Then a defines a unique bounded operator A

$$A : V \to V' \qquad u \longmapsto \big(v \mapsto a(u,v)\big)$$

- Let V be compactly embedded in $L^2(\Omega)$. The form a is said *V-coercive* if

$$(2) \qquad \exists C > 0,\ c > 0 \quad \forall u \in V, \quad \operatorname{Re} a(u,u) \ge c\|u\|^2_{X(\Omega)} - C\|u\|^2_{L^2(\Omega)}$$

- a is said *strongly V-coercive* if C can be taken to 0 in (2).

Theorem S.1

- If a is continuous and strongly V-coercive, then A is an isomorphism $V \to V'$
- If a is continuous and V-coercive, then A is Fredholm of index 0 from V into V'

Solution in weak sense is solution in strong sense

Introduce the $d \times d$ system $L = (L_{ij})$ with

$$L_{ij} = \sum_{|\alpha| \leq 1} \sum_{|\beta| \leq 1} (-1)^{|\beta|} \partial_x^\beta a_{ij}^{\alpha\beta}(x)\, \partial_x^\alpha, \quad x \in \Omega$$

and the conormal $d \times d$ system on the boundary $N = (N_{ij})$ with

$$N_{ij} = \sum_{|\alpha| \leq 1} \sum_{|\beta| = 1} \boldsymbol{n}^\beta(x) a_{ij}^{\alpha\beta}(x)\, \partial_x^\alpha, \quad x \in \partial\Omega$$

where, for $|\beta| = 1$, $\boldsymbol{n}^\beta(x)$ is the component β of the unit outward normal $\boldsymbol{n}$ to $\partial\Omega$ at x.

Assume that $X(\Omega) = H^1(\Omega)$.
Let a be continuous and V-coercive with smooth coeffs on a bounded regular domain Ω

Theorem

Assume that $u \in V$ is solution of $Au = f$ with $f \in L^2(\Omega)$ in weak sense, i.e.

$$\forall v \in V, \;\; a(u, v) = \int_\Omega f\, \bar{v}\, \mathrm{d}x$$

Then u belongs to $\boldsymbol{H}^2(\Omega)$ and solves the BVP in strong sense

$$\left\{ \begin{array}{rcll} Lu & = & f & \text{in } \Omega \\ \Pi_D u & = & 0 & \text{on } \partial\Omega \quad \text{(essential boundary condtions)} \\ (\mathbb{I} - \Pi_D) Nu & = & 0 & \text{on } \partial\Omega \quad \text{(natural boundary condtions)} \end{array} \right.$$

The BVP induced by a coercive form is elliptic

Theorem S.2 → Localization → Resolvent estimates

Under the previous assumptions:

1. The boundary value system $\{L, B\}$ with $B = \{\Pi_D, (\mathbb{I} - \Pi_D)N\}$ is elliptic.
2. Set $f = Au$. If $f \in \boldsymbol{L}^2(\Omega)$, then $u \in \boldsymbol{H}^2(\Omega)$ and exists C independent of u such that

$$\|u\|_{H^2(\Omega)} \leq C\Big(\|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}\Big) \tag{1}$$

3. Let $\mathcal{U}$ and $\mathcal{U}'$ two open sets in $\mathbb{R}^n$ such that $\overline{\mathcal{U}} \subset \mathcal{U}'$. Set $\mathcal{V} = \Omega \cap \mathcal{U}$, $\mathcal{V}' = \Omega \cap \mathcal{U}'$. If $\partial\Omega \cap \mathcal{U}'$ is smooth, and $f \in \boldsymbol{L}^2(\mathcal{V}')$ then $u \in \boldsymbol{H}^2(\mathcal{V})$ and we have the following local a priori estimates with a constant independent of u

$$\|u\|_{H^2(\mathcal{V})} \leq C\Big(\|f\|_{L^2(\mathcal{V}')} + \|u\|_{H^1(\mathcal{V}')}\Big) \tag{2}$$

4. If f belongs to $\boldsymbol{H}^{m-2}(\Omega)$ with $m > 2$, then u belongs to $\boldsymbol{H}^m(\Omega)$ and the estimates (1) and (2) hold with H^m and H^{m-2} norms instead of H^2 and L^2 norms.

Points 2 – 4 are classical results.

For 1, see Chap. 3, Sec. 2 in

M. Costabel, M. Dauge, and S. Nicaise
Corner Singularities and Analytic Regularity for Linear Elliptic Systems. Part I: Smooth domains (2010) https://hal.archives-ouvertes.fr/hal-00453934

Outline

1 The Dirichlet Laplacian

Scalar $\nabla \cdot \nabla$ form (here $d = 1$)

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x \quad = \sum_{|\alpha|=1} \int_\Omega \partial_x^\alpha u \, \partial_x^\alpha v \, \mathrm{d}x \quad = \sum_{\ell=1}^{n} \int_\Omega \partial_{x_\ell} u \, \partial_{x_\ell} v \, \mathrm{d}x$$

The space $X(\Omega)$ is the Sobolev space $H^1(\Omega) = \{u \in L^2(\Omega), \ \nabla u \in L^2(\Omega)^n\}$.

Dirichlet conditions: The projection operator Π^D is the identity on $\partial\Omega$, so

$$V = H_0^1(\Omega) = \mathring{H}^1(\Omega) \ = \{u \in H^1(\Omega), \quad u = 0 \ \text{ on } \ \partial\Omega\}$$

Then the form a is strongly coercive.

If $Au = f$ with $f \in L^2(\Omega)$, then u solves the BVP

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases}$$

Note: The BVP makes also sense for $f \in H^{-1}(\Omega)$, the dual space of $H_0^1(\Omega)$.

2 The Neumann Laplacian

Scalar $\nabla\cdot\nabla$ *form*: $a(u,v) = \int_\Omega \nabla u\cdot\nabla v \,\mathrm{d}x$, with $X(\Omega) = H^1(\Omega)$.

Neumann conditions: The projection operator Π^D is 0 on $\partial\Omega$, so

$$V = H^1(\Omega)$$

Then the form a is coercive (and not strongly coercive).
A is self-adjoint. $\ker A$ is generated by constant functions and $\operatorname{rg} A = \{1\}^\perp$

If $Au = f$ with $f\in L^2(\Omega)$, then u solves the BVP

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \partial_{\boldsymbol{n}} u = 0 & \text{on } \partial\Omega \end{cases}$$

- The identity $Au = f$ makes sense for $f\in H^1(\Omega)'$, and not for $f\in H^{-1}(\Omega)$.
- The BVP does not make sense for general $f\in H^1(\Omega)'$.
- If $g\in H^{-1/2}(\partial\Omega)$ is orthogonal to 1, the problem

$$u\in H^1(\Omega),\quad \forall v\in H^1(\Omega)\quad a(u,v) = \langle g, v\rangle_{H^{-1/2}(\partial\Omega)\,|\,H^{1/2}(\partial\Omega)}$$

has a solution which solves the BVP

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ \partial_{\boldsymbol{n}} u = g & \text{on } \partial\Omega. \end{cases}$$

2' The Laplacian with mixed Dirichlet-Neumann conditions

Scalar $\nabla\cdot\nabla$ *form*: $a(u,v) = \int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x$, with $X(\Omega) = H^1(\Omega)$.

Mixed boundary conditions: Let $\partial_k\Omega$ be the connected components of $\partial\Omega$, $k\in\mathscr{K}$.
Let $\mathscr{K}^{\mathrm{Dir}}\cup\mathscr{K}^{\mathrm{Neu}}$ be a partition of $\mathscr{K}$.
The projection operator Π^D is set to 0 on $\partial_k\Omega$ for $k\in\mathscr{K}^{\mathrm{Neu}}$ and to $\mathbb{I}$ for $k\in\mathscr{K}^{\mathrm{Dir}}$, so

$$V = \{u\in H^1(\Omega),\quad u\big|_{\partial_k\Omega} = 0,\ \ k\in\mathscr{K}^{\mathrm{Dir}}\}$$

Then the form a is coercive (and strongly coercive if $\mathscr{K}^{\mathrm{Dir}}\neq\emptyset$).

If $Au = f$ with $f\in L^2(\Omega)$, then u solves the BVP

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial_k\Omega \text{ for all } k\in\mathscr{K}^{\mathrm{Dir}} \\ \partial_{\boldsymbol{n}} u = 0 & \text{on } \partial_k\Omega \text{ for all } k\in\mathscr{K}^{\mathrm{Neu}} \end{cases}$$

Let

$$\partial_{\mathrm{Dir}}\Omega = \bigcup_{k\in\mathscr{K}^{\mathrm{Dir}}} \partial_k\Omega \quad\text{and}\quad \partial_{\mathrm{Neu}}\Omega = \bigcup_{k\in\mathscr{K}^{\mathrm{Neu}}} \partial_k\Omega$$

It is possible to consider the more general situation where the closures of $\partial_{\mathrm{Dir}}\Omega$ and $\partial_{\mathrm{Neu}}\Omega$ are not disjoint... But this pertains to corner problems!

3 The Lamé system with conditions of simple support

Lamé bilinear form of elasticity defined for $\boldsymbol{u}, \boldsymbol{v} \in H^1(\Omega)^n$

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \Big(2\mu \sum_{i=1}^n \sum_{j=1}^n e_{ij}(\boldsymbol{u})\, e_{ij}(\boldsymbol{v}) + \lambda \operatorname{div} \boldsymbol{u}\ \operatorname{div} \boldsymbol{v} \Big)\, \mathrm{d}x$$

- $\lambda \geq 0$ and $\mu > 0$ are the Lamé coefficients
- $e_{ij}(\boldsymbol{u}) = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ are the components of the strain tensor.

Boundary condition of simple support: The projection Π^D is defined as $\Pi^D \boldsymbol{u} = \boldsymbol{u} - (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n}$, so $\Pi^D \boldsymbol{u} = 0$ means that the tangential component of $\boldsymbol{u}$ is 0 on $\partial\Omega$.

$$V = H_N(\Omega) := \{\boldsymbol{u} \in H^1(\Omega)^n, \quad \boldsymbol{u} - (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n} = 0 \ \text{ on } \ \partial\Omega\}$$

Then the form a is (strongly) coercive.

Set $\boldsymbol{u}_\tau = \boldsymbol{u} - (\boldsymbol{u}\cdot\boldsymbol{n})\boldsymbol{n}$. If $A\boldsymbol{u} = \boldsymbol{f}$ with $\boldsymbol{f} \in L^2(\Omega)^n$, then $\boldsymbol{u}$ solves the system

$$\left\{ \begin{array}{rcll} -(\mu\Delta + (\lambda+\mu)\nabla \operatorname{div})\boldsymbol{u} &=& \boldsymbol{f} & \text{in } \ \Omega \\ \boldsymbol{u}_\tau &=& 0 & \text{on } \ \partial\Omega \\ \mu(\partial_{\boldsymbol{n}}\boldsymbol{u})\cdot\boldsymbol{n} + (\lambda+\mu)\operatorname{div}\boldsymbol{u} &=& 0 & \text{on } \ \partial\Omega \end{array} \right.$$

NB. $-(\mu\Delta + (\lambda+\mu)\nabla \operatorname{div}) = \mu\, \mathbf{curl}\,\mathbf{curl} - (\lambda + 2\mu)\nabla \operatorname{div}$

4 Maxwell system (after regularization) with PCE condition

Maxwell regularized electric form defined for $\boldsymbol{u}, \boldsymbol{v} \in X(\Omega) = H(\mathbf{curl};\Omega) \cap H(\mathrm{div};\Omega)$

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \Big(\mu^{-1}\, \mathbf{curl}\, \boldsymbol{u}\ \ \mathbf{curl}\, \boldsymbol{v} + s\, \mathrm{div}\, \varepsilon \boldsymbol{u}\ \ \mathrm{div}\, \varepsilon \boldsymbol{v} \Big)\, \mathrm{d}x$$

with electric permittivity $\varepsilon > 0$ and magnetic permeability $\mu > 0$. Real parameter $s > 0$.
Electric perfect conductor condition: The projector Π^D is defined as in $\boxed{3}$. Then

$$V = X_N(\Omega) := \{\boldsymbol{u} \in H(\mathbf{curl};\Omega) \cap H(\mathrm{div};\Omega), \quad \boldsymbol{u} - (\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{n} = 0 \ \text{ on } \ \partial\Omega\}$$

NB: If $n = 3$, $\boldsymbol{u} - (\boldsymbol{u} \cdot \boldsymbol{n})\boldsymbol{n} = 0$ if and only if $\boldsymbol{u} \times \boldsymbol{n} = 0$.

Then the form a is (strongly) coercive.

If $A\boldsymbol{u} = \boldsymbol{f}$ with $\boldsymbol{f} \in L^2(\Omega)^n$, then $\boldsymbol{u}$ solves the system (here $\mu = \varepsilon = 1$)

$$\begin{cases} (\mathbf{curl}\,\mathbf{curl}\ - s\nabla\,\mathrm{div})\boldsymbol{u} = \boldsymbol{f} & \text{in } \ \Omega \\ \boldsymbol{u}_\tau = 0 & \text{on } \ \partial\Omega \\ \mathrm{div}\, \boldsymbol{u} = 0 & \text{on } \ \partial\Omega \end{cases}$$

NB: In any open set where $\partial\Omega$ is flat and $\boldsymbol{u}_\tau = 0$, we have $\mathrm{div}\, \boldsymbol{u} = \partial_{\boldsymbol{n}}(\boldsymbol{u} \cdot \boldsymbol{n}) = (\partial_{\boldsymbol{n}} \boldsymbol{u}) \cdot \boldsymbol{n}$
With $\mu = 1$ and $\lambda = s - 2$, the operators L of Lamé and Maxwell coincide.

The sole difference between Lamé and Maxwell lies in variational spaces

Regularized Maxwell system in smooth domains

Assume

- Ω is a bounded open set with a smooth boundary in $\mathbb{R}^n$
- The coefficients μ and ε are smooth positive functions on $\overline{\Omega}$.

Theorem S.3 *cf.* [Amrouche et al, 1998]

Under the previous assumptions:

- The variational space $X_N(\Omega)$ is a subset of $H^1(\Omega)^d$, which means that $X_N(\Omega)$ coincides with the variational space $H_N(\Omega)$ of elasticity.
- The same holds with the “magnetic” spaces $X_T(\Omega)$ and $H_T(\Omega)$ for which the essential boundary condition is $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$.
- There exists a constant C such that for all $\boldsymbol{u} \in X_N(\Omega)$ (or $\boldsymbol{u} \in H_T(\Omega)$)

$$\|\boldsymbol{u}\|_{H^1(\Omega)} \leq C\Big(\|\,\mathbf{curl}\,\boldsymbol{u}\|_{L^2(\Omega)} + \|\,\mathrm{div}\,\boldsymbol{u}\|_{L^2(\Omega)} + \|\boldsymbol{u}\|_{L^2(\Omega)}\Big) \tag{3}$$

- As a consequence of this theorem, the regularized Maxwell form satisfies the assumptions of the general case, i.e. when $X(\Omega) = \boldsymbol{H}^1(\Omega)$.
- So the conclusions of Theorem S.2 apply.

C. AMROUCHE, C. BERNARDI, M. DAUGE, V. GIRAULT
Vector potentials in three-dimensional non-smooth domains.
Math. Methods Appl. Sci. 21 (1998), no. 9, 823–864.

Part II

Dirichlet Laplacian in 2D corner domains

Outline

2D corner domains

A 2D (two-dimensional) smooth domain is defined as follows:

For any $x \in \overline{\Omega}$, one out the two following alternatives is true

1. Near x, the set Ω is loc.[1] isometric to a NBH[2] of 0 in $\mathbb{R}^2$ (then $x \in \Omega$)
2. Near x, the set $\overline{\Omega}$ is loc. smoothly isomorphic to a NBH of 0 in the half-plane $\mathbb{R}^2_+$ (then $x \in \partial\Omega$)

We obtain the class of Lipschitz curvilinear polygons if we add a 3rd model

3. Near x, the set $\overline{\Omega}$ is loc. smoothly isomorphic to a NBH of 0 in a plane sector Γ of opening $\omega \in (0, \pi) \cup (\pi, 2\pi)$ (then x is a corner)

There exist two natural levels of extension to this class:

- **A** Include the sector of opening 2π in order to modelize cracks. The consequence is necessity of introducing the unfolded boundary $\partial_*\Omega$ of Ω to "double" the sides of the crack outside its tip(s).
- **B** Relax the notion of smoothness for local diffeomorphism: Replace smoothness in Cartesian coordinates by smoothness in polar coordinates centered at x. Allows to modelize outward cusps and mild singular points.

---

[1] "loc." for "locally"

[2] "NBH" for "neighborhood"

Polygons

In this part, we consider a model class, the class of polygons, for which all diffeomorphisms are isometries: Either ❶, ❷, or ❸, is true:

1. Near x, the set Ω is loc. isometric to a NBH of 0 in $\mathbb{R}^2$ (then $x \in \Omega$)
2. Near x, the set $\overline{\Omega}$ is loc. isometric to a NBH of 0 in $\mathbb{R}^2_+$ (then $x \in$ a side of Ω)
3. Near x, the set $\overline{\Omega}$ is loc. isometric to a NBH of 0 in a plane sector Γ of opening $\omega \in (0, \pi) \cup (\pi, 2\pi]$ (then x is a corner)

In other words Ω is a bounded open set and its boundary is a finite union of segments. The ends of these segments are the corners of Ω.

Notation

- Denote by $\mathscr{C}$ the set of corners $\boldsymbol{c}$.
- For each $\boldsymbol{c} \in \mathscr{C}$, exists a plane sector $\Gamma_{\boldsymbol{c}}$ that coincides with Ω in a ball $\mathscr{B}(\boldsymbol{c}, R_{\boldsymbol{c}})$.
- Denote by $\omega_{\boldsymbol{c}}$ the *opening* of $\Gamma_{\boldsymbol{c}}$ (i.e. the opening of Ω at the corner $\boldsymbol{c}$) and set

$$\boxed{\kappa_{\boldsymbol{c}} = \frac{\pi}{\omega_{\boldsymbol{c}}}}$$

- Introduce local polar coordinates $(r_{\boldsymbol{c}}, \theta_{\boldsymbol{c}})$ such that

$$\Gamma_{\boldsymbol{c}} = \left\{x \in \mathbb{R}^2, \quad r_{\boldsymbol{c}} > 0, \ \ \theta_{\boldsymbol{c}} \in (0, \omega_{\boldsymbol{c}})\right\}$$

Corner localization

- Let Ω be a *polygon* in the plane $\mathbb{R}^2$.
- Let $f \in L^2(\Omega)$.
- We are going to study in detail solutions of $-\Delta u = f \quad \text{in} \quad \Omega, \quad u \in H^1_0(\Omega)$

Lemma: Regularity outside corners → Theorem L.2

For any smooth cut-off χ with support $\mathscr{U}'$ disjoint from the corners

$$u \in H^2(\mathscr{U} \cap \Omega) \quad \text{with} \quad \mathscr{U} = \chi^{-1}(1)$$

Proof. We have

$$-\Delta(\chi u) = \chi f + 2\nabla\chi \cdot \nabla u + (\Delta\chi)u$$

Therefore, by extension by 0

$$\Delta(\chi u) \in L^2(\mathscr{U}' \cap \Omega) \quad \text{and} \quad \chi u \in H^1_0(\mathscr{U}' \cap \Omega)$$

The lemma is a consequence of Theorem S.2. □

Introduce a new smooth cut-off that localizes near the corner $\boldsymbol{c}$

$$\chi_{\boldsymbol{c}}(x) = \begin{cases} 0 & \text{if} \quad x \notin \mathscr{B}(\boldsymbol{c}, R_{\boldsymbol{c}}) \\ 1 & \text{if} \quad x \in \mathscr{B}(\boldsymbol{c}, R_{\boldsymbol{c}}/2) \end{cases}$$

Then

$$\Delta(\chi_{\boldsymbol{c}} u) \in L^2(\Gamma_{\boldsymbol{c}}) \quad \text{and} \quad \chi_{\boldsymbol{c}} u \in H^1_0(\Gamma_{\boldsymbol{c}})$$

Outline

Dirichlet Laplacian on a sector – Change of variables

A We have reduced our problem on a polygon to problems of the type (we re-baptize as u the localized function $\chi_{\boldsymbol{c}} u$, translate $\boldsymbol{c}$ to 0, and we drop the index $\boldsymbol{c}$)

$$\begin{cases} \Delta u = f \ \text{ in } \ \Gamma, \quad f \in L^2(\Gamma), & \operatorname{supp} f \subset \mathscr{B}(0,R) \\ u \in H^1_0(\Gamma), & \operatorname{supp} u \subset \mathscr{B}(0,R) \end{cases}$$

B Use polar coordinates (r,θ) and transform our problem on the half-strip $\mathbb{R}_+ \times \mathscr{I}$. We have

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta \quad \text{and} \quad \begin{cases} r\partial_r &= r\cos\theta\,\partial_{x_1} + r\sin\theta\,\partial_{x_2} \\ \partial_\theta &= -r\sin\theta\,\partial_{x_1} + r\cos\theta\,\partial_{x_2} \end{cases}$$

Set $\widetilde{u}(r,\theta) = u(x)$ and $\widetilde{g}(r,\theta) = r^2 f(x)$. The equation $\Delta u = f$ is equivalent to

$$\big((r\partial_r)^2 + \partial_\theta^2\big)\widetilde{u} = \widetilde{g} \quad \text{in} \quad \mathbb{R}_+ \times \mathscr{I} \quad \text{with} \quad \boxed{\mathscr{I} = (0,\omega)}$$

C Set $t = \log r$ (i.e. $r = e^t$) – Euler change of variables – and go the strip $\mathbb{R} \times \mathscr{I}$

$$r\partial_r = \partial_t \quad \text{and} \quad \mathrm{d}x = r\,\mathrm{d}r\,\mathrm{d}\theta = e^{2t}\,\mathrm{d}t\,\mathrm{d}\theta$$

Set $\breve{u}(t,\theta) = \widetilde{u}(r,\theta)$ and $\breve{g}(t,\theta) = \widetilde{g}(r,\theta)$. The equation $\Delta u = f$ is equivalent to

$$(\partial_t^2 + \partial_\theta^2)\breve{u} = \breve{g} \quad \text{in} \quad \mathbb{R} \times \mathscr{I}.$$

Dirichlet Laplacian on a sector – Exponential weights

Lemma

(1) If $f \in L^2(\Gamma)$ with $\operatorname{supp} f \in \mathscr{B}(0,R)$, then $\breve{g} = e^{2t}\breve{f}$ satisfies

$$\forall \eta \le 1, \quad e^{-\eta t}\,\breve{g} \in L^2(\mathbb{R}\times\mathscr{I}) \quad \text{and} \quad \|e^{-\eta t}\,\breve{g}\|_{L^2(\mathbb{R}\times\mathscr{I})} \le C\|f\|_{L^2(\Gamma)}$$

(2) If $u \in H^1_0(\Gamma)$ with $\operatorname{supp} u \in \mathscr{B}(0,R)$, then $\breve{u}$ satisfies

$$\forall \eta \le 0, \quad e^{-\eta t}\,\breve{u} \in H^1_0(\mathbb{R}\times\mathscr{I}) \quad \text{and} \quad \|e^{-\eta t}\,\breve{u}\|_{H^1(\mathbb{R}\times\mathscr{I})} \le C\|u\|_{H^1(\Gamma)}$$

The constant C is independent of η, of u and of f.

Proof of (1)

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathscr{I}} |e^{-\eta t}\,\breve{g}(t,\theta)|^2 \,\mathrm{d}t\,\mathrm{d}\theta &= \int_{\mathbb{R}}\int_{\mathscr{I}} |e^{(-\eta+1)t}\,\breve{f}(t,\theta)|^2 \; e^{2t}\,\mathrm{d}t\,\mathrm{d}\theta \\ &= \int_{\Gamma} |r^{-\eta+1}\,f(x)|^2 \,\mathrm{d}x \\ &\le R^{1-\eta}\int_{\Gamma} |f(x)|^2 \,\mathrm{d}x\end{aligned}$$

because of the support condition and $1-\eta \ge 0$. □

Dirichlet Laplacian on a sector – Exponential weights

Proof of (2)

- $$\int_{\mathbb{R}}\int_{\mathscr{I}} |e^{-\eta t}\,\partial_t \breve{u}(t,\theta)|^2 \,\mathrm{d}t\,\mathrm{d}\theta = \int_{\mathbb{R}}\int_{\mathscr{I}} |e^{-\eta t}\,e^{-t}\partial_t \breve{u}(t,\theta)|^2 \;e^{2t}\,\mathrm{d}t\,\mathrm{d}\theta$$
  $$\leq \int_{\Gamma} |r^{-\eta}\,\partial_{x_1} u(x)|^2 + |r^{-\eta}\,\partial_{x_2} u(x)|^2 \,\mathrm{d}x$$
  $$\leq R^{-\eta}\int_{\Gamma} |\nabla u(x)|^2 \,\mathrm{d}x$$
  because of the support condition and $-\eta \geq 0$. The same for $\partial_\theta \breve{u}$.
- For $\breve{u}$, we find
  $$\int_{\mathbb{R}}\int_{\mathscr{I}} |e^{-\eta t}\,\breve{u}(t,\theta)|^2 \,\mathrm{d}t\,\mathrm{d}\theta = \int_{\Gamma} |r^{-\eta-1}\,u(x)|^2 \,\mathrm{d}x$$
  $$\leq R^{-\eta}\int_{\Gamma} |r^{-1}u(x)|^2 \,\mathrm{d}x$$
  and may conclude if we know that $r^{-1}u$ belongs to $L^2(\Gamma)$.
- Dirichlet conditions $\widetilde{u}(r,0) = \widetilde{u}(r,\omega) = 0$ yield the Poincaré inequality
  $$\|\widetilde{u}(r,\cdot)\|_{L^2(\mathscr{I})} \leq C\|\partial_\theta \widetilde{u}(r,\cdot)\|_{L^2(\mathscr{I})}$$
  that implies $\|r^{-1}u\|_{L^2(\Gamma)} \leq C\|\nabla u\|_{L^2(\Gamma)}$. Hence $r^{-1}u \in L^2(\Gamma)$. $\square$

Dirichlet Laplacian on a sector – Laplace and Mellin transform

Definition

- If $r^{-\eta-1}v$ belongs to $L^2(\Gamma)$, we define the Mellin transform $\mathfrak{M}v$ as

$$\mathfrak{M}v[\lambda](\theta) = \int_0^\infty r^{-\lambda}\, \widetilde{v}(r,\theta)\, \frac{\mathrm{d}r}{r}, \quad \lambda \in \mathbb{C},\ \operatorname{Re}\lambda = \eta$$

- If $e^{-\eta t}\breve{v}$ belongs to $L^2(\mathbb{R}\times\mathscr{I})$, we define the Fourier-Laplace transform $\mathfrak{L}(\breve{v})$ as

$$\mathfrak{L}(\breve{v})[\lambda](\theta) = \int_{\mathbb{R}} e^{-\lambda t}\, \breve{v}(t,\theta)\, \mathrm{d}t, \quad \lambda \in \mathbb{C},\ \operatorname{Re}\lambda = \eta$$

Note the equivalence

$$r^{-\eta-1}v \in L^2(\Gamma) \iff e^{-\eta t}\breve{v} \in L^2(\mathbb{R}\times\mathscr{I})$$

and the identities, with $\mathfrak{F}$ the standard Fourier transform,

$$\lambda = \eta + i\xi : \quad \mathfrak{M}v[\lambda] = \mathfrak{L}(\breve{v})[\lambda] = \mathfrak{F}(e^{-\eta t}\breve{v})[\xi]$$

We mainly use it in its Mellin form.

Dirichlet Laplacian on a sector – Laplace and Mellin transform

Properties

Let $\mathfrak{X}$ be a Hilbert space of functions on the interval $\mathscr{I}$, for example $\mathfrak{X} = H^m(\mathscr{I})$.

- The function $e^{-\eta t}\check{v}$ belongs to $L^2(\mathbb{R}, \mathfrak{X})$, if and only if the function
$$\xi \longmapsto \mathfrak{M}v[\eta + i\xi] \quad \text{belongs to } L^2(\mathbb{R}, \mathfrak{X})$$
- Let $\eta_0 < \eta_1$. If $e^{-\eta t}\check{v} \in L^2(\mathbb{R}, \mathfrak{X})$ for $\eta = \eta_0$ and $\eta = \eta_1$, then $e^{-\eta t}\check{v} \in L^2(\mathbb{R}, \mathfrak{X})$ for all $\eta \in [\eta_0, \eta_1]$ and the function
$$\{\lambda \in \mathbb{C}, \quad \operatorname{Re}\lambda \in (\eta_0, \eta_1)\} \ni \lambda \longmapsto \mathfrak{M}v[\lambda] \quad \text{is holomorphic with values in } \mathfrak{X}$$

Recall that $\quad u \in H_0^1(\Gamma), \quad \operatorname{supp} u \subset \mathscr{B}(0, R), \quad \Delta u = f \in L^2(\Gamma).$

With $g := r^2 f$ and $\mathscr{I} = (0, \omega)$ we have

- $\mathfrak{M}u$ is *holomorphic* in the complex half-plane $\operatorname{Re}\lambda < 0$ with values in $H_0^1(\mathscr{I})$
- $\mathfrak{M}g$ is *holomorphic* in the complex half-plane $\operatorname{Re}\lambda < 1$ with values in $L^2(\mathscr{I})$ and $\xi \mapsto \mathfrak{M}g[\eta + i\xi]$ belongs to $L^2(\mathbb{R} \times \mathscr{I})$ for all $\eta \leq 1$.
- Owing to formulas $\mathfrak{L}(\partial_t \check{u})[\lambda] = \lambda \mathfrak{L}(\check{u})[\lambda]$ and $\mathfrak{L}(\partial_\theta \check{u})[\lambda] = \partial_\theta \mathfrak{L}(\check{u})[\lambda]$ we have
$$(\lambda^2 + \partial_\theta^2)\, \mathfrak{M}u[\lambda] = \mathfrak{M}g[\lambda], \quad \forall \lambda, \quad \operatorname{Re}\lambda \leq 0.$$

Dirichlet Laplacian on a sector – Mellin symbol

Definition

The *Mellin symbol* of Δ with Dirichlet conditions is defined for any $\lambda \in \mathbb{C}$ as

$$\begin{array}{rccc} \mathfrak{A}[\lambda]: & H^1_0(\mathscr{I}) & \longrightarrow & H^{-1}(\mathscr{I}) \\ & U & \longmapsto & (\lambda^2 + \partial^2_\theta)\, U \end{array}$$

The symbol $\mathfrak{A}[\lambda]$ can also be considered as acting from $(H^2 \cap H^1_0)(\mathscr{I})$ into $L^2(\mathscr{I})$.

Denote by $\mathfrak{P}$ the positive Laplace operator $-\partial^2_\theta$ from $(H^2 \cap H^1_0)(\mathscr{I})$ into $L^2(\mathscr{I})$.

- The operator $\mathfrak{P}$ is self-adjoint on $L^2(\mathscr{I})$, positive, with compact resolvent. Its spectrum $\sigma(\mathfrak{P})$ is a discrete subset of $(0, +\infty)$. Explicit calculation shows that

$$\sigma(\mathfrak{P}) = \left\{ \mu_\ell := \left(\frac{\ell\pi}{\omega}\right)^2, \quad \ell \in \mathbb{N}_* \right\}$$

- Since
$$\mathfrak{A}[\lambda] = -\mathfrak{P} + \lambda^2$$
we deduce that the spectrum $\sigma(\mathfrak{A})$ of the Mellin symbol $\mathfrak{A}$ is
$$\sigma(\mathfrak{A}) = \left\{ \lambda_\ell := \frac{\ell\pi}{\omega}, \quad \ell \in \mathbb{Z}_* \right\}$$

- The symbol $\lambda \mapsto \mathfrak{A}[\lambda]$ is *holomorphic* and its inverse $\lambda \mapsto \mathfrak{A}[\lambda]^{-1}$ is *meromorphic*.

Dirichlet Laplacian on a sector – Meromorphic extension

Recall that

$$\forall\lambda \ \text{ s.t. } \ \mathrm{Re}\,\lambda \le 0, \qquad \mathfrak{A}[\lambda]\,\mathfrak{M}u[\lambda] = \mathfrak{M}g[\lambda]$$

We define a meromorphic extension $U[\lambda]$ of $\mathfrak{M}u[\lambda]$ by setting

$$\boxed{\forall\lambda \ \text{ s.t. } \ \mathrm{Re}\,\lambda \in (0,1], \qquad U[\lambda] = \mathfrak{A}[\lambda]^{-1}\,\big(\mathfrak{M}g[\lambda]\big)}$$

We are going to prove

Proposition P

(1) Exists u_1 such that $e^{-t}\breve{u}_1 \in H^2(\mathbb{R}\times\mathscr{I})$ and $\mathfrak{M}u_1[\lambda] = U[\lambda]$ for λ s.t. $\mathrm{Re}\,\lambda = 1$

(2) The difference between u and u_1 is given by the residue formula

$$u_1 - u \quad = \sum_{\substack{\lambda_0\in\sigma(\mathfrak{A})\\ \mathrm{Re}\,\lambda_0\in(0,1)}} \operatorname*{Res}_{\lambda=\lambda_0} r^\lambda U[\lambda]$$

Using that $\sigma(\mathfrak{A}) = \frac{\pi}{\omega}\,\mathbb{Z}_*$, we obtain as a corollary

$$\begin{cases} u_1 = u & \text{if } \omega < \pi \\ u_1 - u = \operatorname*{Res}_{\lambda=\frac{\pi}{\omega}} r^\lambda U[\lambda] & \text{if } \omega > \pi \end{cases}$$

Dirichlet Laplacian on a sector – Resolvent estimates

The proof of **Proposition P** is based on estimates of $\mathfrak{A}[\lambda]^{-1}$ in operator norm.
Define the parameter norm $\|U\|_{H^m(\mathscr{I};\lambda)}$ for $m \in \mathbb{N}$ as

$$\boxed{\|U\|^2_{H^m(\mathscr{I};\lambda)} = \sum_{k=0}^{m} (|\lambda|+1)^{2k} \|U\|^2_{H^{m-k}(\mathscr{I})}}$$

Lemma General

Let $\eta_0 \leq \eta_1$ and $\delta > 0$. Define the set

$$\Lambda = \Big\{\lambda \in \mathbb{C},\ \operatorname{Re}\lambda \in [\eta_0, \eta_1]\Big\} \setminus \bigcup_{\lambda_0 \in \sigma(\mathfrak{A})} \mathscr{B}(\lambda_0, \delta).$$

Let $m \geq 2$. Exists a constant C such that

(1) $$\boxed{\forall \lambda \in \Lambda, \quad \forall G \in H^{m-2}(\mathscr{I}), \quad \|\mathfrak{A}[\lambda]^{-1} G\|_{H^m(\mathscr{I};\lambda)} \leq C\|G\|_{H^{m-2}(\mathscr{I};\lambda)}}$$

Proof: Based on two steps.

Step 1 If K is a compact set in $\mathbb{C}$ disjoint from the spectrum of $\mathfrak{A}$ the resolvent estimate (1) holds with a constant C depending on K by continuity of $\mathfrak{A}[\lambda]^{-1}$ with respect to λ.

Step 2 It remains to prove (1) for $|\lambda|$ is large enough in the strip $\{\lambda \in \mathbb{C},\ \operatorname{Re}\lambda \in [\eta_0, \eta_1]\}$

Dirichlet Laplacian on a sector – Resolvent estimates, continued

We have to prove for $\lambda \in \Lambda$ and $|\lambda|$ large enough

(1) $$\forall G \in H^{m-2}(\mathscr{I}), \quad \|\mathfrak{A}[\lambda]^{-1} G\|_{H^m(\mathscr{I};\lambda)} \leq C\|G\|_{H^{m-2}(\mathscr{I};\lambda)}$$

To perform Step 2 of the proof of (1), we choose λ, G, and set $V = \mathfrak{A}[\lambda]^{-1} G$. Then introduce on $\mathbb{R} \times \mathscr{I}$ *(Agmon's method of addition of variable)*

$$\breve{v}(t,\theta) = e^{\lambda t}\, V(\theta) \quad \text{and} \quad \breve{g}(t,\theta) = e^{\lambda t}\, G(\theta)$$

We can check:

- $\breve{v} \in H^1((-2,2) \times \mathscr{I})$, $\breve{v} = 0$ on $\mathbb{R} \times \partial\mathscr{I}$
- $\Delta \breve{v} = \breve{g}$
- Local estimates (Theorem S.2) give

$$\|\breve{v}\|_{H^m((-1,1)\times\mathscr{I})} \leq C\Big(\|\breve{g}\|_{H^{m-2}((-2,2)\times\mathscr{I})} + \|\breve{v}\|_{H^1((-2,2)\times\mathscr{I})}\Big)$$

- Coming back to V and G —here C depends on η_0 and η_1,

$$\|V\|_{H^m(\mathscr{I};\lambda)} \leq C\Big(\|G\|_{H^{m-2}(\mathscr{I};\lambda)} + \|V\|_{H^1(\mathscr{I};\lambda)}\Big)$$

- We conclude because for $|\lambda|$ large enough, $C\|V\|_{H^1(\mathscr{I};\lambda)} \leq \frac{1}{2}\|V\|_{H^m(\mathscr{I};\lambda)}$ $\square$

Dirichlet Laplacian on a sector – Regular part

Proposition P, point (1), rephrased

Let u_1 be defined as the inverse Mellin transform

$$u_1(x) = \frac{1}{2i\pi} \int_{\mathrm{Re}\,\lambda=1} r^\lambda\, U[\lambda](\theta)\, \mathrm{d}\lambda$$

Then $e^{-t}\breve{u}_1 \in H^2(\mathbb{R} \times \mathscr{I})$, which means for u_1:

$$r^{-2+|\alpha|}\, \partial_x^\alpha u_1 \in L^2(\Gamma), \quad |\alpha| \leq 2.$$

- $U[\lambda] = \mathfrak{A}[\lambda]^{-1}\, \mathfrak{M}g[\lambda]$
- The Mellin transform of g satisfies "$\xi \mapsto \mathfrak{M}g[1 + i\xi]$ belongs to $L^2(\mathbb{R}, L^2(\mathscr{I}))$"
- The resolvent estimate with $m = 2$ on the line $\mathrm{Re}\,\lambda = 1$ (disjoint from $\sigma(\mathfrak{A})$!) yields

$$\xi \mapsto |1 + i\xi|^k U[1 + i\xi] \quad \text{belongs to} \quad L^2(\mathbb{R}, H^{2-k}(\mathscr{I})), \quad k = 0, 1, 2$$

- Since
$$\frac{1}{2i\pi} \int_{\mathrm{Re}\,\lambda=1} r^\lambda\, U[\lambda]\, \mathrm{d}\lambda = \frac{1}{2\pi}\, e^t \int_{\mathbb{R}} e^{it\xi}\, U[1 + i\xi]\, \mathrm{d}\xi$$
we find by inverse Fourier transform that $e^{-t}\breve{u}_1 \in H^2(\mathbb{R} \times \mathscr{I})$).
- Back to Cartesian coordinates with $r = e^t$ gives the weighted regularity for u_1 □

Dirichlet Laplacian on a sector – Residue formula

Proposition P, point (2)

$$u_1 - u \quad = \sum_{\substack{\lambda_0 \in \sigma(\mathfrak{A}) \\ \mathrm{Re}\,\lambda_0 \in (0,1)}} \operatorname*{Res}_{\lambda=\lambda_0} r^\lambda U[\lambda]$$

- $U[\lambda] = \mathfrak{A}[\lambda]^{-1}\,\mathfrak{M}g[\lambda]$
- The poles of $U[\lambda]$ are in the set of poles of $\mathfrak{A}[\lambda]^{-1}$ since $\mathfrak{M}g[\lambda]$ is holomorphic
- For any simple rectifiable curve γ that is contained in the open strip $\mathrm{Re}\,\lambda \in (-\infty, 1)$ and surrounds the pole $\lambda_0 = \frac{\pi}{\omega}$ when $\omega > \pi$, we have

$$\frac{1}{2i\pi}\int_\gamma r^\lambda\, U[\lambda](\theta)\,\mathrm{d}\lambda \quad = \sum_{\substack{\lambda_0 \in \sigma(\mathfrak{A}) \\ \mathrm{Re}\,\lambda_0 \in (0,1)}} \operatorname*{Res}_{\lambda=\lambda_0} r^\lambda U[\lambda]$$

- Take γ as the rectangle $\mathrm{Re}\,\lambda = 0$, $\mathrm{Im}\,\lambda = -\xi$, $\mathrm{Re}\,\lambda = 1-\delta$, $\mathrm{Re}\,\lambda = \xi$, with $\delta > 0$ small enough and $\xi > 0$. We may push ξ to infinity and δ to 0 using the resolvent estimates to bound $U[\lambda]$, hence proving **Proposition P**. $\square$

Dirichlet Laplacian on a sector – Description of the residue

If $\omega > \pi$, we have one residue in the relevant region Re $\lambda \in [0, 1]$.

- Recall that $\mathfrak{A}[\lambda] = -\mathfrak{P} + \lambda^2$ with $\mathfrak{P} = -\partial_\theta^2$ on $H_0^1(\mathscr{I})$.
- Let ϕ_ℓ be an orthonormal spectral basis for $\mathfrak{P}$ associated with eigenvalues μ_ℓ.

  Setting $\boxed{\kappa = \frac{\pi}{\omega}}$ we have

$$\mu_\ell = \lambda_\ell^2 \ \text{ with } \ \lambda_\ell = \ell\kappa, \quad \text{and} \quad \phi_\ell(\theta) = \sqrt{\frac{2}{\omega}}\, \sin \ell\kappa\theta, \qquad \boxed{\ell \geq 1}$$

- Then

$$\mathfrak{P}U = \sum_{\ell\in\mathbb{N}_*} \lambda_\ell^2 \langle \phi_\ell, U\rangle\, \phi_\ell \quad \text{hence} \quad (-\mathfrak{P} + \lambda^2)^{-1} G = \sum_{\ell\in\mathbb{N}_*} \frac{1}{\lambda^2 - \lambda_\ell^2} \langle \phi_\ell, G\rangle\, \phi_\ell$$

- With $G[\lambda]$ the Mellin transform of $g = r^2 f$, we find

$$\begin{aligned}\operatorname*{Res}_{\lambda=\lambda_1} r^\lambda U[\lambda] &= \operatorname*{Res}_{\lambda=\lambda_1} r^\lambda (-\mathfrak{P} + \lambda^2)^{-1} G[\lambda] \\ &= r^{\lambda_1} \frac{1}{2\lambda_1} \langle \phi_1, G[\lambda_1]\rangle\, \phi_1 \\ &= \gamma_1\, r^\kappa \ \sin\kappa\theta \qquad \text{with} \qquad \gamma_1 = \frac{1}{\pi}\int_\Gamma r^{-\kappa} \sin\kappa\theta\ f(x)\, \mathrm{d}x\end{aligned}$$

- Finally $u = u_1 - \gamma_1\, r^\kappa\, \sin\kappa\theta$ and we have proved the following theorem...

Dirichlet Laplacian on a sector – Theorem with rhs in L^2

Theorem L.1 → Theorem L.3

Let Γ be a plane sector of opening $\omega \in (0,\pi)\cup(\pi,2\pi)$ and $u \in H^1_0(\Gamma)$ with compact support such that

$$\Delta u = f \ \text{ in } \ \Gamma, \quad f \in L^2(\Gamma).$$

- If $\omega < \pi$, then u belongs to $H^2(\Gamma)$ and moreover satisfies $r^{-2}u,\ r^{-1}\nabla u$ in $L^2(\Gamma)$
- If $\omega > \pi$, then with the ground frequency $\kappa = \frac{\pi}{\omega}$:

(L.1) $$\boxed{u = u_1 - \gamma_1\, r^{\kappa}\, \sin\kappa\theta \quad \text{with} \quad \gamma_1 = \frac{1}{\pi}\int_\Gamma r^{-\kappa} \sin\kappa\theta\ f(x)\,\mathrm{d}x}$$

and u_1 satisfying $$\boxed{r^{-2+|\alpha|}\,\partial_x^\alpha u_1 \in L^2(\Gamma), \quad |\alpha| \le 2}$$

Case of a crack Optimal statement using injectivity modulo polynomials

If $\omega = 2\pi$, Theorem L.1 does not apply because the Mellin symbol has a pole in $\lambda = 1$. Nevertheless it is possible to prove that (L.1) holds with the weaker regularity for u_1

$$\partial_x^\alpha u_1 \in L^2(\Gamma),\ \ |\alpha| = 2, \quad \text{and} \quad r^{-2+\delta+|\alpha|}\,\partial_x^\alpha u_1 \in L^2(\Gamma),\ \ |\alpha| \le 1,\ \delta > 0.$$

Outline

Dirichlet Laplacian on a polygon – Theorem with rhs in L^2

Theorem L.2 → Theorem L.4

Let Ω be a plane polygon with corners set $\mathscr{C} \ni \boldsymbol{c}$ associated with openings $\omega_{\boldsymbol{c}} \in (0,\pi) \cup (\pi, 2\pi)$. Let $\chi_{\boldsymbol{c}}$ smooth cut-off functions separating the corners.

- Let $u \in H^1_0(\Omega)$ be solution of

$$\Delta u = f \;\text{ in }\; \Omega, \quad f \in L^2(\Omega).$$

- Then, with $\kappa_{\boldsymbol{c}} = \frac{\pi}{\omega_{\boldsymbol{c}}}$:

$$\boxed{u = u_{1,\mathrm{reg}} + \sum_{\boldsymbol{c}\in\mathscr{C},\ \omega_{\boldsymbol{c}}>\pi} \chi_{\boldsymbol{c}}(x)\,\gamma_{\boldsymbol{c}}\, r_{\boldsymbol{c}}^{\kappa_{\boldsymbol{c}}} \sin \kappa_{\boldsymbol{c}}\theta_{\boldsymbol{c}}} \tag{L.2}$$

with

- constants $\gamma_{\boldsymbol{c}}$ depending continuously on $f \in L^2(\Omega)$
- a regular part $u_{1,\mathrm{reg}} \in H^2(\Omega)$ satisfying

$$\boxed{r_{\boldsymbol{c}}^{-2+|\alpha|}\,\partial_x^\alpha u_1 \in L^2(\Omega), \quad |\alpha| \le 2,\ \boldsymbol{c} \in \mathscr{C}}$$

Proof: Localize around each corner as specified in Section 4
Then apply Theorem L.1 at each corner, multiply the expansion (L.1) by $\chi_{\boldsymbol{c}}$ and sum on $\boldsymbol{c} \in \mathscr{C}$.

Weighted Sobolev spaces of Kondrat'ev type

The same tools allow to prove results in a general class of weighted Sobolev spaces (those introduced by Kondrat'ev, with a different notation).

Definition

Let $m \in \mathbb{N}$ and $\beta \in \mathbb{R}$.

- Let Γ be a plane sector, with r the distance to its vertex.

$$K^m_\beta(\Gamma) = \{u \in L^2_{\mathrm{loc}}(\Gamma), \quad r^{\beta+|\alpha|}\partial^\alpha_x u \in L^2(\Gamma), \ \forall\alpha, \ |\alpha| \le m\}$$

- Let Ω be a polygon with corner set $\mathscr{C}$, and $r_{\boldsymbol{c}} = r_{\boldsymbol{c}}(x)$ the distance of x to $\boldsymbol{c}$. Let ρ be the minimum of the $r_{\boldsymbol{c}}$, $c \in \mathscr{C}$., i.e. the distance function to the set of corners.

$$K^m_\beta(\Omega) = \{u \in L^2_{\mathrm{loc}}(\Omega), \quad \rho^{\beta+|\alpha|}\partial^\alpha_x u \in L^2(\Omega), \ \forall\alpha, \ |\alpha| \le m\}$$

Properties:

- If $u \in K^m_\beta(\Omega)$, then $\chi_{\boldsymbol{c}} u \in K^m_\beta(\Gamma_{\boldsymbol{c}})$.
- $K^m_\beta(\Gamma) \subset K^{m-1}_\beta(\Gamma) \subset \cdots \subset K^0_\beta(\Gamma)$
- $K^m_\beta(\Omega) \subset K^{m'}_{\beta'}(\Omega)$ for $m' \le m$ and $\beta' \ge \beta$
- ∂^α_x is continuous $K^m_\beta(\Gamma) \to K^{m-|\alpha|}_{\beta+|\alpha|}(\Gamma)$ and Δ is continuous $K^m_\beta(\Gamma) \to K^{m-2}_{\beta+2}(\Gamma)$
- $K^m_{-m}(\Omega) \subset H^m(\Omega) \subset K^m_0(\Omega)$
- $H^1_0(\Omega) \subset K^1_{-1}(\Omega)$

Weighted Sobolev spaces and Mellin transform

Lemma

Let $m \in \mathbb{N}$ and $\beta \in \mathbb{R}$. Let Γ be a plane sector of opening ω, and $\mathscr{I} = (0, \omega)$. Set

$$\eta = -\beta - 1$$

The following three assertions are equivalent

1. $u \in K^m_\beta(\Gamma)$
2. $e^{-\eta t}\breve{u} \in H^m(\mathbb{R} \times \mathscr{I})$, with $u(x) = \breve{u}(t, \theta)$.
3. The Mellin transform $U[\lambda] := \mathfrak{M}u[\lambda]$ is well defined for $\operatorname{Re}\lambda = \eta$ and the function

$$\xi \longmapsto \|U[\eta + i\xi]\|_{H^m(\mathscr{I};|\xi|)} \quad \text{belongs to } L^2(\mathbb{R}) \tag{1}$$

Moreover, if the function U satisfies (1), then the inverse Mellin transform

$$u(x) = \frac{1}{2i\pi} \int_{\operatorname{Re}\lambda = \eta} r^\lambda \, U[\lambda](\theta) \, \mathrm{d}\lambda$$

defines an element of $K^m_\beta(\Gamma)$.

We have already seen this in the particular case $m = 2$, $\beta = -2$.

Dirichlet Laplacian in weighted Sobolev spaces on a sector

The same tools as used for Theorem L.1, including the general form of our resolvent estimates, allows to prove:

Theorem L.3 → Theorem L.5 in dim. n

Let Γ be a plane sector of opening ω. Let $m \geq 2$ and $\beta < -1$. Set $\eta = -\beta - 1$. We assume that

The line $\mathrm{Re}\,\lambda = \eta$ *is disjoint from the Mellin spectrum* $\sigma(\mathfrak{A})$

(here this means that $\eta \notin \frac{\pi}{\omega}\mathbb{Z}_*$)

- Let $u \in H_0^1(\Gamma)$ with compact support such that

$$\Delta u = f \text{ in } \Gamma, \quad f \in K^{m-2}_{\beta+2}(\Gamma).$$

- Then, with $\kappa = \frac{\pi}{\omega}$

(L.3) $$\boxed{u = u_\eta + \sum_{\ell \in \mathbb{N}_*,\ \ell\kappa < \eta} \gamma_\ell\, r^{\ell\kappa} \sin \ell\kappa\theta} \quad \text{and} \quad \boxed{u_\eta \in K^m_\beta(\Gamma)}$$

with $$\gamma_\ell = \frac{1}{\pi} \int_\Gamma r^{-\ell\kappa} \sin \ell\kappa\theta\ f(x)\,\mathrm{d}x$$

Dirichlet Laplacian in weighted Sobolev spaces on a polygon

Theorem L.4

→ Theorem L.2

Let Ω be a plane polygon with corners set $\mathscr{C} \ni \boldsymbol{c}$. Let $\chi_{\boldsymbol{c}}$ smooth cut-off functions separating the corners.

- Let $m \geq 2$ and $\beta < -1$. Set $\eta = -\beta - 1$. We assume that

  $$\forall \boldsymbol{c} \in \mathscr{C}, \textit{ the line } \ \operatorname{Re}\lambda = \eta \ \textit{ is disjoint from the Mellin spectrum } \ \sigma(\mathfrak{A}_{\boldsymbol{c}})$$

  (here this means that $\eta \notin \frac{\pi}{\omega_{\boldsymbol{c}}}\mathbb{Z}_*$ for all $\boldsymbol{c} \in \mathscr{C}$)
- Let $u \in H^1_0(\Omega)$ such that $\Delta u = f$ in Ω, $f \in K^{m-2}_{\beta+2}(\Omega)$.
- Then, with $\kappa_{\boldsymbol{c}} = \frac{\pi}{\omega_{\boldsymbol{c}}}$

$$\text{(L.4)} \qquad \boxed{u = u_{\text{reg}} + \sum_{\boldsymbol{c}\in\mathscr{C}} \chi_{\boldsymbol{c}}(x) \left\{ \sum_{\ell\in\mathbb{N}_*,\ \ell\kappa_{\boldsymbol{c}}<\eta} \gamma_{\boldsymbol{c},\ell}\, r_{\boldsymbol{c}}^{\ell\kappa_{\boldsymbol{c}}} \ \sin\left(\ell\kappa_{\boldsymbol{c}}\,\theta_{\boldsymbol{c}}\right) \right\}}$$

with $\boxed{u_{\text{reg}} \in K^m_\beta(\Omega)}$

Relations with standard Sobolev spaces

If $f \in K^{m-2}_{-m+2}(\Omega)$, then $u_{\text{reg}} \in H^m(\Omega)$. But the case $f \in H^{m-2}(\Omega)$ is not straightforward.

Part III

Model elliptic BVP in regular cones

Outline

7. **General regular cones and weighted spaces**
   - Model objects
   - Results
8. General regular cones and spaces without weight
   - Taylor expansions
   - Results

nD cones

1. An open subset Γ of $\mathbb{R}^n$ is called a cone if $\forall x \in \Gamma,\ \forall \rho > 0,\ \rho x \in \Gamma$.
2. Denote for $x \neq 0$: $\boxed{r = |x| \quad \text{and} \quad \hat{x} = \frac{x}{r}}$
3. For a cone Γ, define its *section* by $\boxed{\hat{\Gamma} = \Gamma \cap \mathbb{S}^{n-1}}$
4. Γ is said a *regular cone* if $\hat{\Gamma}$ is a smooth submanifold with boundary of $\mathbb{S}^{n-1}$.

NB in dimension $n = 2$, any sector of positive opening $\omega < 2\pi$ is a regular cone.

Lemma. Let $m \in \mathbb{N}$ and $\beta \in \mathbb{R}$. Let Γ be a cone of section $\hat{\Gamma}$. Set $\boxed{\eta = -\beta - \frac{n}{2}}$

With same definition of spaces K^m_β as above, the following 3 assertions are equivalent

1. $u \in K^m_\beta(\Gamma)$
2. $e^{-\eta t}\breve{u} \in H^m(\mathbb{R} \times \hat{\Gamma})$, with $\breve{u}(\log|x|, \hat{x}) = u(x)$.
3. The Mellin transform $U[\lambda] := \mathfrak{M}u[\lambda]$ is well defined for $\operatorname{Re}\lambda = \eta$ and the function

$$\xi \longmapsto \|U[\eta + i\xi]\|_{H^m(\hat{\Gamma};|\xi|)} \quad \text{belongs to } L^2(\mathbb{R}) \tag{1}$$

Dirichlet Laplacian in regular cones

The variable $\hat{x} \in \hat{\Gamma} \subset \mathbb{S}^{n-1}$ plays the same role as the angle $\theta \in (0, \omega) = \mathscr{I}$:

$$\Delta \quad \text{becomes} \quad e^{-2t}\big(\partial_t^2 + (n-2)\partial_t - \mathfrak{L}_{n-1}\big)$$

with $\mathfrak{L}_{n-1}$ the (positive) Laplace-Beltrami operator on $\mathbb{S}^{n-1}$. Let (μ_ℓ, ϕ_ℓ) be its Dirichlet eigenpairs on $\hat{\Gamma}$. The Mellin symbol of the Dirichlet Laplacian on Γ is

$$\mathfrak{A}[\lambda] : U \longmapsto \big(\lambda^2 + (n-2)\lambda - \mathfrak{L}_{n-1}\big)U \qquad (H^2 \cap H^1_0)(\hat{\Gamma}) \longrightarrow L^2(\hat{\Gamma})$$

The spectrum $\sigma(\mathfrak{A})$ coincides with $\{\lambda_\ell,\ \ell \in \mathbb{Z}_*\}$ with

$$\lambda_{\pm\ell} = 1 - \tfrac{n}{2} \pm \sqrt{\mu_\ell + (\tfrac{n}{2} - 1)^2}, \quad \ell \in \mathbb{N}_*$$

Theorem L.5 → Theorem L.3 for $n = 2$

Let Γ be a regular cone in $\mathbb{R}^n$. Let $m \geq 2$ and $\beta < -1$. Set $\eta = -\beta - \frac{n}{2}$. Assume

The line $\operatorname{Re}\lambda = \eta$ *is disjoint from the Mellin spectrum* $\sigma(\mathfrak{A})$

- Let $u \in H^1_0(\Gamma)$ with compact support such that $\Delta u \in K^{m-2}_{\beta+2}(\Gamma)$
- Then

(L.5) $$u = u_\eta + \sum_{\ell \in \mathbb{Z}_*,\ 1-\frac{n}{2} < \lambda_\ell < \eta} \gamma_\ell\, r^{\lambda_\ell}\, \phi_\ell(\hat{x}) \quad \text{with} \quad u_\eta \in K^m_\beta(\Gamma)$$

Model elliptic BVP in regular cones

Let P a PDE operator of order μ in $\mathbb{R}^n$. We say that P is homogeneous of order μ with radially constant coefficients if in polar Euler variables $t = \log|x|$ and $\hat{x} = x/|x|$, P takes the form

$$(*) \qquad P(x;\partial_x) = e^{-\mu t}\mathscr{P}(\hat{x};\partial_t,\partial_{\hat{x}}) \quad \text{with} \quad \mathscr{P}(\hat{x};\partial_t,\partial_{\hat{x}}) = \sum_{|\alpha|\le\mu} \mathscr{P}^{\alpha}(\hat{x})\,(\partial_t,\partial_{\hat{x}})^{\alpha}.$$

Examples

- If P is homogeneous with constant coefficients, it satisfies the condition above with the sum reduced to $|\alpha| = \mu$.
- If P is homogeneous with coefficients depending on $\hat{x}$ and not $|x|$, the same holds. The normal derivative around a regular cone satisfies this at order $\mu = 1$.
- A Schrödinger op. with singular potential $-\Delta + |x|^{-2}$ satisfies $(*)$ at order $\mu = 2$.

Let $A = \{L, B\}$ be an elliptic system of order 2. Assume that L is homogeneous of order 2 with radially constant coefficients, and the same for the boundary operators B_ℓ with the order μ_ℓ equal to 0 or 1. Then the Mellin symbol $\mathfrak{A}(\lambda)$ is generated by the BVP

$$\{\mathscr{L}(\hat{x};\lambda,\partial_{\hat{x}}),\ \mathscr{B}(\hat{x};\lambda,\partial_{\hat{x}})\}$$

If $\mathscr{B}(\hat{x};\lambda,\partial_{\hat{x}})$ is constant in λ,

$$\mathfrak{A}[\lambda] : U \longmapsto \mathscr{L}(\hat{x};\lambda,\partial_{\hat{x}})U \qquad \{U \in \boldsymbol{H}^2(\hat{\Gamma}),\ \mathscr{B}U\big|_{\partial\hat{\Gamma}} = 0\} \longrightarrow \boldsymbol{L}^2(\hat{\Gamma})$$

If not, we replace the condition $\mathscr{B}U\big|_{\partial\hat{\Gamma}} = 0$ by the action of $\mathscr{B}$ from $\boldsymbol{H}^2(\hat{\Gamma})$ into trace spaces $H^{3/2}(\partial\hat{\Gamma})$ and $H^{1/2}(\partial\hat{\Gamma})$.

Mellin symbol, meromorphic resolvent

Theorem 2D Laplace

Let $A = \{L, B\}$ be an elliptic system of order 2. Assume that L is homogeneous of order 2 with radially constant coefficients, and the same for the boundary operators B_ℓ with the order μ_ℓ equal to 0 or 1.

Then the Mellin symbol $\mathfrak{A}$ associated with A has a meromorphic resolvent $\lambda \mapsto \mathfrak{A}[\lambda]^{-1}$ that satisfies the uniform parameter estimates

$$(1) \qquad \forall\lambda \in \Lambda, \quad \forall G \in \boldsymbol{H}^{m-2}(\hat\Gamma), \quad \|\mathfrak{A}[\lambda]^{-1} G\|_{H^m(\hat\Gamma;\lambda)} \le C_\Lambda \|G\|_{H^{m-2}(\hat\Gamma;\lambda)}$$

on any set Λ of the form

$$\Lambda = \{\lambda \in \mathbb{C},\ \operatorname{Re}\lambda \in [\eta_0, \eta_1]\} \setminus \bigcup_{\lambda_0 \in \sigma(\mathfrak{A})} \mathscr{B}(\lambda_0, \delta)$$

The proof is based on local elliptic estimates Theorem S.2, a parametrix construction, and the Analytic Fredholm Theorem.

See also the early classical reference

M. S. Agranovich, M. I. Vishik.
Elliptic problems with a parameter and parabolic problems of general type.
Russian Math. Surveys 19 (1964) 53?157.

Model elliptic BVP in regular cones: Fredholm properties

Let $A = \{L, B\}$ be an ellptic BVP, homogeneous with radially constant coefficients. Assume that L is a $d \times d$ system of order 2 and set, for a cone Γ

$$\boldsymbol{K}^m_\beta(\Gamma) = K^m_\beta(\Gamma)^d \quad \text{and} \quad \boldsymbol{K}^m_\beta(\Gamma; B) = \{u \in \boldsymbol{K}^m_\beta(\Gamma), \quad Bu\big|_{\partial\Gamma} = 0\}$$

Let $\mathfrak{A}$ be its Mellin symbol and $\sigma(\mathfrak{A})$ the spectrum of $\mathfrak{A}$.

Theorem G.1

1. Let Γ be a regular cone in $\mathbb{R}^n$. Let $m \geq 2$ and $\beta \in \mathbb{R}$. Set $\eta = -\beta - \frac{n}{2}$. Assume

   The line $\operatorname{Re}\lambda = \eta$ *is disjoint from the Mellin spectrum* $\sigma(\mathfrak{A})$

   Then A defines an isomorphismt from $\boldsymbol{K}^m_\beta(\Gamma; B)$ onto $\boldsymbol{K}^{m-2}_{\beta-2}(\Gamma)$.

2. Let $\beta' > \beta$, so that $\eta' := -\beta' - \frac{n}{2}$ is $< \eta$. If, moreover

   The line $\operatorname{Re}\lambda = \eta'$ *is disjoint from the Mellin spectrum* $\sigma(\mathfrak{A})$

   then A defines a Fredholm operator from $(\boldsymbol{K}^m_\beta \cap \boldsymbol{K}^m_{\beta'})(\Gamma; B)$ onto $(\boldsymbol{K}^{m-2}_{\beta-2} \cap \boldsymbol{K}^{m-2}_{\beta'-2})(\Gamma)$

3. If moreover

   The strip $\operatorname{Re}\lambda \in [\eta', \eta]$ *is disjoint from the Mellin spectrum* $\sigma(\mathfrak{A})$

   then A defines an isomorphism from $(\boldsymbol{K}^m_\beta \cap \boldsymbol{K}^m_{\beta'})(\Gamma; B)$ onto $(\boldsymbol{K}^{m-2}_{\beta-2} \cap \boldsymbol{K}^{m-2}_{\beta'-2})(\Gamma)$

Model elliptic BVP in regular cones: Residue formulas

Theorem G.2

ⓐ Let Γ be a regular cone in $\mathbb{R}^n$. Let $m \geq 2$ and $\beta \in \mathbb{R}$. Set $\eta = -\beta - \frac{n}{2}$. Assume

The line $\operatorname{Re}\lambda = \eta$ *is disjoint from the Mellin spectrum* $\sigma(\mathfrak{A})$

ⓑ Let $\beta' > \beta$, so that $\eta' := -\beta' - \frac{n}{2}$ is $< \eta$. Assume that $u \in \boldsymbol{K}^m_{\beta'}(\Gamma; B)$ satifies

$$Au =: f \in \boldsymbol{K}^{m-2}_{\beta-2}(\Gamma)$$

Then the Mellin transform $G[\lambda]$ of $g := r^2 f$ is holomorphic in the strip $\operatorname{Re}\lambda \in [\eta', \eta]$.

ⓒ Then there exists a unique $u_\eta \in \boldsymbol{K}^m_\beta(\Gamma; B)$ such that $Au_\eta = f$ and we have

$$u_\eta - u = \sum_{\substack{\lambda_0 \in \sigma(\mathfrak{A}) \\ \operatorname{Re}\lambda_0 \in [\eta', \eta)}} \operatorname*{Res}_{\lambda=\lambda_0} r^\lambda \mathfrak{A}[\lambda]^{-1} G[\lambda]$$

Denote as Φ_{λ_0} the residue at $\lambda = \lambda_0$:

$$\Phi_{\lambda_0} := \operatorname*{Res}_{\lambda=\lambda_0} r^\lambda \mathfrak{A}[\lambda]^{-1} G[\lambda]$$

Structure of residues

Lemma

The singular functions Φ_{λ_0} have the general form

$$\Phi_{\lambda_0}(x) = \sum_{q=0}^{Q} r^{\lambda_0} \log^q r \;\; \phi_p(\hat{x}) \quad \text{with} \quad \phi_p \in \mathscr{C}^\infty(\,\bar{\hat{\Gamma}}\,)$$

and satisfy

$$L\,\Phi_{\lambda_0} = 0 \quad \text{in} \quad \Gamma \quad \text{and} \quad B\,\Phi_{\lambda_0} = 0 \quad \text{on} \quad \partial\Gamma$$

Proof. We have, for a suitable closed contour γ around λ_0

$$\Phi_{\lambda_0} = \frac{1}{2i\pi} \int_\gamma r^\lambda\, \mathfrak{A}[\lambda]^{-1}\, G[\lambda]\, \mathrm{d}\lambda$$

Therefore

$$\begin{aligned} L\,\Phi_{\lambda_0} &= \frac{1}{2i\pi} \int_\gamma r^{-2} \mathscr{L}(\hat{x}; r\partial_r, \partial_{\hat{x}}) \Big(r^\lambda\, \mathfrak{A}[\lambda]^{-1}\, G[\lambda] \Big)\, \mathrm{d}\lambda \\ &= \frac{1}{2i\pi} \int_\gamma r^{-2} r^\lambda \mathscr{L}(\hat{x}; \lambda, \partial_{\hat{x}})\, \mathfrak{A}[\lambda]^{-1}\, G[\lambda]\, \mathrm{d}\lambda \\ &= \frac{1}{2i\pi} \int_\gamma r^{-2} r^\lambda \mathfrak{A}[\lambda]\, \mathfrak{A}[\lambda]^{-1}\, G[\lambda]\, \mathrm{d}\lambda \quad = \quad 0 \end{aligned}$$

Outline

Choice of functional spaces to describe corner regularity

a To prepare for local analysis in general corner domains, we consider data f with support in a compact NBH $\mathcal{U}$ of the vertex of a regular cone $\Gamma \subset \mathbb{R}^n$.

b The previous theorems apply if f belong to some weighted spaces $K^\ell_\beta(\Gamma)$, and describe the solution u in weighted spaces.

c We have to relate weighted spaces to the standard scale of Sobolev spaces H^m on $\Gamma \cap \mathcal{U}$. The natural embeddings

$$K^m_{-m}(\Gamma \cap \mathcal{U}) \subset H^m(\Gamma \cap \mathcal{U}) \subset K^m_0(\Gamma \cap \mathcal{U})$$

are rough and not sufficient (except for $m = 0$) do get relevant information.

d Note that $K^m_{-m}(\Gamma)$ is a space of functions that are *flat* in 0:

$$u \in K^m_{-m}(\Gamma) \quad \Longleftrightarrow \quad r^{|\alpha|-m}\partial^\alpha_x u \in L^2(\Gamma), \ \forall |\alpha| \le m.$$

Check that:

If $u \in K^m_{-m}(\Gamma)$, for any $\alpha \in \mathbb{N}^n$ with $|\alpha| < m - \frac{n}{2}$ there holds $\partial^\alpha_x u(0) = 0$.

e Comparison between the two scales of spaces K^m_{-m} and H^m on $\Gamma \cap \mathcal{U}$ involves Taylor expansion at 0.

Taylor expansion and weighted spaces

Generalize weighted spaces K^m_β to positive non-integer exponents: Set

$$K^s_{-s}(\Gamma) = \Big\{u, \quad r^{|\alpha|-s}\partial^\alpha_x u \in L^2(\Gamma), \quad \forall|\alpha| < s$$
$$\int_\Gamma\int_\Gamma \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^2}{|x-y|^{2(s-[s])+n}}\,dxdy, \quad \forall|\alpha| = [s]\Big\}$$

and $K^s_{\gamma-s}(\Gamma)$ is the space of functions u such that $r^\gamma u \in K^s_{-s}(\Gamma)$.

Theorem [Dauge88, Th. (AA.7)]

Let Γ be an open cone in $\mathbb{R}^n$, $\mathcal{U}$ be a compact NBH of 0, and let $s > 0$. Set

$$H^s_*(\Gamma\cap\mathcal{U}) = \Big\{u \in H^s(\Gamma\cap\mathcal{U}), \quad \partial^\alpha_x u(0) = 0 \quad \forall|\alpha| < s - \tfrac{n}{2}\Big\}$$

Then, if $s - \frac{n}{2}$ is not an integer

$$H^s_*(\Gamma\cap\mathcal{U}) = K^s_{-s}(\Gamma\cap\mathcal{U})$$

and if $s - \frac{n}{2}$ is an integer, then

$$H^s_*(\Gamma\cap\mathcal{U}) \subset K^s_{\gamma-s}(\Gamma\cap\mathcal{U}) \quad \forall\gamma > 0.$$

Hardy's inequality

The prrevious theorem partly relies on Hardy's inequality:

Hardy's inequality

For any $f \in \mathscr{C}_0^\infty(\mathbb{R}_+)$ and $\gamma \neq 1$:

$$\int_0^\infty r^{\gamma-2}|f(r)|^2\,\mathrm{d}r \leq \frac{4}{(1-\gamma)^2}\int_0^\infty r^\gamma|f'(r)|^2\,\mathrm{d}r.$$

- When $\gamma < 1$, this inequality still holds for any function f which is zero at 0 *"Hardy's inequality at zero"* in the following sense

$$\forall R > 0,\quad f' \in L^1(0,R) \quad \text{and} \quad f(r) = \int_0^r f'(s)\,\mathrm{d}s$$

- When $\gamma > 1$, this inequality still holds for any function f which is zero at infinity *"Hardy's inequality at infinity"* in the following sense

$$\forall R > 0,\quad f' \in L^1(R,\infty) \quad \text{and} \quad f(r) = \int_r^\infty f'(s)\,\mathrm{d}s$$

Mellin transform and Taylor expansion

Let $u \in H^s(\Gamma)$ with compact support. As $u \in K_0^s(\Gamma)$, and $\in K_\beta^s(\Gamma)$ for all $\beta \geq 0$, the Mellin transform $U[\lambda]$ well defined for $\operatorname{Re}\lambda = -\frac{n}{2}$, and is holomorphic in the half-plane $\operatorname{Re}\lambda < -\frac{n}{2}$. A much better result holds

Theorem [Dauge88, Th. (AA.29)]

Let Γ be an open cone in $\mathbb{R}^n$, $\mathscr{U}$ be a compact NBH of 0, and let $s > 0$.
Let $u \in H^s(\Gamma)$ with compact support and $U = \mathfrak{M}u$ its Mellin transform.
Then U has a meromrphic extension for $\operatorname{Re}\lambda < s - \frac{n}{2}$ and its poles are non-negative integers $0 \leq k < s - \frac{n}{2}$

$$\operatorname*{Res}_{\lambda=k} r^\lambda U[\lambda] = -\sum_{|\alpha|=k} \frac{\partial_x^\alpha u(0)}{\alpha!}\, x^\alpha$$

Moreover, if $s - \frac{n}{2}$ is not an integer, the function

$$\xi \longmapsto \|U[s - \tfrac{n}{2} + i\xi]\|_{H^s(\hat{\Gamma};|\xi|)} \quad \text{belongs to } L^2(\mathbb{R})$$

Corner expansion

Let $A = \{L, B\}$ be an ellptic BVP, homogeneous with radially constant coefficients with L a system of order 2. We have the following variant/improvement of Theorem G.2

Theorem G.2'

a Let Γ be a regular cone in $\mathbb{R}^n$. Let $s \geq 2$. Assume

$$\eta := s - \tfrac{n}{2} \notin \mathbb{N} \quad \text{and} \quad \textit{the line} \ \operatorname{Re}\lambda = \eta \ \textit{is disjoint from the Mellin spectrum} \ \sigma(\mathfrak{A})$$

b Let $f \in \boldsymbol{H}^{s-2}(\Gamma)$ with compact support. Then the Mellin transform $G[\lambda]$ of $g := r^2 f$ is meromorphic in the strip $\operatorname{Re}\lambda \in (-\infty, \eta]$.

c Let β', so that $\eta' := -\beta' - \frac{n}{2}$ is $< \eta$. Assume that $u \in \boldsymbol{K}^2_{\beta'}(\Gamma; B)$ satifies $Au = f$. Then there exists a unique $u_\eta \in \boldsymbol{K}^s_{-s}(\Gamma; B)$ such that $Au_\eta = f$ and we have

$$u_\eta - u \quad = \sum_{\substack{\lambda_0 \in \sigma(\mathfrak{A}) \cup \mathbb{N} \\ \operatorname{Re}\lambda_0 \in [\eta', \eta)}} \operatorname*{Res}_{\lambda = \lambda_0} r^\lambda \mathfrak{A}[\lambda]^{-1} G[\lambda]$$

Note the differences Th. G.2 versus Th. G.2':

- f belongs to an unweighted Sobolev space
- The regularity exponent does not need to be real
- The initial regularity of u does not need to be at exponent s.
- Residues may appear on a larger set.

Injectivity modulo polynomials

What is not satisfactory with Th. G.2':

- a. Cannot handle integer s and even dim. (e.g. $n = 2$) "curse of even dimensions"
- b. The residues can be polynomial function in x, thus regular in Cartesian coord.

Definition

Let Γ be a regular cone in $\mathbb{R}^n$, with $\hat{\Gamma}$ its section. Let $\lambda \in \mathbb{C}$.

1. Let S^λ be the space of quasi-homogeneous d-component functions

$$S^\lambda(\Gamma) = \Big\{\Psi = r^\lambda \sum_{q=0}^{Q} \log^q r\, \psi_q(\hat{x}), \quad \psi_q \in \mathscr{C}^\infty(\overline{\Gamma})^d\Big\}$$

   We mention zero boundary conditions $B\Psi = 0$ as usual: $\Psi \in S^\lambda(\Gamma; B)$.

2. Let P^λ be the space of homogeneous polynomials in Cartesian var. x, of deg. λ
3. The system $A = \{L, B\}$ said to be *injective modulo polynomials* if

   (1) $\qquad \Psi \in S^\lambda(\Gamma; B)$ and $L\Psi \in P^{\lambda-2} \quad \Longrightarrow \quad \Psi \in P^\lambda$

   This condition can be written as

$$L : S^\lambda(\Gamma; B)/P^\lambda(\Gamma; B) \longrightarrow S^{\lambda-2}(\Gamma)/P^{\lambda-2} \qquad \text{injective}$$

4. Let $\sigma_\star(A)$ be the set of $\lambda \in \mathbb{C}$ such that (1) is not satisfied.

Injectivity modulo polynomials and corner expansion

Improvement of Theorem G.2' using the new set $\sigma_\star(A)$:

Theorem G.3 → remark on cracks

(a) Let Γ be a regular cone in $\mathbb{R}^n$. Let $s \geq 2$. Assume

The line $\operatorname{Re}\lambda = \eta := s - \frac{n}{2}$ *is disjoint from the set* $\sigma_\star(A)$

(b) Let $f \in \boldsymbol{H}^{s-2}(\Gamma)$ with compact support, and $G[\lambda] = \mathfrak{M}[g := r^2 f]$.

(c) Let β', so that $\eta' := -\beta' - \frac{n}{2}$ is $< \eta$. Assume that $u \in \boldsymbol{K}^2_{\beta'}(\Gamma; B)$ satisfies $Au = f$. Then we have

$$u \quad + \sum_{\substack{\lambda_0 \in \sigma_\star(A) \\ \operatorname{Re}\lambda_0 \in [\eta', \eta)}} \operatorname*{Res}_{\lambda=\lambda_0} r^\lambda \mathfrak{A}[\lambda]^{-1} G[\lambda] =: u_\eta \in \boldsymbol{H}^s(\Gamma \cap \mathcal{U})$$

If the system $A = \{L, B\}$ ensues from a coercive variational pb with $V \subset X = \boldsymbol{H}^1$:

Theorem G.3' (a) & (b) same as above

(c) Assume that $u \in V$ satisfies $Au = f$. Then we have

$$u \quad + \sum_{\substack{\lambda_0 \in \sigma_\star(A) \\ \operatorname{Re}\lambda_0 \in (1-\frac{n}{2}, s-\frac{n}{2})}} \operatorname*{Res}_{\lambda=\lambda_0} r^\lambda \mathfrak{A}[\lambda]^{-1} G[\lambda] =: u_\eta \in \boldsymbol{H}^s(\Gamma \cap \mathcal{U})$$

Part IV

Applications and particular cases