Sobolev Spaces, Integral Equations, and Scattering on non-Lipschitz and Fractal Sets

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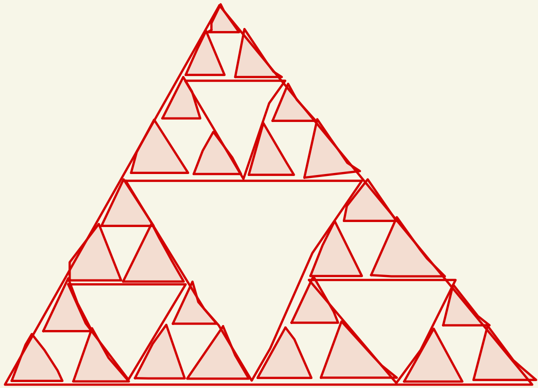
University of Reading



Part 5 Coda

back to original Qs, more open problems, refs

Given $g \in L^2(D)$, $\operatorname{supp}(g) \subset D$ & compact, find $u \in \tilde{H}^1(D)$ s.t.

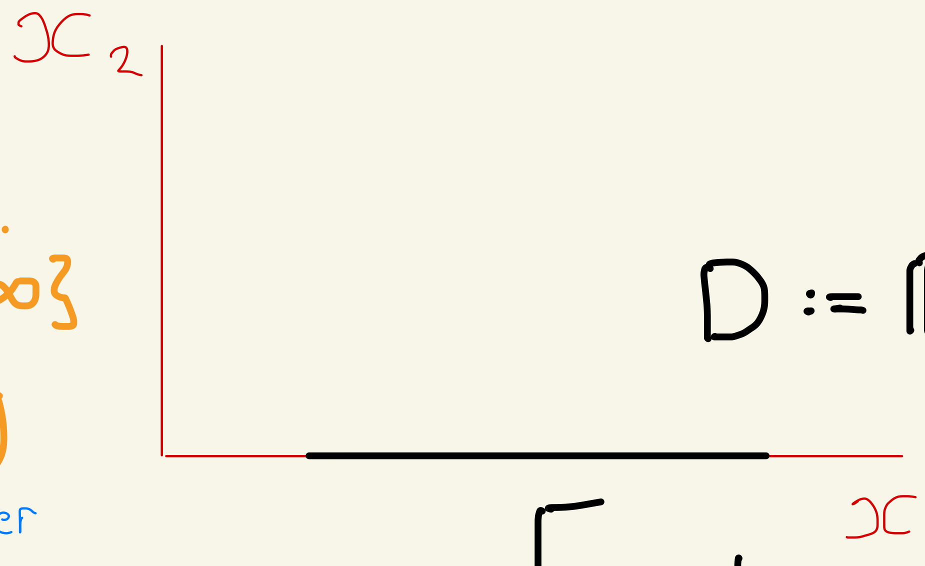
$$\Delta u + k^2 u = g \quad \text{in } D$$

$$\tilde{H}^1(D) = \overline{C_0^\infty(D)}^{H^1(\mathbb{R}^2)}$$

$$H^1(\mathbb{R}^2) = \{ v \in L^2(\mathbb{R}^2).\ \|v\| < \infty \}$$

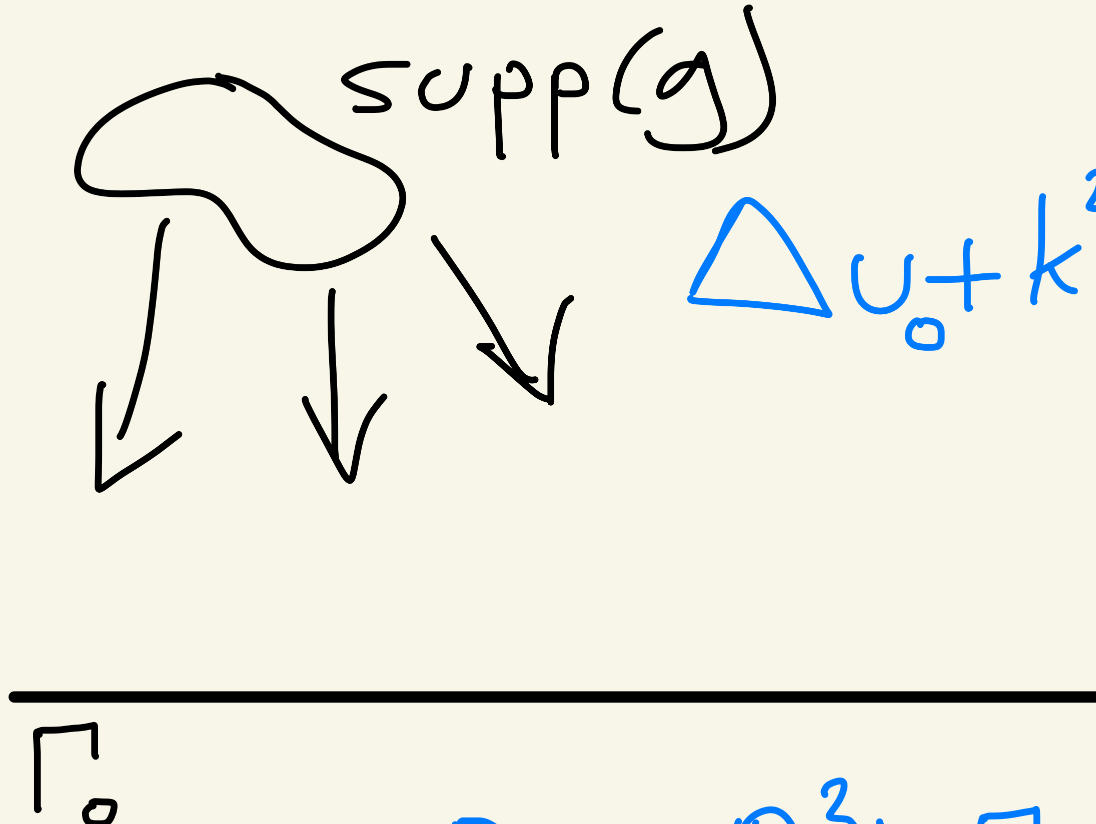
$$\|v\|^2 = \int_D \left(|v|^2 + |\nabla v|^2 \right)$$

dist der



$D := \mathbb{R}^2 \setminus \Gamma$

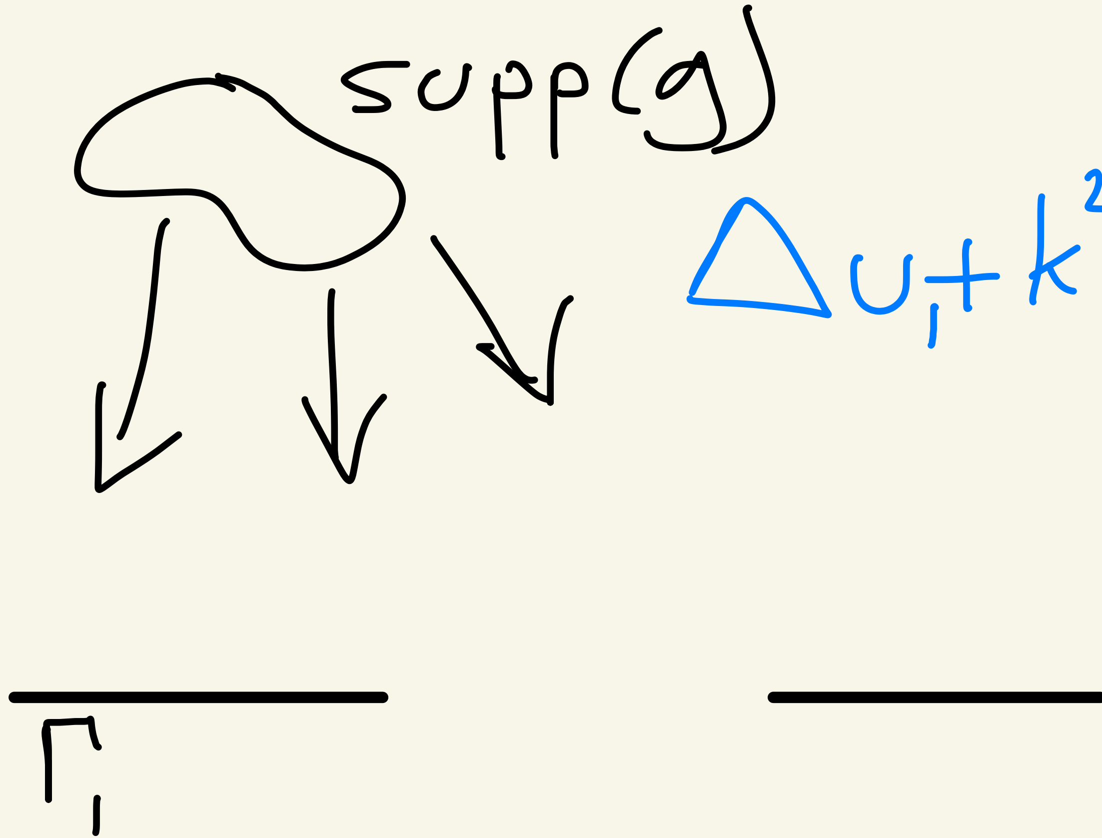
Γ, closed



$$\Delta u_0 + k^2 u_0 = g$$

$$j = 0$$

$$D_0 := \mathbb{R}^2 \setminus \Gamma_0$$

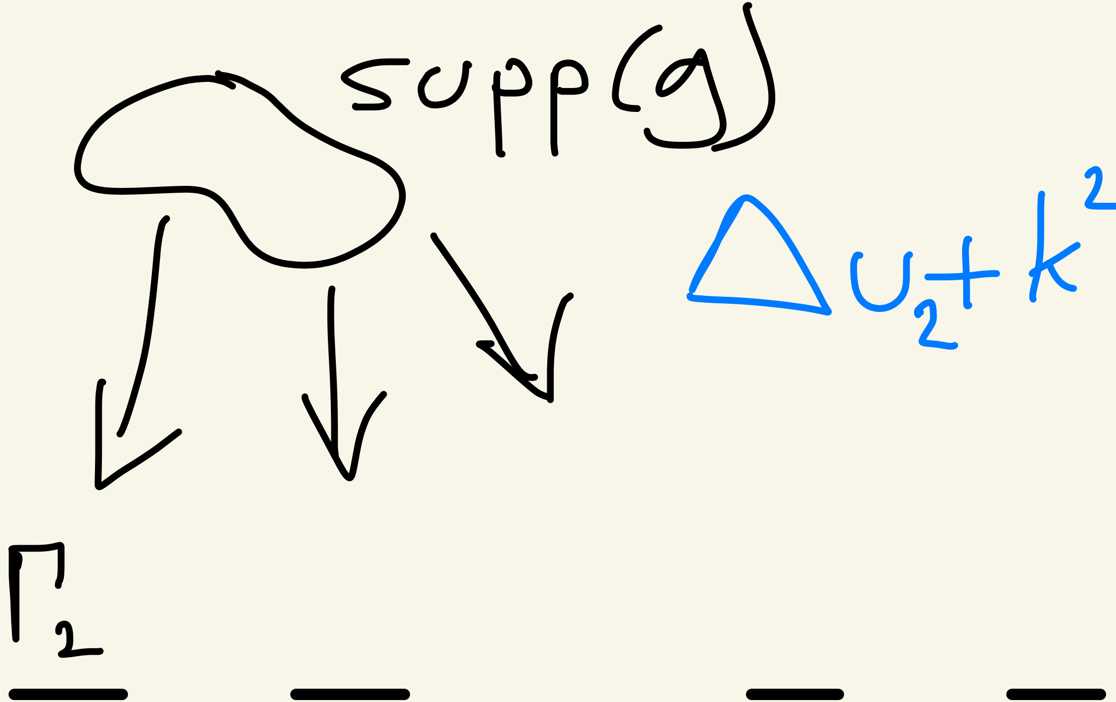


$$\Delta u_1 + k^2 u_1 = g$$

$j = 1$

D_1

D_2



$$\Delta u_2 + k^2 u_2 = g$$

$j = 2$

What is the Cantor set limit,

$$u := \lim_{j\to\infty} u_j \ ?$$

Coercive Formulation in D

Find $u_j \in \tilde{H}^1(D_j)$ s t

$$a(u_j, v_j) = \langle g, \bar{v}_j \rangle, \quad \forall v_j \in \tilde{H}^1(D_j)$$

$D_1 \subset D_2 \subset \dots$ so $\tilde{H}^1(D_j) \xrightarrow{M} \tilde{H}^1(D)$,

where $D = \bigcup_j D_j = \mathbb{R}^2 \setminus \Gamma$, with $\Gamma = \bigcap_j \Gamma_j$

Coercive Formulation in D

Find $u_j \in \tilde{H}^1(D_j)$ s t

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Cantor set

Coercive Formulation in D

Find $u_j \in \tilde{H}^1(D_j)$ s t

$$a(u_j, v_j) = \langle g, \bar{v}_j \rangle, \quad \forall v_j \in \tilde{H}^1(D_j)$$

$D_1 \subset D_2 \subset \ldots$ so $\tilde{H}^1(D_j) \xrightarrow{\Pi} \tilde{H}^1(D)$,

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Cantor set

So $u_j \to u \in \tilde{H}^1(D)$ given by

$$a(u, v) = \langle g, \bar{v} \rangle, \quad \forall v \in \tilde{H}^1(D).$$

Coercive Formulation on Γ

Find $\phi_j \in H^{-1}_{\Gamma_j}$ s.t.

$$A(\phi_j, \psi_j) = -\langle \bar{\psi}_j, \varsigma g \rangle, \quad \forall \psi_j \in H^{-1}_{\Gamma_j}$$

$\Gamma_1 \supset \Gamma_2 \supset \ldots$ so $H^{-1}_{\Gamma_j} \xrightarrow{M} H^{-1}_{\Gamma}$

$\Gamma = \bigcap_j \Gamma_j$

Cantor set

Coercive Formulation on Γ

Find $\phi_j \in H^{-1}_{\Gamma_j}$ s t

$$A(\phi_j, \psi_j) = -\langle \bar{\psi}_j, \mathcal{G} g \rangle, \quad \forall \psi_j \in H^{-1}_{\Gamma_j}$$

$\Gamma_1 \supset \Gamma_2 \supset \ldots$ so $H^{-1}_{\Gamma_j} \xrightarrow{M} H^{-1}_{\Gamma}$

$\Gamma = \bigcap_j \Gamma_j$

Cantor set

So $\phi_j \to \phi \in H^{-1}_{\Gamma}$ given by

$$A(\phi, \psi) = -\langle \bar{\psi}, \mathcal{G} g \rangle, \quad \forall \psi \in H^{-1}_{\Gamma}$$

The limiting solution for the Cantor set

$u \in \tilde{H}^1(D)$ st $a(u,v) = \langle g, \bar{v} \rangle, \ \forall v \in \tilde{H}^1(D).$

$\phi \in H^{-1}_{\Gamma}$ st $A(\phi,\psi) = -\langle \bar{\psi}, \gamma g \rangle, \ \forall \psi \in H^{-1}_{\Gamma}$

$D = \mathbb{R}^2 \setminus \Gamma, \qquad \Gamma = \text{Cantor set}$

The limiting solution for the Cantor set

$u \in \tilde{H}^1(D)$ st $a(u,v) = \langle g, \bar{v} \rangle, \ \forall v \in \tilde{H}^1(D)$.

$\phi \in H^{-1}_{\Gamma}$ st $A(\phi, \psi) = -\langle \bar{\psi}, G g \rangle, \ \forall \psi \in H^{-1}_{\Gamma}$

$D = \mathbb{R}^2 \setminus \Gamma, \qquad \Gamma = \text{Cantor set}$

These are same soln,

$u = G\phi + Gg, \quad \phi = (\Delta + k^2) u - g.$

The limiting solution for the Cantor set

$u \in \tilde{H}^1(D)$ st $a(u,v) = \langle g, \bar{v} \rangle, \ \forall v \in \tilde{H}^1(D).$

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effect of Γ

soln without Γ

The limiting solution for the Cantor set

$u \in \tilde{H}^1(D)$ st $a(u,v) = \langle g, \bar{v} \rangle, \ \forall v \in \tilde{H}^1(D)$.

$\phi \in H^{-1}_{\Gamma}$ st $A(\phi,\psi) = -\langle \bar{\psi}, Gg \rangle, \ \forall \psi \in H^{-1}_{\Gamma}$

These are same soln,

$u = G\phi + Gg, \quad \phi = (\Delta + k^2)u - g.$

effect of Γ — soln without Γ

$\tilde{H}^1(D) \neq H^1(\mathbb{R}^2) \iff H^{-1}_{\Gamma} \neq \{0\} \Leftarrow \dim_H \Gamma > 0$

The limiting solution for the Cantor set

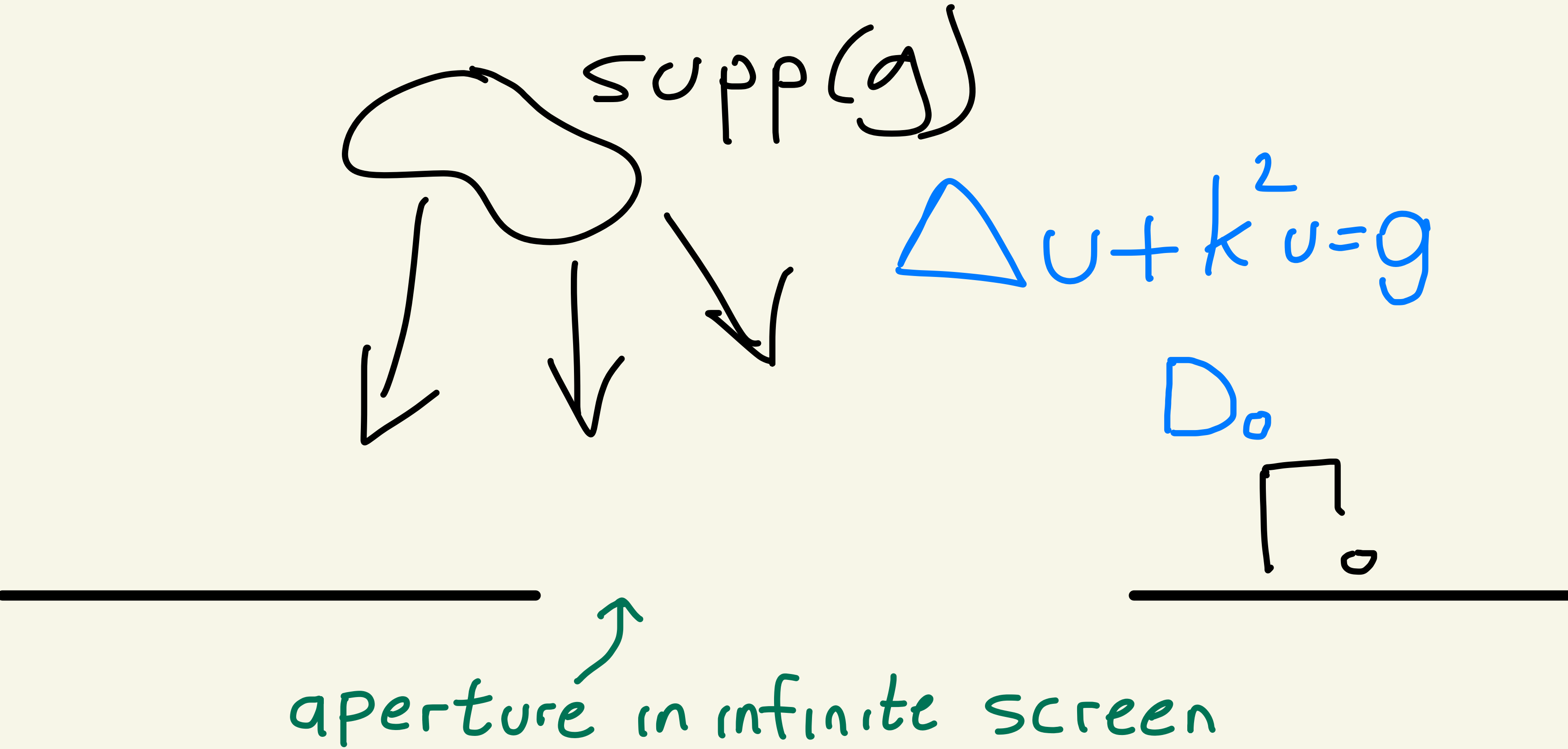
$u \in \tilde{H}^1(D)$ st $a(u,v) = \langle g, \bar{v} \rangle$, $\forall v \in \tilde{H}^1(D)$.

$\phi \in H^{-1}_{\Gamma}$ st $A(\phi,\psi) = -\langle \bar{\psi}, Gg \rangle$, $\forall \psi \in H^{-1}_{\Gamma}$

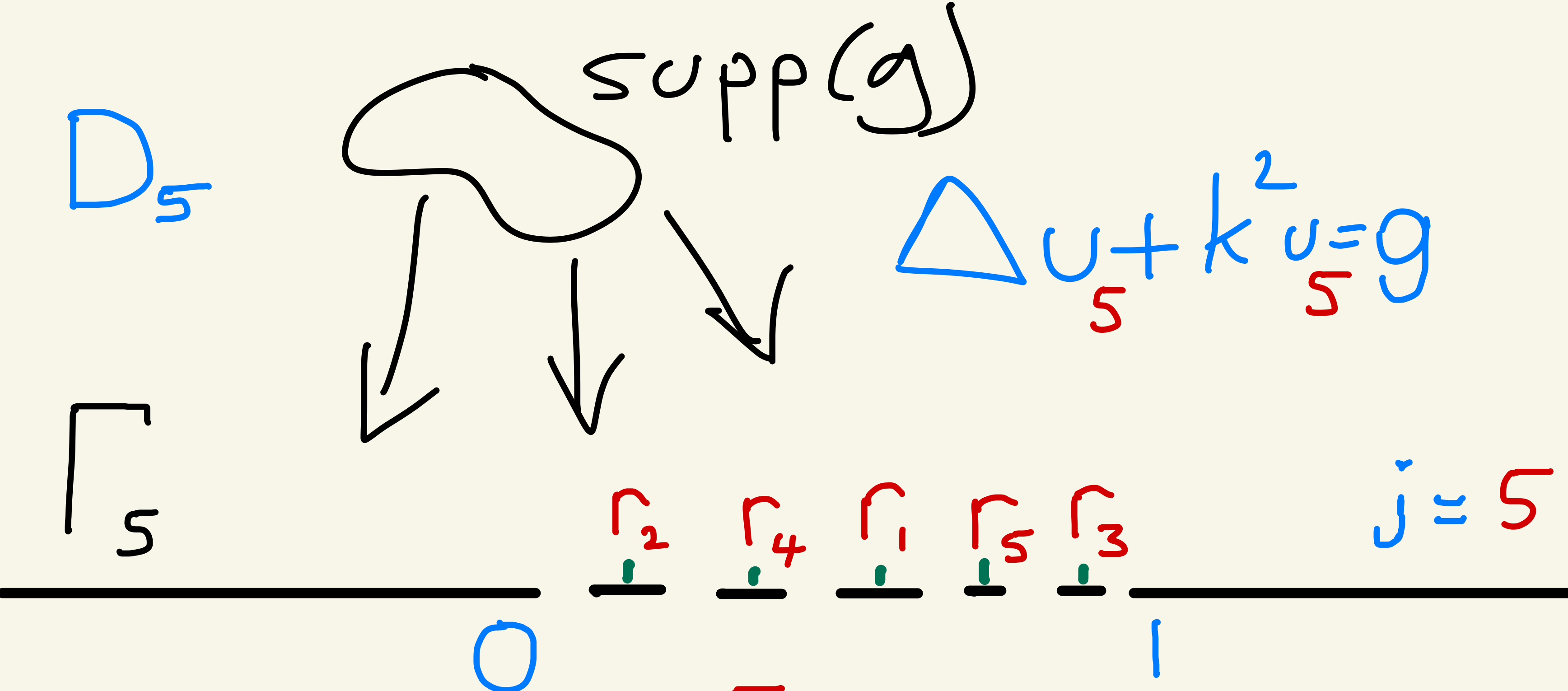
$u = G\phi + Gg$, $\phi = (\Delta + k^2)u - g$.

effect of Γ soln without Γ

$\boxed{\text{Thm}}$ (C-W, Hewett, SIMA, 2018) If $g \in L^2(D)$, $\mathrm{supp}(g) \subset D$, and $Gg(x) \neq 0$ on Γ, then $\phi \neq 0$ and $G\phi \neq 0$

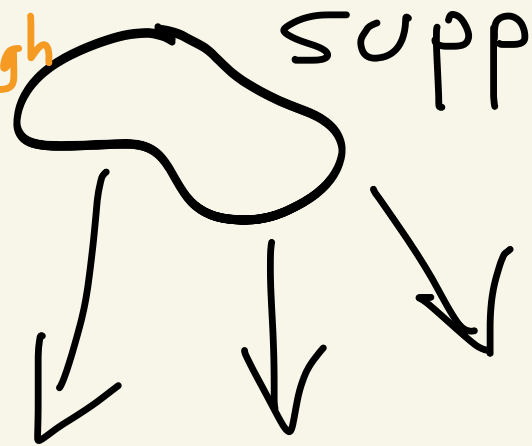

supp(g)
$\Delta u + k^2 u = g$
D_0
Γ_0
aperture in infinite screen

$$\Delta u_5 + k^2 u_5 = g$$



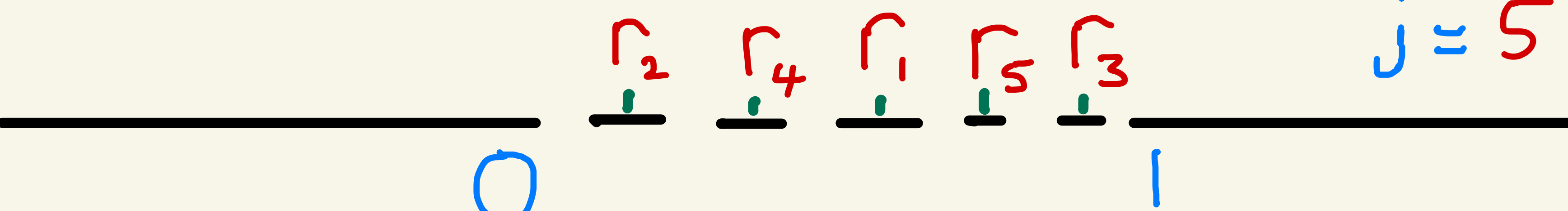
At step j add $[r_j - \epsilon_j, r_j + \epsilon_j]$, centred on jth rational $r_j \in (0,1)$

Does any sound get through in limit $j \to \infty$?

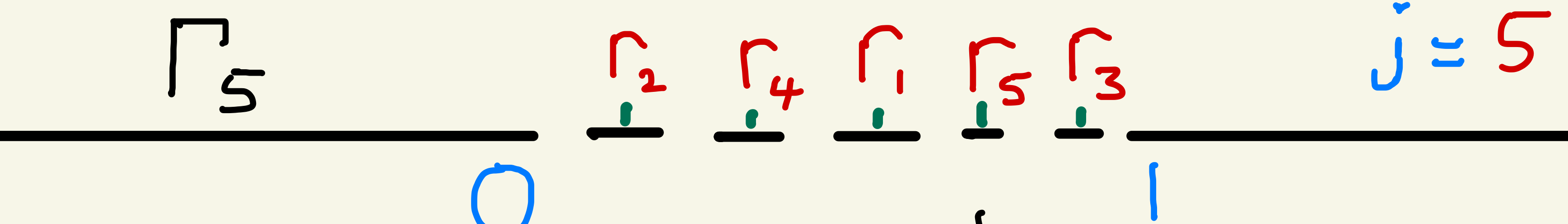


$$\Delta u_5 + k^2 u_5 = g$$

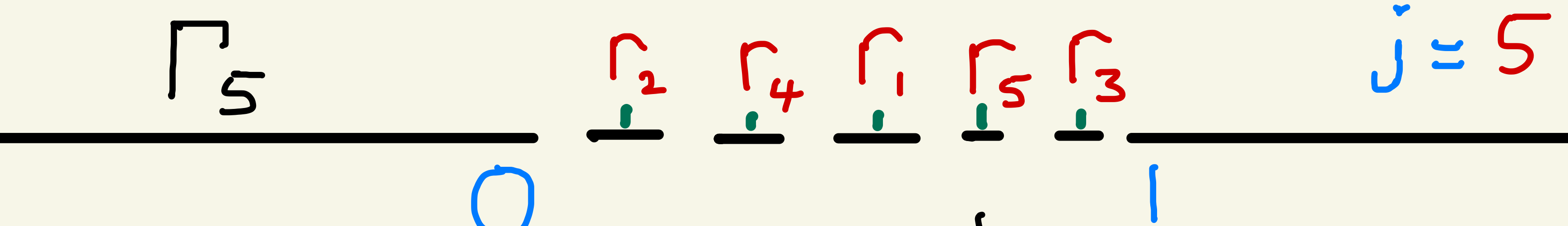
$j = 5$



At step j add $[r_j - \epsilon_j, r_j + \epsilon_j]$, centred on jth rational $r_j \in (0,1)$



$$\Gamma_j = (-\infty, 0] \cup [1, \infty) \cup \bigcup_{m=1}^{j} [r_m - \epsilon_m, r_m + \epsilon_m]$$

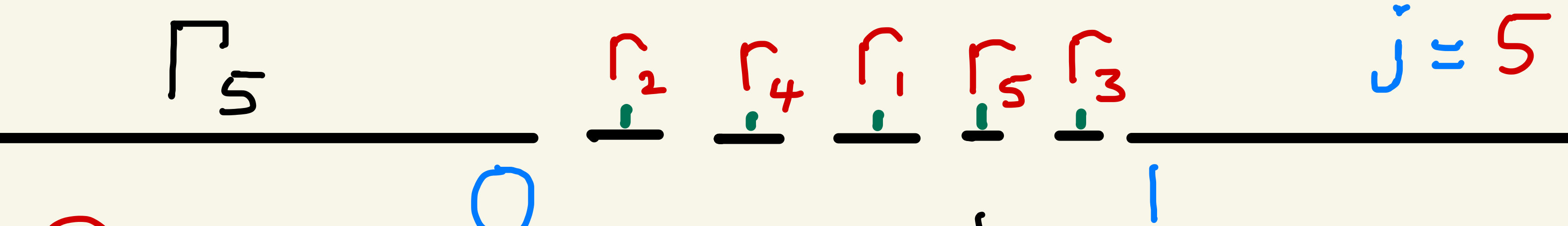


$$\Gamma_j = (-\infty, 0] \cup [1, \infty) \cup \bigcup_{m=1}^{j} [r_m - \epsilon_m, r_m + \epsilon_m]$$

$u_j = G\phi_j + Gg = G\gamma^* \tilde{\phi}_j + Gg$, where

$\tilde{\phi}_j \in H^{-1/2}_{\Gamma_j} \subset H^{-1/2}(\mathbb{R})$,

$$\langle \tilde{\psi}_j, S\tilde{\phi}_j \rangle = \langle \tilde{\psi}_j, \gamma G g \rangle, \quad \tilde{\psi}_j \in H^{-1/2}_{\Gamma_j}$$

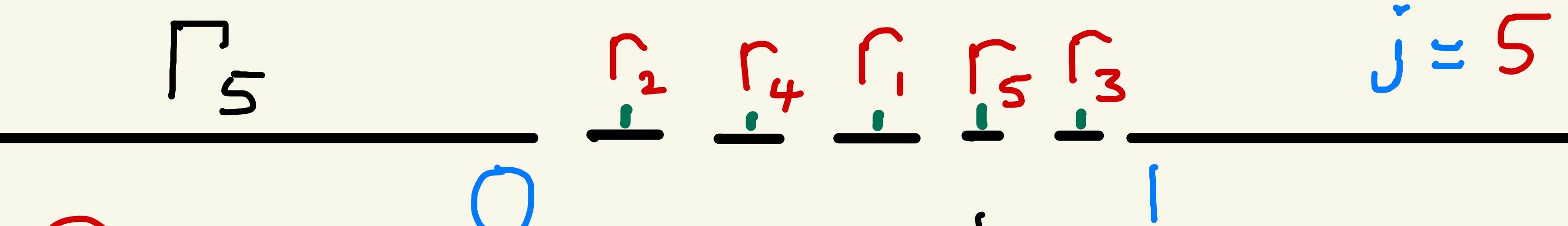


$$\Omega_j = (-\infty,0)\cup(1,\infty)\cup\bigcup_{m=1}^{j}(r_m-\epsilon_m,\, r_m+\epsilon_m)$$

$$u_j = G\phi_j + Gg = G\gamma^*\tilde{\phi}_j + Gg, \text{ where}$$

$$\tilde{\phi}_j \in H^{-1/2}_{\Gamma_j} = \tilde{H}^{-1/2}(\Omega_j)$$

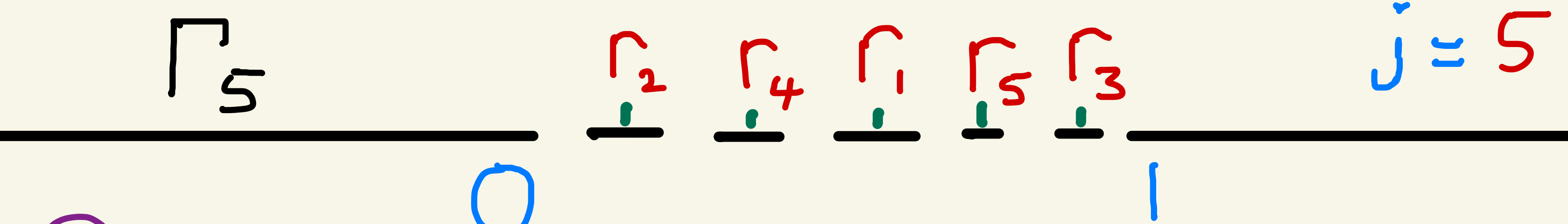
$$\langle \tilde{\psi}_j, S\tilde{\phi}_j\rangle = \langle \tilde{\psi}_j, \gamma G g\rangle, \quad \tilde{\psi}_j \in \tilde{H}^{-1/2}(\Omega_j)$$



$$\Omega_j = (-\infty, 0) \cup (1, \infty) \cup \bigcup_{m=1}^{j} (r_m - \epsilon_m, r_m + \epsilon_m)$$

$$u_j = G\gamma^* \tilde{\phi}_j + Gg, \quad \tilde{\phi}_j \in \tilde{H}^{-1/2}(\Omega_j)$$

$\Omega_1 \subset \Omega_2 \subset$ so $\tilde{H}^{-1/2}(\Omega_j) \xrightarrow{M} \tilde{H}^{-1/2}(\Omega)$



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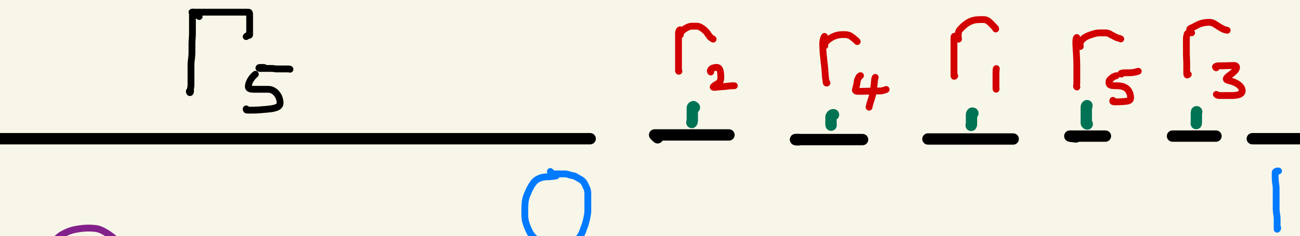


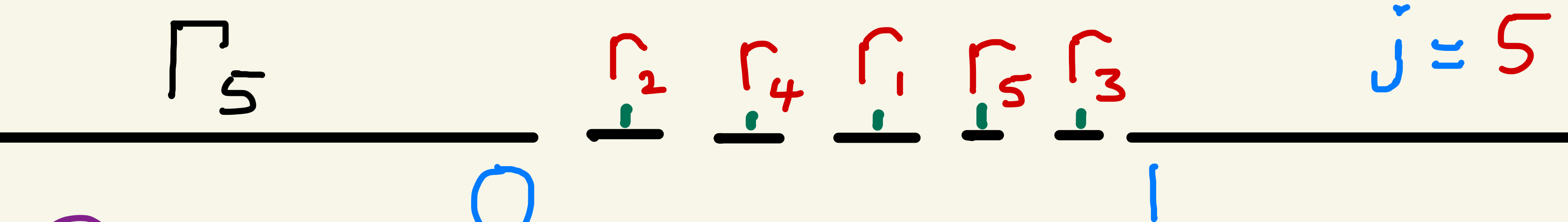
$j = 5$

$$\Omega = (-\infty, 0) \cup (1, \infty) \cup \bigcup_{m=1}^{\infty} (r_m - \epsilon_m, r_m + \epsilon_m)$$

$$u_j = G\gamma^* \tilde{\phi}_j + Gg, \quad \tilde{\phi}_j \in \tilde{H}^{-1/2}(\Omega_j)$$

$\Omega_1 \subset \Omega_2 \subset$ so $\tilde{H}^{-1/2}(\Omega_j) \xrightarrow{M} \tilde{H}^{-1/2}(\Omega)$,

where $\Omega = \bigcup_j \Omega_j$, so $\tilde{\phi}_j \to \tilde{\phi}$, $u_j \to u$



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$$u = G\gamma^* \tilde{\phi} + Gg, \quad \tilde{\phi} \in \tilde{H}^{-1/2}(\Omega),$$

$$\langle \tilde{\psi}, S\tilde{\phi} \rangle = \langle \tilde{\psi}, \gamma G g \rangle, \quad \tilde{\psi} \in \tilde{H}^{-1/2}(\Omega)$$

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$\overline{\Omega} = \mathbb{R}$ Is $\tilde{H}^{-1/2}(\Omega) \neq H^{-1/2}(\mathbb{R})$?

$$\Omega = (-\infty,0) \cup (1,\infty) \cup \bigcup_{m=1}^{\infty} (r_m-\epsilon_m, r_m+\epsilon_m)$$

$$u = G\gamma^* \tilde{\phi} + Gg, \quad \tilde{\phi} \in \tilde{H}^{-1/2}(\Omega),$$

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$\overline{\Omega} = \mathbb{R}$ Is $\tilde{H}^{-1/2}(\Omega) \neq H^{-1/2}(\mathbb{R})$?

(If not u is soln for infinite solid screen: no sound gets through)

$$\Omega = (-\infty, 0) \cup (1, \infty) \cup \bigcup_{m=1}^{\infty} (r_m - \epsilon_m, r_m + \epsilon_m)$$

$$\overline{\Omega} = \mathbb{R} \qquad \text{Is } \tilde{H}^{-1/2}(\Omega) \neq H^{-1/2}(\mathbb{R})\,?$$

$$\Omega = (-\infty,0) \cup (1,\infty) \cup \bigcup_{m=1}^{\infty} (r_m-\epsilon_m, r_m+\epsilon_m)$$

$\bar{\Omega} = \mathbb{R}$ Is $\tilde{H}^{-1/2}(\Omega) \neq H^{-1/2}(\mathbb{R})$?

Let

$$F = \mathbb{R} \setminus \Omega = [0,1] \setminus \bigcup_{m=1}^{\infty} (r_m-\epsilon_m, r_m+\epsilon_m)$$

$$\Omega = (-\infty,0) \cup (1,\infty) \cup \bigcup_{m=1}^{\infty} (r_m-\epsilon_m, r_m+\epsilon_m)$$

$$\overline{\Omega} = \mathbb{R} \qquad \text{Is } \underbrace{\tilde{H}^{-1/2}(\Omega) \neq H^{-1/2}(\mathbb{R})}_{(+)} \text{ ?}$$

Let

$$F := \mathbb{R} \setminus \Omega = [0,1] \setminus \bigcup_{m=1}^{\infty} (r_m-\epsilon_m, r_m+\epsilon_m)$$

$$(+) \Leftrightarrow H_F^{1/2} \neq \{0\}$$

$$F = [0,1] \setminus \bigcup_{m=1}^{\infty} (r_m - \epsilon_m, r_m + \epsilon_m)$$

Is $H_F^{1/2} \neq \{0\}$?

$$F = [0,1] \setminus \bigcup_{m=1}^{\infty} (r_m - \epsilon_m, r_m + \epsilon_m)$$

Is $H_F^{1/2} \neq \{0\}$?

$$H_F^{1/2} \neq \{0\} \Longrightarrow H_F^{0} \neq \{0\}$$

$$\Longleftrightarrow m(F) > 0$$

$$\Longleftarrow 1 - 2\sum_{m=1}^{\infty} \epsilon_m > 0$$

$$F = [0,1] \setminus \bigcup_{m=1}^{\infty} (r_m - \epsilon_m, r_m + \epsilon_m)$$

Is $H^{1/2}_F \neq \{0\}$?

For $\epsilon > 0$,

$$H^{1/2+\epsilon}_F \subseteq \{\phi \in C(\mathbb{R}) : \operatorname{supp}(\phi) \subset F\} = \{0\}$$

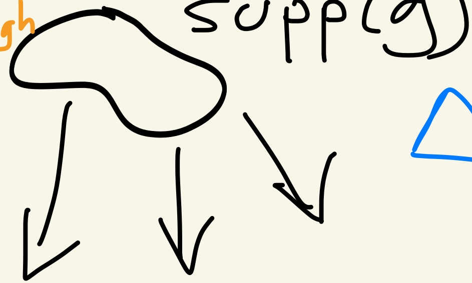
Sobolev embedding thm

$$F = [0,1] \setminus \bigcup_{m=1}^{\infty} (r_m - \epsilon_m, r_m + \epsilon_m)$$

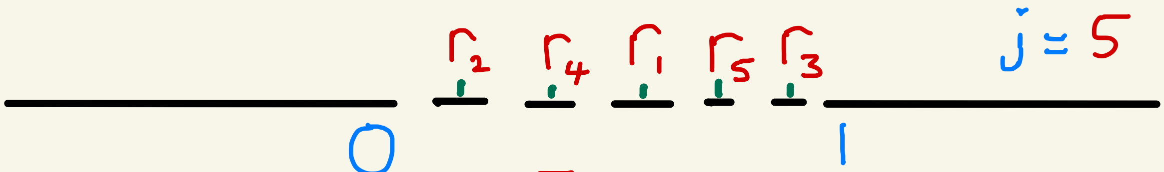
Is $H_F^{1/2} \neq \{0\}$?

$\boxed{\text{Thm}}$ (Polking, 1972) $H_F^{1/2} \neq \{0\}$ if $\sum_{m=1}^{\infty} [\log(2/\epsilon_m)]^{-1}$ is sufficiently small

Does any sound get through in limit $j \to \infty$?



$$\Delta u_5 + k^2 u_5 = g$$



At step j add $[r_j - \epsilon_j, r_j + \epsilon_j]$, centred on jth rational $r_j \in (0,1)$

$\boxed{\text{Cor}}$ As $j \to \infty$, $u_j \to u$ and u is not zero in $\{x_2 < 0\}$ if $\sum_{m=1}^{\infty} [\log(2/\epsilon_m)]^{-1}$ is sufficiently small

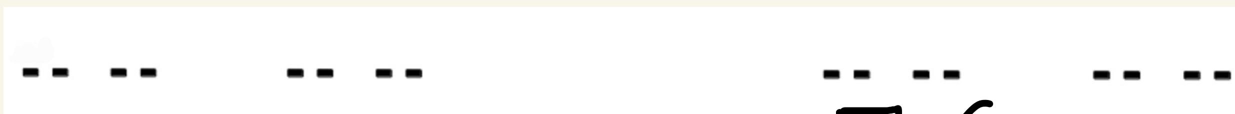
Open Problems

Given $g \in L^2(D)$

find $u \in \tilde{H}^1(D)$ s t

$$\Delta u + k^2 u = g$$

$D = \mathbb{R}^2 \setminus \Gamma$



Γ (fractal)

OP (Regularity)

For which $s \geq 1$

is $u \in H^s(\mathbb{R}^2)$?

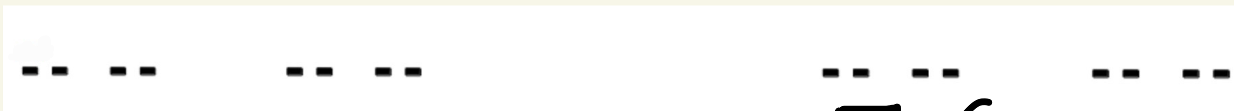
Open Problems

Given $g \in L^2(D)$

find $u \in \tilde{H}^1(D)$ s t

$$\Delta u + k^2 u = g$$

$D = \mathbb{R}^2 \setminus \Gamma$



Γ (fractal)

OP (Regularity)

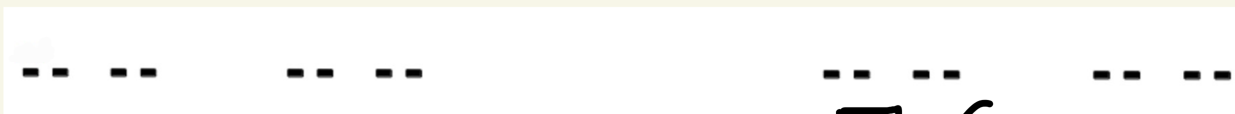
For which $s \geqslant 1$ is $u \in H^s(\mathbb{R}^2)$?

Conjecture $s < 1 + \frac{\dim_H \Gamma}{2}$

Open Problems

Given $g \in L^2(D)$
find $u \in \tilde{H}^1(D)$ s t
$$\Delta u + k^2 u = g$$

$$D = \mathbb{R}^2 \setminus \Gamma$$



Γ (fractal)

$\boxed{\text{OP}}$ (Regularity)
For which $s \geq 1$
is $u \in H^s(\mathbb{R}^2)$?
Conjecture $s < 1 + \frac{\dim_H \Gamma}{2}$

$\boxed{\text{OP}}$ Rate of convergence
$\|u - u_j\| \to 0$?

MOSCO CONVERGENCE AND APPLICATION IN PDE

[0] U Mosco, *Convergence of convex sets and of solutions to variational inequalities*, Advances in Mathematics, 1969 **[The original and best!]**

[1] D Danvers, *Dirichlet problems on varying domains*, J Diff. Eq., 2003

[2] G Menegatti and L. Rondi, *Stability for the acoustic scattering problem for sound-hard scatterers*, Inverse Probl. Imaging, 2013.

[3] J M Arrieta, P D Lamberti, *Higher order elliptic operators on variable domains. Stability results and boundary oscillations for intermediate problems,* J Diff. Eq., 2017.

[4] S N C-W & D P Hewett, *Well-posed PDE and integral equation formulations for scattering by fractal screens*, SIAM J. Math. Anal., 2018.

[5] S N C-W, D P Hewett, A Moiola & J Besson, *Boundary element methods for acoustic scattering by fractal screens*, to appear in Numerische Mathematik.

SOBOLEV SPACES

[6] J C Polking, Leibniz formula for some differentiation operators of fractional order, Indiana Univ. Math. J., 1972.

[7] W McLean, *Strongly Elliptic Systems and Boundary Integral Equations*, CUP, 2000. **[Contains most of what you need for these lectures.]**

[8] V G Maz'ya, *Sobolev Spaces with Applications to Elliptic Partial Differential Equations*, 2nd Ed., Springer, 2011.

[9] S N Chandler-Wilde, D P Hewett & A Moiola, *Sobolev spaces on non-Lipschitz subsets of R^n with application to boundary integral equations on fractal screens*, Integral Eq. Oper. Theory, 2017.

[10] A Caetano, D P Hewett, A Moiola, *Density results for Sobolev, Besov, and Triebel-Lizorkin spaces on rough sets*, arXiv:1904.05420, 2019.

INTEGRAL EQUATIONS FOR SCATTERING PROBLEMS INCLUDING FRACTAL

[11] D L Colton & R Kress, Integral Equation Methods in Scattering Theory, Wiley, 1983. **[Smooth domains, classical function spaces.]**

Reference [7] above (book by McLean). **[Modern Sobolev space setting, strongly-elliptic systems, Lipschitz domains]**

[12] S N C-W, I G Graham, S Langdon, & E A Spence, *Numerical-asymptotic boundary integral methods in high-frequency acoustic scattering*, Acta Numerica, 2012. **[Focussed on acoustic scattering.]**

[13] S N C-W & D P Hewett, *Wavenumber-explicit continuity and coercivity estimates in acoustic scattering by planar screens*, Integral Eq. Oper. Theory, 2015, and references [4,5,9] above. **[Scattering by screens rougher than Lipschitz, e.g. fractal or fractal boundary, Dirichlet and Neumann boundary conditions, Im(k)>0 and k>0.]**