University of Padua, Italy (June, 2017)

Mini-Course

Boundary-Domain Integral and Integro-Differential Equations for Elliptic BVPs with variable coefficients

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Lecture Topics

- **Classical potential method for constant coefficient boundary value problems: Indirect and direct boundary integral equations method.**
- **Scalar BVPs with one variable coefficient (isotropic case): Parametrix based BDIE approach.**
- **Scalar BVPs with matrix variable coefficient (anisotropic case): Localized harmonic parametrix based BDIE approach.**
- **Applications of BDIE method to transmission problems of acoustic scattering by inhomogeneous anisotropic obstacles.**

LECTURE 1

CLASSICAL POTENTIAL METHOD

FOR CONSTANT COEFFICIENT

BOUNDARY VALUE PROBLEMS

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1. **Classical formulation of the BVPs**
2. **Green's formulas and Integral Representation of solutions**
3. **Weak formulation of the problems, uniqueness theorems**
4. **Properties of potentials**
5. **Reduction to BIEs: Direct and Indirect methods**
6. **Existence and regularity of solutions (based on the Fredholm-Riesz Theory and the Theory of pseudodifferential equations)**

Classical Potential Method - Regular Case

N.M. Günter, Potential Theory and its application to the basic problems of mathematical physics. Fizmatgiz, Moscow 1953 (Russian). Translation in English: Frederick Ungar Publishing, New York, 1967.

K. Miranda, Partial differential equations of elliptic type. Springer, Berlin-Heidelberg, 2-nd edition, 1970.

R. Dautray and J.L. Lions, Mathematical analysis and numerical methods for science and technology. Vol. 4. Integral equations and numerical methods. Springer-Verlag, Berlin, 1990.

G. Hsiao and W. Wendland, Boundary integral equations. Springer, Berlin - Heidelberg, 2008.

Theory of pseudodifferential equations on manifolds with boundary

G. Eskin, Boundary value problems for elliptic pseudodifferential equations. Translation of Mathematical Monographs, vol. 52. American Mathematical Society: Providence, RI, 1981.

E. Shargorodsky, An L_p analogue of the Vishik-Eskin theory. Memoirs on Differential Equations and Mathematical Physics, 2 (1994), 41-146.

CLASSICAL SETTING OF BVPs AND UNIQUENESS THEOREMS

The Laplace equation in $\Omega \subset \mathbb{R}^3$:

$$A(\partial)u(x) \equiv \Delta u(x) = \Big[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\Big] u(x) = 0; \quad (1)$$

The Poisson equation in $\Omega \subset \mathbb{R}^3$:

$$A(\partial)u(x) \equiv \Delta u(x) = \Big[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\Big] u(x) = \Phi(x); \quad (2)$$

u – is an unknown function in $\Omega \in \{\Omega^+,\ \Omega^-\}$

Φ – is a given function in Ω

Ω^+ - a bounded domain in $\mathbb{R}^3$ with a simply connected boundary $\partial\Omega^+ = S$;

$\overline{\Omega^+} = \Omega^+ \cup S$; $\qquad \Omega^- := \mathbb{R}^3 \setminus \overline{\Omega^+}$;

$S \in C^\infty$ or $S \in \mathrm{Lip}$ or $S \in C^{k,\beta}$, $0 < \beta \leqslant 1$, $k \in \mathbb{N}$;

Dissection of the boundary surface $S = \overline{S}_D \cup \overline{S}_N$, $S_D \cap S_N = \varnothing$,

$\ell = \overline{S}_D \cap \overline{S}_N$;

The symbols $\{\,\cdot\,\}^\pm_S$ denote one-sided limits (traces) on S from $\Omega^\pm$;

$n = (n_1, n_2, n_3)$ - outward unit normal vector to S;

$$\partial_n := \frac{\partial}{\partial n} = n_1 \frac{\partial}{\partial x_1} + n_2 \frac{\partial}{\partial x_2} + n_3 \frac{\partial}{\partial x_3}$$ – normal derivative.

$\boldsymbol{C^{k,\alpha}}$, $\boldsymbol{L_p}$, $\boldsymbol{W_p^r}$, $\boldsymbol{H_p^s}$, $\boldsymbol{B_{p,q}^s}$ – **the well-known Höder, Lebesgue, Sobolev–Slobodetskii, Bessel potential, and Besov function spaces**
($\boldsymbol{r \geq 0}$, $\boldsymbol{s \in \mathbb{R}}$, $\boldsymbol{p \in (1, \infty)}$, $\boldsymbol{1 \leq q \leq \infty}$**).**

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$H_2^r = W_2^r = B_{2,2}^r$ for $r \geq 0$, $H_2^s = B_{2,2}^s$ for any $s \in \mathbb{R}$,
$B_{p,p}^t = W_p^t$ and $H_p^k = W_p^k$ for any positive and non-integer t, for any non-negative integer k and for any $p > 1$.

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Let S_1 be an open proper submanifold of S with smooth boundary:

$$\widetilde{\mathbb{X}}(S_1) := \{f : \ f \in \mathbb{X}(S), \ \operatorname{supp} f \subset \overline{S_1}\},$$

$$\mathbb{X}(S_1) := \{r_{S_1} f : \ f \in \mathbb{X}(S)\},$$

($r_{\mathcal{M}}$ stands for the restriction operator onto $\mathcal{M}$)

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($r_{\mathcal{M}}$ stands for the restriction operator onto $\mathcal{M}$)

A regular functions in $\Omega \in \{\Omega^+,\ \Omega^-\}$: $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$.

THE BASIC BVPs: **Find a regular solution u to the equation**

$$A(\partial)u = \Phi, \quad x \in \Omega^+, \tag{3}$$

satisfying one of the following boundary conditions:

The Dirichlet problem (D)$^+$:

$$\{u(x)\}^+ = f(x), \quad x \in S; \tag{4}$$

The Neumann problem (N)$^+$:

$$\{\partial_n u(x)\}^+ = F(x), \quad x \in S; \tag{5}$$

The mixed type problem (M)$^+$:

$$\{u(x)\}^+ = f^*(x), \;\; x \in S_D, \tag{6}$$

$$\{\partial_n u(x)\}^+ = F^*(x), \;\; x \in S_N. \tag{7}$$

GREEN's FORMULAS:

$$\int_{\Omega^+} \Delta u \; v \, dx = -\int_{\Omega^+} \nabla u \cdot \nabla v \, dx + \int_S \{\partial_n u\}^+ \, \{v\}^+ \, dS \qquad \textbf{(G1)}$$

$$\int_{\Omega^+} \left[\Delta u \; v - u \; \Delta v\right] dx = \int_S \left[\{\partial_n u\}^+ \{v\}^+ - \{u\}^+ \{\partial_n v\}^+\right] dS \qquad \textbf{(G2)}$$

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FUNDAMENTAL SOLUTION:

$$A(\partial)\Gamma(x) = \delta(x) \qquad \Rightarrow \qquad \Gamma(x) = -\frac{1}{4\pi\,|x|}, \qquad v(y) = \Gamma(x-y)$$

GENERAL INTEGRAL REPRESENTATION ($\forall\; u \in C^2(\overline{\Omega})$):

$$u(x) = \int_{\Omega^+} \Gamma(x-y) \Delta u(y) dy - \int_S \Gamma(x-y) \{\partial_n u(y)\}^+ \, dS + \\ + \int_S \big[\partial_{n(y)} \Gamma(x-y)\big] \{u(y)\}^+ dS_y, \qquad x \in \Omega; \qquad \textbf{(G3)}$$

GREEN's FORMULAS:

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$$+ \int_S \big[\partial_{n(y)} \Gamma(x-y)\big] \{u(y)\}^+ dS_y, \qquad x \in \Omega; \qquad \textbf{(G3)}$$

Green's formulas can be extended to the space

$$W_p^{1,0}(\Omega^+, \Delta) \equiv H_p^{1,0}(\Omega^+, \Delta) := \{\, u \in H_p^1(\Omega^+) \; : \; \Delta u \in L_p(\Omega^+) \,\}.$$

VOLUME AND LAYER POTENTIALS:

$$N_\Omega(\Psi)(x) = \int_\Omega \Gamma(x-y)\,\Psi(y)dy \tag{8}$$

$$V(g)(x) = \int_S \Gamma(x-y)\,g(y)dS_y \tag{9}$$

$$W(h)(x) = \int_S \left[\partial_{n(y)}\Gamma(x-y)\right] h(y)dS_y \tag{10}$$

VOLUME AND LAYER POTENTIALS:

$$N_{\Omega}(\Psi)(x) = \int_{\Omega} \Gamma(x-y)\, \Psi(y) dy \tag{11}$$

$$V(g)(x) = \int_{S} \Gamma(x-y)\, g(y) dS_y \tag{12}$$

$$W(h)(x) = \int_{S} \left[\partial_{n(y)} \Gamma(x-y)\right] h(y) dS_y \tag{13}$$

$N_{\Omega}(\Phi)$ is a particular solution of Poisson's equation:
If $\Phi \in L_p(\Omega)$ with $p > 1$ then $N_{\Omega}(\Phi) \in W_p^2(\Omega)$ and

$$A(\partial) N_{\Omega}(\Phi) = \Phi \ \text{ almost everywhere in } \ \Omega. \tag{14}$$

VOLUME AND LAYER POTENTIALS:

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The layer potentials $V(g)$ and $W(h)$ belong to $C^{\infty}(\Omega^{\pm})$ and are solutions of the homogeneous equation $A(\partial)u = 0$:

$$A(\partial)V(g)(x) = A(\partial)W(h)(x) = 0, \quad x \in \Omega^{\pm}. \tag{19}$$

In the case of **weak formulation** of the above BVPs we look for **weak solutions** in the spaces $H^{1,0}_p(\Omega^+,\Delta)$, $p>1$.

The differential equation $A(\partial)u=\Phi$ is understood in the distributional sense in Ω^+.

The Dirichlet type condition is understood in the usual trace sense in the space $B^{1-\frac{1}{p}}_{p,p}(S)$;

The generalized trace of normal derivative is understood in the functional sense $\{\partial_n u\}^+ \in B^{-\frac{1}{p}}_{p,p}(S) = [B^{\frac{1}{p}}_{p',p'}(S)]^*$ defined with the help of Green's first identity:

$$\big\langle \{\partial_n u\}^+ \,,\, \{v\}^+ \big\rangle_S := \int_{\Omega^+} \big[\,(\Delta u)\,v + \nabla u\cdot\nabla v\,\big]\,dx\,, \qquad (20)$$

$$u\in H^{1,0}_p(\Omega^+,\Delta),\quad v\in H^1_{p'}(\Omega^+),\quad \frac{1}{p}+\frac{1}{p'}=1.$$

The symbol $\langle\cdot,\,\cdot\rangle_S$ denotes bilinear duality brackets between the mutually adjoint spaces $B^{-\frac{1}{p}}_{p,p}(S)$ and $B^{\frac{1}{p}}_{p',p'}(S)$;

UNIQUENESS THEOREM

Let the manifolds S, S_D, and S_N be Lipschitz.

The BVPs (D)$^+$ and (M)$^+$ possess at most one weak solution in the space $W_2^1(\Omega^+)$, while the general solution of the homogeneous Neumann problem (N)$^+$ is a constant.

UNIQUENESS THEOREM

The homogeeous BVPs (D)$^+$ and (M)$^+$ possess only the trivial weak solution in the space $W_2^1(\Omega^+)$, while the general solution of the homogeneous Neumann problem (N)$^+$ is a constant.

Proof follows from Green's first formula:

$$\int_{\Omega^+} \Delta u \; u \, dx = - \int_{\Omega^+} |\nabla u|^2 \, dx + \langle \{\partial_n u\}^+ , \{u\}^+ \rangle_S \quad \textbf{(G1)}$$

PROPERTIES OF VOLUME POTENTIALS

N_{Ω^+} is a pseudodifferential (smoothing) operator of order -2.

The following operators are continuous

$$N_{\Omega^+} : C^{0,\alpha}(\overline{\Omega^+}) \to C^{2,\alpha}(\overline{\Omega^+}), \quad 0 < \alpha < 1, \tag{21}$$

$$: \widetilde{H}_p^s(\Omega^+) \to H_p^{s+2}(\Omega^+), \quad s \in \mathbb{R}, \tag{22}$$

$$: H_p^s(\Omega^+) \to H_p^{s+2}(\Omega^+), \quad s > -1 + \frac{1}{p}. \tag{23}$$

PROPERTIES OF LAYER POTENTIALS

The boundary operators generated by the single and double layer potentials

$$(\mathcal{H}\,g)(x) := \int_S \Gamma(x-y)\,g(y)\,dS_y\,,\; x \in S, \tag{24}$$

$$(\mathcal{K}\,g)(x) := \int_S \big[\,\partial_{n(x)}\Gamma(x-y)\,\big]\,g(y)\,dS_y,\; x \in S, \tag{25}$$

$$(\widetilde{\mathcal{K}}\,h)(x) := \int_S \big[\,\partial_{n(y)}\,\Gamma(x-y)\,\big]\,h(y)dS_y,\; x \in S, \tag{26}$$

$$(\mathcal{L}h)(x) := \big\{\partial_{n(x)}W(h)(x)\big\}^+,\; x \in S. \tag{27}$$

The boundary operators $\mathcal{H}$ and $\mathcal{L}$ are pseudodifferential operators of order -1 and 1, respectively, while the operators $\mathcal{K}$ and $\widetilde{\mathcal{K}}$ are mutually adjoint weakly singular integral operators.

$\mathcal{H}$ – is a (smoothing) weakly singular integral operator;
$\mathcal{L}$ – is a singular integro-differential operator;

JUMP RELATIONS

$$\{V(g)\}^{+} = \{V(g)\}^{-} = \mathcal{H}\, g \quad \text{on} \quad S \tag{28}$$

$$\{\partial_n V(g)\}^{\pm} = \mp\frac{1}{2}\, g + \mathcal{K} g \equiv [\mp 2^{-1} I + \mathcal{K}]\, g \quad \text{on} \quad S \tag{29}$$

$$\{W(h)\}^{\pm} = \pm\frac{1}{2}\, h + \widetilde{\mathcal{K}} h \equiv [\pm 2^{-1} I + \widetilde{\mathcal{K}}]\, h \quad \text{on} \quad S \tag{30}$$

$$\{\partial_n W(h)\}^{+} = \{\partial_n W(h)\}^{-} = \mathcal{L}\, h \quad \text{on} \quad S \tag{31}$$

$$g \in B_{p,p}^{-\frac{1}{p}}(S), \quad h \in B_{p,p}^{1-\frac{1}{p}}(S), \quad p > 1. \tag{32}$$

MAPPING PROPERTIES OF LAYER POTENTIALS

Let S be C^∞-smooth and $k \in \mathbb{N}$, $0 < \alpha < 1$, $1 < p < \infty$, $1 \le t \le \infty$, $s \in \mathbb{R}$. Then the operators are continuous:

$$
\begin{aligned}
V \quad &: \quad C^{k,\alpha}(S) \to C^{k+1,\alpha}(\overline{\Omega^+}) \\
&: \quad B^{s}_{p,p}(S) \to H_p^{s+1+\frac{1}{p}}(\Omega^+) \\
W \quad &: \quad C^{k,\alpha}(S) \to C^{k,\alpha}(\overline{\Omega^+}) \\
&: \quad B^{s}_{p,p}(S) \to H_p^{s+\frac{1}{p}}(\Omega^+)
\end{aligned}
$$

Let S **be** C^∞**-smooth and** $k \in \mathbb{N}$, $0 < \alpha < 1$, $1 < p < \infty$, $1 \le t \le \infty$, $s \in \mathbb{R}$. **Then**

$$\begin{aligned} \mathcal{H} &: C^{k-1,\alpha}(S) \to C^{k,\alpha}(S), \\ \pm 2^{-1} I + \mathcal{K},\ \pm 2^{-1} I + \widetilde{\mathcal{K}} &: C^{k,\alpha}(S) \to C^{k,\alpha}(S), \\ \mathcal{L} &: C^{k,\alpha}(S) \to C^{k-1,\alpha}(S) \end{aligned}$$

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$$\begin{aligned} \mathcal{H} &: H^s_p(S) \to H^{s+1}_p(S) \qquad \left[B^s_{p,t}(S) \to B^{s+1}_{p,t}(S)\right], \\ \pm 2^{-1} I + \mathcal{K} &: H^s_p(S) \to H^s_p(S) \qquad \left[B^s_{p,t}(S) \to B^s_{p,t}(S)\right], \\ \pm 2^{-1} I + \widetilde{\mathcal{K}} &: H^s_p(S) \to H^s_p(S) \qquad \left[B^s_{p,t}(S) \to B^s_{p,t}(S)\right], \\ \mathcal{L} &: H^{s+1}_p(S) \to H^s_p(S) \qquad \left[B^{s+1}_{p,t}(S) \to B^s_{p,t}(S)\right] \end{aligned}$$

These operators are Fredholm operators with zero index.

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$$\begin{aligned} \mathcal{H} &: C^{k-1,\alpha}(S) \to C^{k,\alpha}(S), \\ \pm 2^{-1} I + \mathcal{K},\ \pm 2^{-1} I + \widetilde{\mathcal{K}} &: C^{k,\alpha}(S) \to C^{k,\alpha}(S), \\ \mathcal{L} &: C^{k,\alpha}(S) \to C^{k-1,\alpha}(S) \end{aligned}$$

$$\begin{aligned} \mathcal{H} &: H_p^s(S) \to H_p^{s+1}(S) && \left[B_{p,t}^s(S) \to B_{p,t}^{s+1}(S)\right], \\ \pm 2^{-1} I + \mathcal{K} &: H_p^s(S) \to H_p^s(S) && \left[B_{p,t}^s(S) \to B_{p,t}^s(S)\right], \\ \pm 2^{-1} I + \widetilde{\mathcal{K}} &: H_p^s(S) \to H_p^s(S) && \left[B_{p,t}^s(S) \to B_{p,t}^s(S)\right], \\ \mathcal{L} &: H_p^{s+1}(S) \to H_p^s(S) && \left[B_{p,t}^{s+1}(S) \to B_{p,t}^s(S)\right] \end{aligned}$$

These operators are Fredholm operators with zero index.

The principal homogeneous symbols of the operators $-\mathcal{H}$ and $\mathcal{L}$ are positive: $\mathfrak{S}_0(\mathcal{H};\xi) = 1/(2|\xi|)$, $\mathfrak{S}_0(\mathcal{L};\xi) = |\xi|/2$, $\xi \in \mathbb{R}^2 \setminus \{0\}$.

Reduction of BVPs to BIEs: INDIRECT METHOD

Indirect BIE method (Potential method) for the Dirichlet problem:

$$\Delta u = 0 \text{ in } \Omega^+, \qquad u \in W_p^1(\Omega^+), \tag{33}$$

$$\{u\}^+ = f \text{ on } S = \partial\Omega^+, \quad f \in B_{p,p}^{1-\frac{1}{p}}(S), \quad p > 1. \tag{34}$$

Look for a solution as a double layer potential,

$$u(x) = W(g)(x), \quad x \in \Omega^+, \tag{35}$$

$$g \in B_{p,p}^{1-\frac{1}{p}}(S)\,. \tag{36}$$

The Dirichlet boundary condition and jump relations for the double layer potential lead to the integral equation for g,

$$\mathcal{D}\, g \equiv (\, 2^{-1}\, I + \widetilde{\mathcal{K}}\,)\, g = f \text{ on } S. \tag{37}$$

For a smooth boundary S, the operator $\widetilde{\mathcal{K}}$ with weakly singular kernel generates a compact operator and therefore the operator

$$\mathcal{D} \equiv 2^{-1} I + \widetilde{\mathcal{K}} \; : \; L_2(S) \to L_2(S) \tag{38}$$

is a Fredholm-Riesz operator with zero index. Therefore the injectivity property implies invertibility of the operator.

For a smooth boundary S, the operator $\widetilde{\mathcal{K}}$ with weakly singular kernel generates a compact operator and therefore the operator

$$\mathcal{D} \equiv 2^{-1} I + \widetilde{\mathcal{K}} \; : \; L_2(S) \to L_2(S) \tag{38}$$

is a Fredholm-Riesz operator with zero index. Therefore the injectivity property implies invertibility of the operator.

Injectivity easily follows from the uniqueness results:

$$(2^{-1} I + \widetilde{\mathcal{K}})g = 0, \quad g \in L_2(S) \tag{39}$$

$$\Rightarrow \quad \text{(bootstrap arguments)} \quad \Rightarrow \quad g \in C^{1,\alpha}(S) \tag{40}$$

$$\Rightarrow \quad W(g) \in C^{1,\alpha}(\overline{\Omega^{\pm}}) \;\; \& \;\; \{W(g)\}_S^{+} = 0 \tag{41}$$

$$\Rightarrow \quad W(g)(x) = 0, \quad x \in \overline{\Omega^{+}} \tag{42}$$

$$\Rightarrow \quad \{\partial_n W(g)\}_S^{+} = \{\partial_n W(g)\}_S^{-} = 0 \tag{43}$$

$$\Rightarrow \quad W(g)(x) = 0, \quad x \in \overline{\Omega^{-}} \tag{44}$$

$$\Rightarrow \quad \{W(g)\}_S^{+} - \{W(g)\}_S^{-} = g = 0 \tag{45}$$

Consequently, the operator (38) is invertible.

Due to the general theory of pseudodifferential operators, it then follows that the operators

$$\mathcal{D} \equiv 2^{-1} I + \widetilde{\mathcal{K}} \; : \; H_p^t(S) \to H_p^t(S) \tag{46}$$

$$: \; B_{p,p}^t(S) \to B_{p,p}^t(S) \tag{47}$$

are invertible as well for arbitrary $t \in \mathbb{R}$, $p > 1$ and $1 \leqslant q \leqslant \infty$.

This leads to the following existence result.

THEOREM 1. The Dirichlet problem (D)$^+$ with arbitrary boundary function $f \in B_{p,p}^{1-\frac{1}{p}}(S)$ is uniquely solvable in the space $W_p^1(\Omega^+)$, $p > 1$, and the solution is representable as a double layer potential

$$u(x) = W(g)(x), \quad x \in \Omega^+, \tag{48}$$

where the density vector function $g \in B_{p,p}^{1-\frac{1}{p}}(S)$ is defined by the uniquely solvable integral equation

$$\mathcal{D}\, g \equiv (\, 2^{-1}\, I + \widetilde{\mathcal{K}}\,)\, g = f \;\text{ on }\; S. \tag{49}$$

THEOREM 1. The Dirichlet problem (D)$^+$ with arbitrary boundary function $f \in B^{1-\frac{1}{p}}_{p,p}(S)$ is uniquely solvable in the space $W^1_p(\Omega^+)$, $p > 1$, and the solution is representable as a double layer potential

$$u(x) = W(g)(x), \quad x \in \Omega^+, \tag{48}$$

where the density vector function $g \in B^{1-\frac{1}{p}}_{p,p}(S)$ is defined by the uniquely solvable integral equation

$$\mathcal{D}\, g \equiv (\, 2^{-1}\, I + \widetilde{\mathcal{K}}\,)\, g = f \ \text{ on } \ S. \tag{49}$$

In addition, if $f \in B^{t-\frac{1}{p}}_{p,p}(S)$ for $t \geqslant 1$ and $p > 1$, then $g \in B^{t-\frac{1}{p}}_{p,p}(S)$ and $u \in B^t_{p,p}(\Omega^+)$.

In particular, $f \in C^\infty(S) \quad \Rightarrow \quad u \in C^\infty(\overline{\Omega^+})$.

Single layer approach: If we look for a solution to the same Dirichlet problem as a single layer potential,

$$u(x) = V(g)(x), \quad x \in \Omega^{+}. \tag{50}$$

we arrive at the integral equation for g,

$$\mathcal{H}\, g = f \ \text{ on } \ S. \tag{51}$$

The operator

$$-\mathcal{H} \ : \ H_2^{-\frac{1}{2}}(S) \to H_2^{\frac{1}{2}}(S) \tag{52}$$

is strongly coercive, i.e. $\langle -\mathcal{H}\varphi \,,\, \varphi\rangle_S \geqslant C\, \|\varphi\|^2_{H_2^{-\frac{1}{2}}(S)}$, implying that the operators

$$\mathcal{H} \ : \ H_p^{t}(S) \to H_p^{t+1}(S) \tag{53}$$

$$: \ B_{p,q}^{t}(S) \to B_{p,q}^{t+1}(S) \tag{54}$$

are invertible for $t \in \mathbb{R}, \quad p > 1, \quad 1 \leqslant q \leqslant \infty$.

Any solution u of the Laplace equation of the class $W_p^1(\Omega^+)$ with $p > 1$ and $t \geqslant 1$ can be uniquely represented as a single layer potential:

$$u(x) = V\big(\mathcal{H}^{-1} f\big)(x) \quad \text{in} \quad \Omega^+ \quad \text{with} \quad f := \{u\}_S^+. \tag{55}$$

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Steklov-Poincaré operator:

$$(55) \quad \Rightarrow \quad \{\partial_n u(x)\}^+ = (-2^{-1}\, I + \mathcal{K})\, \mathcal{H}^{-1}\{u\}^+ \tag{56}$$

$$\mathcal{A} \equiv (-2^{-1}\, I + \mathcal{K})\, \mathcal{H}^{-1} \tag{57}$$

$$\mathcal{A} : B_{p,q}^{t}(S) \to B_{p,q}^{t-1}(S) \tag{58}$$

$$\forall \psi \in H^{\frac{1}{2}}(S) : \ \langle \mathcal{A}\psi \,,\, \psi \rangle_S \geqslant C_1\, \|\psi\|_{H_2^{\frac{1}{2}}(S)} - C_2\, \|\psi\|_{H_2^0(S)} \tag{59}$$

$\mathcal{A}$ is a pseudodifferential operator of order $+1$ with positive principal homogeneous symbol and with index equal to zero.

This operator plays a crucial role in the study of mixed BVPs.

Indirect BIE method (Potential method) for the Neumann problem:

$$\Delta u = 0 \text{ in } \Omega^+, \qquad \{\partial_n u\}^+ = F \text{ on } S = \partial\Omega^+. \tag{60}$$

Look for a solution as a single layer potential,

$$u(x) = V(h)(x), \quad x \in \Omega^+. \tag{61}$$

The Neumann boundary condition and jump relations for the single layer potential lead to the Fredholm-Riesz integral equation for h,

$$\mathcal{N}\, h \equiv (-2^{-1}\, I + \mathcal{K})\, h = F \text{ on } S. \tag{62}$$

$$\text{dim Ker}\,(-2^{-1}\, I + \mathcal{K}) = \text{dim Ker}\,(-2^{-1}\, I + \widetilde{\mathcal{K}}) = 1$$

$$\text{Ker}\,(-2^{-1}\, I + \widetilde{\mathcal{K}}) = \{1\} \quad \text{and} \quad \text{Ker}\,(-2^{-1}\, I + \mathcal{K}) = \{\mathcal{H}^{-1} 1\}$$

Necessary and sufficient condition for solvability of integral equation (62) and of the interior Neumann problem read as:

$$\int_S F(y)\, dS = 0 \text{ on } S. \tag{63}$$

THEOREM 2. The Neumann problem (N)$^+$ with a boundary function $F \in B_{p,p}^{-\frac{1}{p}}(S)$ satisfying the necessary orthogonality condition (63), is solvable in the space $W_p^1(\Omega^+)$, $p > 1$, modulo a constant and solutions are representable in the form of single layer potential

$$u(x) = V(h)(x), \quad x \in \Omega^+, \tag{64}$$

where the density vector function $h \in B_{p,p}^{-\frac{1}{p}}(S)$ is defined modulo the summand $h_0 = const\,(\,\mathcal{H}^{-1}1\,)$ by the integral equation

$$\mathcal{N}\,h \equiv (\,2^{-1}\,I + \mathcal{K}\,)\,h = F \;\; \text{on} \;\; S. \tag{65}$$

If $F \in B_{p,p}^{t-1-\frac{1}{p}}(S)$ for $t \geqslant 1$ and $p > 1$, then $u \in W_p^t(\Omega^+)$.

In particular, $F \in C^\infty(S) \quad \Rightarrow \quad u \in C^\infty(\overline{\Omega^+})$.

Indirect BIE method (Potential method) for the Mixed problem:

$$\Delta u = 0 \ \text{ in } \ \Omega^+, \qquad u \in W_p^1(\Omega^+), \tag{66}$$

$$\{u\}^+ = f^* \ \text{ on } \ S_D, \qquad f^* \in B_{p,p}^{1-\frac{1}{p}}(S), \tag{67}$$

$$\{\partial_n u\}^+ = F^* \ \text{ on } \ S_N, \qquad F^* \in B_{p,p}^{-\frac{1}{p}}(S). \tag{68}$$

Let f_e be some fixed extension of the function f^* from S_D onto the whole of S preserving the space:

$$f_e \in B_{p,p}^{1-\frac{1}{p}}(S), \quad r_{S_D} f_e = f^*. \tag{69}$$

Look for a solution as a single layer potential:

$$u(x) = V\big(\mathcal{H}^{-1}(f_e + \varphi)\big)(x) \quad \text{with} \quad \varphi \in \widetilde{B}_{p,p}^{1-\frac{1}{p}}(S_N)\,, \tag{70}$$

where φ is a sought for function.
Conditions (66) and (67) are satisfied automatically, while the Neumann condition (68) leads to the pseudodifferential equation on S_N:

$$r_{_{S_N}} \mathcal{A} \varphi = Q^* \quad \text{on} \quad S_N, \tag{71}$$

$$\varphi \in \widetilde{B}^{1-\frac{1}{p}}_{p,p}(S_N), \quad Q^* = F^* - r_{_{S_N}} \mathcal{A} f_e \in B^{-\frac{1}{p}}_{p,p}(S_N) \tag{72}$$

where $\mathcal{A} := (-2^{-1} I + \mathcal{K}) \mathcal{H}^{-1}$ is the Steklov-Poincaré operator.

$$r_{S_N} \mathcal{A} \varphi = Q^* \quad \text{on} \quad S_N, \tag{71}$$

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where $\mathcal{A} := (-2^{-1} I + \mathcal{K}) \mathcal{H}^{-1}$ is the Steklov-Poincaré operator.

The following operator is continuous for $s \in \mathbb{R}$, $p > 1$, $1 \leqslant q \leqslant \infty$:

$$r_{S_N} \mathcal{A} \ : \ \widetilde{B}^{s}_{p,q}(S_N) \to B^{s-1}_{p,q}(S_N). \tag{73}$$

$$r_{S_N}\mathcal{A}\varphi = Q^* \text{ on } S_N, \tag{71}$$

$$\varphi \in \widetilde{B}^{1-\frac{1}{p}}_{p,p}(S_N), \quad Q^* = F^* - r_{S_N}\mathcal{A}f_e \in B^{-\frac{1}{p}}_{p,p}(S_N) \tag{72}$$

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$$r_{S_N}\mathcal{A} \; : \; \widetilde{B}^{s}_{p,q}(S_N) \to B^{s-1}_{p,q}(S_N). \tag{73}$$

The operator (73) is invertible if [Vishik-Eskin; Shargorodski]

$$\frac{1}{p} - 1 < s - \frac{1}{2} < \frac{1}{p}. \tag{74}$$

$$r_{S_N}\mathcal{A}\varphi = Q^* \text{ on } S_N, \tag{71}$$

$$\varphi \in \widetilde{B}^{1-\frac{1}{p}}_{p,p}(S_N), \quad Q^* = F^* - r_{S_N}\mathcal{A}f_e \in B^{-\frac{1}{p}}_{p,p}(S_N) \tag{72}$$

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(74) $\Rightarrow$ $r_{S_N}\mathcal{A} : \widetilde{B}^{1-\frac{1}{p}}_{p,p}(S_N) \to B^{-\frac{1}{p}}_{p,p}(S_N)$ is invertible and (71) is uniquely solvable if

$$\frac{4}{3} < p < 4\,. \tag{75}$$

Pseudodifferential operators on manifolds with boundary:

[Vishik-Eskin; Shargorodski]

THEOREM Let $\overline{S}_1 \in C^\infty$ be a compact, 2-dimensional, non–self–intersecting, two–sided surface with boundary $\partial S_1 \in C^\infty$, and $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$. Further, let $\mathcal{B}$ be a pseudo–differential operator of order $\alpha \in \mathbb{R}$ on S_1 having a uniformly positive principal homogeneous symbol, i.e., $\mathfrak{S}(\mathcal{B}; y, \xi) \geq c_0 > 0$ for $y \in \overline{S}_1$, $\xi \in \mathbb{R}^2$ with $|\xi| = 1$, where c_0 is a constant.

Then the operators

$$\mathcal{B}: \quad \widetilde{H}^s_p(S_1) \to H^{s-\alpha}_p(S_1) \qquad \left[\, \widetilde{B}^s_{p,q}(S_1) \to B^{s-\alpha}_{p,q}(S_1) \,\right] \tag{76}$$

are Fredholm operators of index zero if

$$1/p - 1 < s - \alpha/2 < 1/p. \tag{77}$$

Moreover, the null-spaces of operators (76) are the same (for all values of the parameters $q \in [1, +\infty]$ and p, s) provided p and s satisfy inequality (77).

EXISTENCE THEOREM. Let $4/3 < p < 4$, and $f^* \in B_{p,p}^{1-\frac{1}{p}}(S_D)$, $F^* \in B_{p,p}^{-\frac{1}{p}}(S_N)$. The mixed problem $(\mathrm{M})^+$ is uniquely solvable in the space $W_p^1(\Omega^+)$ and the solution is representable as a single potential

$$u(x) = V\big(\mathcal{H}^{-1}(f_e + \varphi)\big)(x), \tag{89}$$

where $f_e \in \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^3$ is some fixed extension of the function f^* from S_D onto the whole of S, while $\varphi \in \widetilde{B}_{p,p}^{1-\frac{1}{p}}(S_N)$ is defined by the uniquely solvable elliptic pseudodifferential equation

$$r_{_{S_N}} \mathcal{A} \varphi = Q^* \ \text{ on } \ S_N, \tag{90}$$

where $\mathcal{A}$ is the Steklov-Poncaré operator and $Q^* = F^* - r_{_{S_N}} \mathcal{A} f_e$.

EXISTENCE THEOREM. Let $4/3 < p < 4$, and $f^* \in B_{p,p}^{1-\frac{1}{p}}(S_D)$, $F^* \in B_{p,p}^{-\frac{1}{p}}(S_N)$. The mixed problem $(\mathrm{M})^+$ is uniquely solvable in the space $W_p^1(\Omega^+)$ and the solution is representable as a single potential

$$u(x) = V\big(\mathcal{H}^{-1}(f_e + \varphi)\big)(x), \qquad (89)$$

where $f_e \in \big[B_{p,p}^{1-\frac{1}{p}}(S)\big]^3$ is some fixed extension of the function f^* from S_D onto the whole of S, while $\varphi \in \widetilde{B}_{p,p}^{1-\frac{1}{p}}(S_N)$ is defined by the uniquely solvable elliptic pseudodifferential equation

$$r_{S_N}\mathcal{A}\varphi = Q^* \ \text{ on } \ S_N, \qquad (90)$$

where $\mathcal{A}$ is the Steklov-Poncaré operator and $Q^* = F^* - r_{S_N}\mathcal{A}f_e$.

If, in addition,

$$f^* \in C^\alpha(\overline{S_D}), \quad F^* \in C^\alpha(\overline{S_N}), \quad \alpha > 0, \qquad \mathbf{(78)}$$

then $U \in \bigcap\limits_{\beta<\kappa} [\,C^\beta(\,\overline{\Omega^+}\,)\,]^6$, with $\kappa = \min\{\alpha, \frac{1}{2}\} > 0$.

DIRECT METHOD FOR THE DIRICHLET PROBLEM

$$\Delta u = \Phi \quad \text{in} \quad \Omega^+, \qquad \Phi \in L_2(\Omega^+) \quad \Rightarrow \quad u \in H_2^{1,0}(\Omega^+, \Delta), \quad \text{(D1)}$$

$$\{u\}_S^+ = f \quad \text{on} \quad S, \qquad f \in H_2^{\frac{1}{2}}(S). \qquad \text{(D2)}$$

Green's third formula in Ω^+ and its trace on S:

$$u = N_{\Omega^+}(\Delta u) + W(\{u\}^+) - V(\{\partial_n u\}^+) \quad \text{in} \quad \Omega^+, \qquad \text{(G3)}$$

$$[-2^{-1}I + \widetilde{\mathcal{K}}]\{u\}^+ - \mathcal{H}\{\partial_n u\}^+ = -\{N_{\Omega^+}(\Delta u)\}_S^+ \quad \text{on} \quad S. \quad \text{(GB)}$$

Substitute here the data of the Dirichlet problem

$$u + V(\psi) = N_{\Omega^+}(\Phi) + W(f) \quad \text{in} \quad \Omega^+, \qquad \text{(E1)}$$

$$\mathcal{H}\psi = \{N_{\Omega^+}(\Phi)\}_S^+ + [-2^{-1}I + \widetilde{\mathcal{K}}]f \quad \text{on} \quad S, \qquad \text{(E2)}$$

where $\psi = \{\partial_n u\}^+$.

Consider (E1)-(E2) as a system with respect to segregated pair of unknowns $(u, \psi) \in H_2^1(\Omega^+) \times H_2^{-\frac{1}{2}}(S)$.

$$\begin{cases} \boldsymbol{u} + \boldsymbol{V}(\psi) = \boldsymbol{N}_{\Omega^+}(\boldsymbol{\Phi}) + \boldsymbol{W}(\boldsymbol{f}) \quad \text{in} \quad \boldsymbol{\Omega}^+, & \textbf{(E1)} \\ \mathcal{H}\psi = \{\boldsymbol{N}_{\Omega^+}(\boldsymbol{\Phi})\}_S^+ + [\,-\mathbf{2}^{-1}\,\boldsymbol{I} + \widetilde{\mathcal{K}}\,]\boldsymbol{f} \quad \text{on} \quad \boldsymbol{S}. & \textbf{(E2)} \end{cases}$$

$$\begin{cases} u + V(\psi) = N_{\Omega^+}(\Phi) + W(f) \ \text{ in } \ \Omega^+, & \text{(E1)} \\ \mathcal{H}\psi = \{N_{\Omega^+}(\Phi)\}_S^+ + [\,-2^{-1}\,I + \widetilde{\mathcal{K}}\,]f \quad \text{on } \ S. & \text{(E2)} \end{cases}$$

The Dirichlet problem is equivalent to system (E1)-(E2):

(i) If $u \in H_2^{1,0}(\Omega^+, \Delta)$ solves the Dirichlet problem, then the pair (u, ψ) with $\psi = \{\partial_n u\}^+ \in H_2^{-\frac{1}{2}}(S)$ solves system (E1)-(E2);

(ii) If a pair $(u, \psi) \in H_2^{1,0}(\Omega^+, \Delta) \times H_2^{-\frac{1}{2}}(S)$ solves system (E1)-(E2), then $\psi = \{\partial_n u\}^+$ and u solves the Dirichlet problem.

$$\begin{cases} u + V(\psi) = N_{\Omega^+}(\Phi) + W(f) \ \text{ in } \ \Omega^+, & \text{(E1)} \\ \mathcal{H}\psi = \{N_{\Omega^+}(\Phi)\}_S^+ + [\,-2^{-1}\, I + \widetilde{\mathcal{K}}\,]f \quad \text{on } \ S. & \text{(E2)} \end{cases}$$

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Proof follows from the properties of potentials:

$$\text{(i)} \quad u = N_{\Omega^+}(\Delta u) + W(\{u\}^+) - V(\{\partial_n u\}^+) \quad \text{in } \ \Omega^+; \qquad \textbf{(G3)}$$

$$\begin{cases} u + V(\psi) = N_{\Omega^+}(\Phi) + W(f) \ \text{in} \ \Omega^+, & \text{(E1)} \\ \mathcal{H}\psi = \{N_{\Omega^+}(\Phi)\}^+_S + [-2^{-1} I + \widetilde{\mathcal{K}}]f \quad \text{on} \ S. & \text{(E2)} \end{cases}$$

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(i) If $u \in H_2^{1,0}(\Omega^+, \Delta)$ solves the Dirichlet problem, then the pair (u, ψ) with $\psi = \{\partial_n u\}^+ \in H_2^{-\frac{1}{2}}(S)$ solves system (E1)-(E2);

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Proof follows from the properties of potentials:

(i) $u = N_{\Omega^+}(\Delta u) + W(\{u\}^+) - V(\{\partial_n u\}^+)$ in Ω^+; **(G3)**

(ii) **(E1)** $\Rightarrow$ $\Delta u = \Phi$ with $\Phi \in L_2(\Omega^+)$;

(E1)$^+$−(E2) $\Rightarrow$ $\{u\}^+ = f$ on S;

(G3)−(E1) $\Rightarrow$ $V(\{\partial_n u\}^+ - \psi) = 0$ in Ω^+ $\Rightarrow$ $\psi = \{\partial_n u\}^+$ on S.

Introduce the operator generated by the left hand side expressions of system (E1)-(E2):

$$\mathfrak{D} := \begin{bmatrix} I & V \\ 0 & \mathcal{H} \end{bmatrix}_{2\times 2} \qquad (79)$$

We have just shown that the operator

$$\mathfrak{D} : H_2^{1,0}(\Omega^+, \Delta) \times H_2^{-\frac{1}{2}}(S) \to$$
$$\to H_2^{1,0}(\Omega^+, \Delta) \times H_2^{\frac{1}{2}}(S) \qquad (80)$$

is invertible.

It is easy to see that the following operators are invertible as well

$$\mathfrak{D} : H_2^{1}(\Omega^+) \times H_2^{-\frac{1}{2}}(S) \to H_2^{1}(\Omega^+) \times H_2^{\frac{1}{2}}(S) \qquad (81)$$

$$\mathfrak{D} : H_2^{r}(\Omega^+) \times H_2^{r-\frac{1}{2}}(S) \to H_2^{r}(\Omega^+) \times H_2^{r+\frac{1}{2}}(S) \qquad (82)$$

THANK YOU!