Weighted norm inequalities for integral transforms with splitting kernels

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Given an integral transform

$$Tf(y)=\int_{\mathbb{R}_+} f(x)K(x,y)\,dx,$$

we will say that the kernel K is splitting if there exist nonnegative functions s_j and w_j, $j=1,2$ such that

$$|K(x,y)|\lesssim\begin{cases}s_1(x)w_1(y), & \text{if } x<\varphi(y),\\ s_2(x)w_2(y), & \text{if } x>\varphi(y),\end{cases}\tag{1}$$

for some bijective C^1 function $\varphi:(0,\infty)\to(0,\infty)$ (typically $\varphi(t)=t$ or $\varphi(t)=1/t$).

We discuss necessary/sufficient conditions for the weighted norm inequality

$$\left(\int_{\mathbb{R}_+} u(y)|Tf(y)|^q\,dy\right)^{1/q}\leq C\left(\int_{\mathbb{R}_+} v(x)|f(x)|^p\,dx\right)\tag{2}$$

to hold, where $1<p\leq q<\infty$, C does not depend on f, and T has a splitting kernel (u and v are nonnegative functions defined on $\mathbb{R}_+$). These conditions will depend on the functions u,v,s, and w, and the sharpness of the obtained results is directly related to the sharpness of the estimate (1). In particular, if $K(x,y)\asymp s(x)w(y)$, then the sufficient conditions for the fulfilment of (2) that we obtain are also necessary. The fact that the kernel is splitting allows to find these sufficient conditions in an easy way through Hardy's inequalities.

Examples of transforms with splitting kernels for which we can obtain sharp (or almost sharp) estimates (1) are the Hardy and Bellman's transforms, the Stieltjes transform, the Struve transform, or the sine and Hankel transforms.