LECTURE 2

SCALAR BVPs WITH ONE VARIABLE COEFFICIENT (ISOTROPIC CASE)

DAVID NATROSHVILI

Georgian Technical University

Tbilisi, GEORGIA

1. **Formulation of mixed type BVPs**
2. **Global parametrix and parametrix based potentials**
3. **Greens formulas and parametrix based integral representation**
4. **Reduction to boundary-domain integral equations (BDIE)**
5. **Equivalence theorems. Two basic lemmas**
6. **Investigation of the BDIE systems for mixed problem**

O. Chkadua, S. Mikhailov, and D. Natroshvili, Analysis of direct boundary-domain integral equations for a mixed BVP with variable coefficient, Part I: Equivalence and invertibility. J. Integral Equations Appl. 21(2009), No. 4, 499–542.

O. Chkadua, S. Mikhailov, and D. Natroshvili, Analysis of direct boundary-domain integral equations for a mixed BVP with variable coefficient, Part II: Solution regularity and asymptotics. J. Integral Equations Appl. 22(2010), No. 1, 19–37.

FORMULATION OF THE MIXED BOUNDARY VALUE PROBLEM

$\Omega = \Omega^+$ - a bounded domain in $\mathbb{R}^3$ with a simply connected boundary $\partial\Omega^+ = S$;

$\overline{\Omega^+} = \Omega^+ \cup S; \qquad \Omega^- := \mathbb{R}^3 \setminus \overline{\Omega^+};$

For simplicity we assume: $S \in C^\infty$;

Dissection of the boundary surface $S = \overline{S}_D \cup \overline{S}_N$, $S_D \cap S_N = \varnothing$,

$\ell = \overline{S}_D \cap \overline{S}_N \in C^\infty$;

The symbols $\{u\}_S^\pm \equiv [u]^\pm \equiv u^\pm$ denote one-sided limits (traces) on S from $\Omega^\pm$;

$n = (n_1, n_2, n_3)$ - outward unit normal vector to S;

$W_2^r = W^r = H_2^r$ and $H_2^s = H^s$ are L_2 based Sobolev–Slobodetskii and Bessel potential function spaces $(r \geq 0, \quad s \in \mathbb{R})$;

Introduce also the spaces:

$$\widetilde{H}^s(S_1) := \big\{ f : \ f \in H^s(S), \ \mathrm{supp}\, f \subset \overline{S_1} \big\},$$

$$H^s(S_1) := \big\{ r_{_{S_1}} f : \ f \in H^s(S) \big\},$$

where S_1 is an open proper submanifold of S;

Let $a \in C^\infty(\mathbb{R}^3)$, $a(x) > 0$ for $x \in \mathbb{R}^3$, and consider the following scalar elliptic differential equation ("isotropic case")

$$A(x, \partial_x)u(x) := \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\Big(a(x)\,\frac{\partial u(x)}{\partial x_i}\Big) = f(x), \quad x \in \Omega, \qquad (1)$$

where u is an unknown function and f is a given function in Ω.

When $a = 1$, the operator $A(x, \partial_x)$ is the Laplace operator Δ.

Let $a \in C^\infty(\mathbb{R}^3)$, $a(x) > 0$ for $x \in \mathbb{R}^3$, and consider the following scalar elliptic differential equation ("isotropic case")

$$A(x, \partial_x)u(x) := \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\Big(a(x)\,\frac{\partial u(x)}{\partial x_i}\Big) = f(x), \quad x \in \Omega, \qquad (2)$$

where u is an unknown function and f is a given function in Ω.

When $a = 1$, the operator $A(x, \partial_x)$ is the Laplace operator Δ.

Introduce the space

$$H^{1,0}(\Omega; A) := \{v \in H^1(\Omega) \;:\; A\,v \in L_2(\Omega)\}$$

with the graph norm $\|v\|^2_{H^{1,0}(\Omega;A)} := \|v\|^2_{H^1(\Omega)} + \|Av\|^2_{L_2(\Omega)}$.

The co–normal derivative operator on S for $u \in H^s(\Omega)$, $s > \frac{3}{2}$,

$$T^{\pm} u(x) \equiv [T\, u(x)]^{\pm}_S := a(x)\, n_i(x) \{\partial_i u(x)\}^{\pm} =$$

$$= a(x) \left\{ \frac{\partial u(x)}{\partial n} \right\}^{\pm} = a(x) \{\partial_n u(x)\}^{\pm}, \quad x \in S. \qquad \textbf{(3)}$$

The co–normal derivative operator on S for $u \in H^s(\Omega)$, $s > \frac{3}{2}$,

$$T^{\pm}\, u(x) \equiv [T\, u(x)]_S^{\pm} := a(x)\, n_i(x) \{\partial_i u(x)\}^{\pm} =$$

$$= a(x) \left\{ \frac{\partial u(x)}{\partial n} \right\}^{\pm} = a(x) \left\{\partial_n u(x)\right\}^{\pm}, \quad x \in S. \qquad \text{(4)}$$

For $u \in H^{1,0}(\Omega; A)$ the generalized trace of co–normal derivative $T^+ u \in H^{-\frac{1}{2}}(S)$ is correctly defined by Green's first formula,

$$\left\langle T^+ u\,,\, w^+ \right\rangle_S := \int_\Omega \left[\, Au\; w + E(u, w) \,\right] dx \quad \forall\; w \in H^1(\Omega), \qquad \text{(5)}$$

where

$$E(u, w) := a(x)\, \partial_i u(x)\, \partial_i w(x),$$

and $\langle \ \cdot \ , \ \cdot \ \rangle_S$ denotes the duality brackets between the spaces $H^{-\frac{1}{2}}(S)$ and $H^{\frac{1}{2}}(S)$ which extends the usual $L_2(S)$ scalar product.

THE MIXED BVP: Find a function $u \in H^{1,0}(\Omega; A) = W^{1,0}(\Omega; A)$ satisfying the conditions

$$A(x, \partial_x)\, u = f \quad \text{in} \quad \Omega^+, \tag{6}$$

$$r_{S_D}\, u^+ = \varphi_0 \quad \text{on} \quad S_D, \tag{7}$$

$$r_{S_N}\, T^+ u = \psi_0 \quad \text{on} \quad S_N, \tag{8}$$

where $\varphi_0 \in H^{\frac{1}{2}}(S_D), \quad \psi_0 \in H^{-\frac{1}{2}}(S_N)$ and $f \in L_2(\Omega)$.

Equation (6) is understood in the distributional sense, condition (7) is understood in the trace sense, while equality (8) is understood in the generalized functional sense.

THE MIXED BVP: Find a function $u \in H^{1,0}(\Omega; A) = W^{1,0}(\Omega; A)$ satisfying the conditions

$$A(x, \partial_x)\, u = f \quad \text{in} \quad \Omega^+, \tag{6}$$

$$r_{S_D}\, u^+ = \varphi_0 \quad \text{on} \quad S_D, \tag{7}$$

$$r_{S_N}\, T^+ u = \psi_0 \quad \text{on} \quad S_N, \tag{8}$$

where $\varphi_0 \in H^{\frac{1}{2}}(S_D), \quad \psi_0 \in H^{-\frac{1}{2}}(S_N)$ and $f \in L_2(\Omega)$.

Equation (6) is understood in the distributional sense, condition (7) is understood in the trace sense, while equality (8) is understood in the generalized functional sense.

THEOREM 1. The homogeneous version of BVP (6)-(8), i.e. with $f = 0$, $\varphi_0 = 0$, $\psi_0 = 0$, has only the trivial solution.

PARAMETRIX BASED INTEGRAL REPRESENTATION

A function $P(x,y)$ of two variables $x, y \in \mathbb{R}^3$ is a parametrix for the operator $A(x, \partial_x)$ if

$$A(x,\partial_x)\, P(x,y) = \delta(x-y) + R(x,y), \tag{9}$$

where $\delta(\cdot)$ is the Dirac distribution and $R(x,y)$ possesses a weak singularity at $x = y$, i.e., $R(x,y) = \mathcal{O}\left(|x-y|^{-\varkappa}\right)$ with $\varkappa < 3$.

For the operator $A(x,\partial_x)$, the function

$$P(x,y) = -\frac{1}{4\pi\, a(y)\, |x-y|}\,, \quad x, y \in \mathbb{R}^3, \tag{10}$$

is a parametrix (Levi function), with the remainder

$$R(x,y) = \sum_{i=1}^{3} \frac{x_i - y_i}{4\pi\, a(y)\, |x-y|^3}\, \frac{\partial a(x)}{\partial x_i} = \mathcal{O}\left(|x-y|^{-2}\right). \tag{11}$$

PARAMETRIX BASED INTEGRAL REPRESENTATION

Green's identity for the operator $A(x, \partial_x)$ and $u, v \in H^{1,0}(\Omega^+; A)$,

$$\int_{\Omega^+} \left[\, v\, A(x, \partial_x)u - u\, A(x, \partial_x)v \,\right] dx =$$

$$= \left\langle\, T^+u\,,\, v^+ \right\rangle_S - \left\langle\, T^+v\,,\, u^+ \,\right\rangle_S\,. \qquad \textbf{(G2)}$$

PARAMETRIX BASED INTEGRAL REPRESENTATION

Green's identity for the operator $A(x, \partial_x)$ and $u, v \in H^{1,0}(\Omega^+; A)$,

$$\int_{\Omega^+} \left[v\, A(x, \partial_x)u - u\, A(x, \partial_x)v \right] dx =$$

$$= \left\langle T^+ u \,,\, v^+ \right\rangle_s - \left\langle T^+ v \,,\, u^+ \right\rangle_s . \qquad \text{(G2)}$$

Substitution $v(x) := P(x, y)$ in Green's second identity for the domain $\Omega^+ \setminus \overline{B(y, \varepsilon)}$ and the standard limiting procedures leads to Green's third identity,

$$u + \mathcal{R}u - VT^+u + Wu^+ = \mathcal{P}Au \quad \text{in} \quad \Omega^+, \qquad \text{(G3)}$$

where V, W, $\mathcal{P}$ and $\mathcal{R}$ are parametrix based single, double, volume and remainder operators:

PARAMETRIX BASED VOLUME AND SURFACE POTENTIALS

$$Vg(y) := -\int_S P(x,y)\,g(x)\,dS_x =$$

$$= \frac{1}{4\pi a(y)} \int_S \frac{1}{|x-y|}\,g(x)\,dS_x = \frac{1}{a(y)}\,V_\Delta g(y), \tag{12}$$

$$Wg(y) := -\int_S \left[\,T(x, n(x), \partial_x)P(x,y)\,\right]\; g(x)dS_x, \tag{13}$$

$$\mathcal{P}g(y) := \int_\Omega P(x,y)\,g(x)\,dx =$$

$$= -\frac{1}{4\pi a(y)} \int_\Omega \frac{1}{|x-y|}\,g(x)\,dx = \frac{1}{a(y)}\,\mathcal{P}_\Delta g(y), \tag{14}$$

$$\mathcal{R}g(y) := \int_{\Omega^+} R(x,y)\,g(x)\,dx. \tag{15}$$

where $T(x, n(x), \partial) = a(x)\partial_n(x)$.

Introduce also the boundary integral operators on S generated by the single and double layer potentials:

$$\mathcal{V}\, g(y) := -\int\limits_{S} P(x,y)\, g(x)\, dS_x, \tag{16}$$

$$\mathcal{W}'\, g(y) := -\int\limits_{S} \big[\, T(y, n(y), \partial_y)\, P(x,y)\, \big]\, g(x)\, dS_x, \tag{17}$$

$$\mathcal{W}\, g(y) := -\int\limits_{S} \big[\, T(x, n(x), \partial_x)\, P(x,y)\, \big]\; g(x)\, dS_x, \tag{18}$$

$$\mathcal{L}^{\pm} g(y) := T^{\pm} W g(y), \tag{19}$$

where $y \in S$.

Due to the structure of the parametrix

$$P(x,y) = -\frac{1}{4\pi\, a(y)\, |x-y|}\,,$$

the parametrix-based potentials and classical harmonic and volume potentials are related by the equations:

$$\mathcal{P}g = a^{-1}\,\mathcal{P}_{\Delta}\, g, \qquad \mathcal{R}g = -a^{-1}\partial_j\big[\mathcal{P}_{\Delta}\big(g\partial_j a\big)\big], \tag{20}$$

$$Vg = a^{-1}V_{\Delta}g, \qquad Wg = a^{-1}W_{\Delta}(ag), \tag{21}$$

$$\mathcal{V}g = a^{-1}\mathcal{V}_{\Delta}g, \qquad \mathcal{W}g = a^{-1}\mathcal{W}_{\Delta}(ag), \tag{22}$$

$$\mathcal{W}'g = \mathcal{W}'_{\Delta}g - \big[a^{-1}\partial_n a\big]\,\mathcal{V}_{\Delta}g, \tag{23}$$

$$\mathcal{L}^{\pm}g = \mathcal{L}_{\Delta}(ag) - \big[a^{-1}\partial_n a\big]\,W^{\pm}_{\Delta}(ag) \tag{24}$$

where the subscript Δ means that the corresponding surface and volume potentials are constructed by means of the harmonic fundamental solution $P_{\Delta}(x-y) = \Gamma(x-y) = -(4\,\pi\,|x-y|)^{-1}$.

Due to the relations

$$\mathcal{P}g = a^{-1}\,\mathcal{P}_\Delta\, g, \qquad \mathcal{R}g = -a^{-1}\partial_j\big[\mathcal{P}_\Delta\big(g\partial_j a\big)\big] \tag{25}$$

and the mapping properties of the classical Newtonian potential $\mathcal{P}_\Delta$ the following parametrix based volume and remainder operators are continuous

$$\mathcal{P} \;:\; \widetilde{H}^s(\Omega^+) \to H^{s+2}(\Omega^+), \quad s \in \mathbb{R}\,, \tag{26}$$

$$:\; H^s(\Omega^+) \to H^{s+2}(\Omega^+), \quad s > -\frac{1}{2}\,; \tag{27}$$

$$\mathcal{R} \;:\; \widetilde{H}^s(\Omega^+) \to H^{s+1}(\Omega^+), \quad s \in \mathbb{R}, \tag{28}$$

$$:\; H^s(\Omega^+) \to H^{s+1}(\Omega^+), \quad s > -\frac{1}{2}\,; \tag{29}$$

$\mathcal{P}$ is a pseudodifferential operator of order -2,

$\mathcal{R}$ is a pseudodifferential operator of order -1.

THEOREM 2. **Let $s \in \mathbb{R}$. The following operators are continuous**

$$\begin{aligned} V &: H^s(S) \to H^{s+\frac{3}{2}}(\Omega^+), \\ W &: H^s(S) \to H^{s+\frac{1}{2}}(\Omega^+). \end{aligned}$$

THEOREM 2. Let $s \in \mathbb{R}$. The following operators are continuous

$$\begin{aligned} V \quad &: \quad H^s(S) \to H^{s+\frac{3}{2}}(\Omega^+)\,, \\ W \quad &: \quad H^s(S) \to H^{s+\frac{1}{2}}(\Omega^+). \end{aligned}$$

THEOREM 3. Let $s \in \mathbb{R}$. The following pseudodifferential operators are continuous

$$\begin{aligned} \mathcal{V} \quad &: \quad H^s(S) \to H^{s+1}(S), \\ \mathcal{W},\ \mathcal{W}' \quad &: \quad H^s(S) \to H^{s+1}(S), \\ \mathcal{L}^{\pm} \quad &: \quad H^s(S) \to H^{s-1}(S). \end{aligned}$$

THEOREM 2. **Let $s \in \mathbb{R}$. The following operators are continuous**

$$\begin{aligned} V &: H^s(S) \to H^{s+\frac{3}{2}}(\Omega^+)\,, \\ W &: H^s(S) \to H^{s+\frac{1}{2}}(\Omega^+). \end{aligned}$$

THEOREM 3. **Let $s \in \mathbb{R}$. The following pseudodifferential operators are continuous**

$$\begin{aligned} \mathcal{V} &: H^s(S) \to H^{s+1}(S), \\ \mathcal{W},\ \mathcal{W}' &: H^s(S) \to H^{s+1}(S), \\ \mathcal{L}^{\pm} &: H^s(S) \to H^{s-1}(S). \end{aligned}$$

THEOREM 4. **Let $g_1 \in H^{-\frac{1}{2}}(S)$, and $g_2 \in H^{\frac{1}{2}}(S)$. Then the following jump relations hold on S,**

$$[Vg_1(y)]^{\pm} = \mathcal{V}g_1(y), \quad y \in S, \tag{30}$$

$$T^{\pm}Vg_1(y) = \pm 2^{-1}g_1(y) + \mathcal{W}'g_1(y), \quad y \in S, \tag{31}$$

$$[Wg_2(y)]^{\pm} = \mp 2^{-1}g_2(y) + \mathcal{W}g_2(y), \quad y \in S. \tag{32}$$

Fredholm properties of ΨDO on manifolds with boundary

[Vishik-Eskin]

Let $\overline{S}_1 \in C^\infty$ be a compact, 2-dimensional, non–self–intersecting, two–sided surface with boundary ∂S_1. Further, let $\mathcal{B}$ be a pseudodifferential operator of order $\alpha \in \mathbb{R}$ on S_1 having a uniformly positive principal homogeneous symbol, i.e., $\mathfrak{S}_0(\mathcal{B}; y, \xi) \geq c_0 > 0$ for $y \in \overline{S}_1$, $\xi \in \mathbb{R}^2$ with $|\xi| = 1$, where c_0 is a constant.

Then the operator

$$\mathcal{B} \ : \ \widetilde{H}^t(S_1) \to H^{t-\alpha}(S_1) \tag{33}$$

is Fredholm operator of index zero if

$$-1/2 < t - \alpha/2 < 1/2. \tag{34}$$

THEOREM 2. **Let S_1 be a nonempty, simply connected sub–manifold of S with infinitely smooth boundary curve, and $0 < s < 1$. Then the operator**

$$r_{S_1}\,\mathcal{V} \quad : \quad \widetilde{H}^{s-1}(S_1) \to H^s(S_1) \tag{35}$$

is invertible.

THEOREM 3. Let S_1 and $S\setminus\overline{S}_1$ be nonempty, open, simply connected sub–manifolds of S with an infinitely smooth boundary curve, and $0 < s < 1$. Then

$$\mathcal{L}^+ g = \widehat{\mathcal{L}} g + (\partial_n \ln a)\left(-2^{-1} I + \mathcal{W}\right) \quad \text{on} \quad S, \tag{36}$$

where

$$\widehat{\mathcal{L}} g := \mathcal{L}^+_{\triangle}(ag) = \mathcal{L}^-_{\triangle}(ag) \tag{37}$$

and $\widehat{\mathcal{L}}$ is the principal singular part of the operator $\mathcal{L}^+$.

The pseudodifferential operator

$$r_{S_1} \widehat{\mathcal{L}} \equiv \mathcal{L}^+_{\triangle}(a\,\bullet) \quad : \quad \widetilde{H}^s(S_1) \to H^{s-1}(S_1) \tag{38}$$

is invertible.

The operator

$$r_{S_1}(\mathcal{L}^+ - \widehat{\mathcal{L}}) \equiv (\partial_n \ln a)\left(-2^{-1} I + \mathcal{W}\right) : \widetilde{H}^s(S_1) \to H^{s-1}(S_1) \tag{39}$$

is compact.

TWO AUXILIARY LEMMAS

Return to the parametric based Green's third identity,

$$u + \mathcal{R}u - VT^{+}u + Wu^{+} = \mathcal{P}Lu \quad \text{in} \quad \Omega^{+} \qquad \text{(G3)}$$

and consider its counterpart equation for some functions f, Ψ, Φ:

$$u + \mathcal{R}u - V\Psi + W\Phi = \mathcal{P}f \quad \text{in} \quad \Omega^{+}. \qquad (40)$$

AUXILIARY LEMMA 1. Let

$$f \in L_2(\Omega^{+}), \quad \Psi \in H^{-\frac{1}{2}}(S), \quad \Phi \in H^{\frac{1}{2}}(S), \qquad (41)$$

and let $u \in H^1(\Omega^{+})$ solve integral equation (40).

Then $u \in H^{1,0}(\Omega^{+}; A)$, $Au = f$ in Ω^{+}, and

$$V(\Psi - T^{+}u)(y) - W(\Phi - u^{+})(y) = 0, \qquad y \in \Omega^{+}. \qquad (42)$$

AUXILIARY LEMMA 2. Let $S = \bar{S}_1 \cup \bar{S}_2$, where $S_1 \cap S_2 = \varnothing$, and

$$\Psi \in \widetilde{H}^{-\frac{1}{2}}(S_1), \quad \Phi \in \widetilde{H}^{\frac{1}{2}}(S_2). \tag{43}$$

If

$$V\Psi(y) - W\Phi(y) = 0, \qquad y \in \Omega^+, \tag{44}$$

then $\Psi = 0$ and $\Phi = 0$ on S.

Proof of AUXILIARY LEMMA 1. First of all let us prove that

$$u \in H^{1,0}(\Omega^+; A).$$

Indeed, since

$$Au = \Delta(au) - \sum \partial_i (u \partial_i a),$$

and the last term belongs to $L_2(\Omega^+)$, we need only to show that $\Delta(au) \in L_2(\Omega^+)$.

We have

$$au = a\mathcal{P}f - a\mathcal{R}u + aV\Psi - aW\Phi = \mathcal{P}_\Delta f - a\mathcal{R}u + V_\Delta \Psi - W_\Delta(a\Phi).$$

Note that the last two terms in the right-hand side are harmonic functions, $\mathcal{R}u \in H^2(\Omega)$ for $u \in H^1(\Omega)$ and $\Delta[\mathcal{P}_\Delta(f)] = f \in L_2(\Omega^+)$. Therefore $Au \in L_2(\Omega^+)$. So, $u \in H^{1,0}(\Omega^+; A)$ and we can write Green's third identity. Thus we have two equations:

$$u + \mathcal{R}u - VT^+u + Wu^+ = \mathcal{P}Au \quad \text{in} \quad \Omega^+, \tag{45}$$

$$u + \mathcal{R}u - V\Psi + W\Phi = \mathcal{P}f \quad \text{in} \quad \Omega^+, \tag{46}$$

implying

$$-V\Psi^* + W\Phi^* = \mathcal{P}[Au - f] \quad \text{in} \quad \Omega^+, \tag{47}$$

where $\Psi^* := T^+u - \Psi$, $\Phi^* := u^+ - \Phi$. Multiplying equality (47) by $a(y)$ we get

$$-V_\Delta\Psi^* + W_\Delta(a\Phi^*) = \mathcal{P}_\Delta[Au - f] \quad \text{in} \quad \Omega^+. \tag{48}$$

which implies equations $Au - f = 0$ and (42). ■

Proof of AUXILIARY LEMMA 2. The items (i) and (ii) are trivial. The item(iii) is equivalent to the equation (by multiplying equation (44) by $a(y)$)

$$V_\Delta \Psi - W_\Delta (a\Phi) = 0 \quad \text{in} \quad \Omega^+.$$

Take the traces of this equation and its normal derivative on S_1 and S_2, respectively, to obtain

$$\begin{cases} r_{S_1} \mathcal{V}_\Delta \Psi - r_{S_1} \mathcal{W}_\Delta \hat{\Phi} = 0 & \text{on} \quad S_1, \\ r_{S_2} \mathcal{W}'_\Delta \Psi - r_{S_2} \mathcal{L}^+_\Delta \hat{\Phi} = 0 & \text{on} \quad S_2, \end{cases} \tag{49}$$

where $\hat{\Phi} = a\Phi$. We put

$$\mathcal{K} := \begin{bmatrix} r_{S_1} \mathcal{V}_\Delta & -r_{S_1} \mathcal{W}_\Delta \\ r_{S_2} \mathcal{W}'_\Delta & -r_{S_2} \mathcal{L}^+_\Delta \end{bmatrix}, \qquad X = \begin{bmatrix} \Psi \\ \hat{\Phi} \end{bmatrix}.$$

Equation (49) then can be written as

$$\mathcal{K} X = 0. \tag{50}$$

The operators

$$r_{_{S_1}}\mathcal{V}_\Delta \;:\; \widetilde{H}^{-\frac{1}{2}}(S_1) \to H^{\frac{1}{2}}(S_1), \quad -r_{_{S_2}}\mathcal{L}^+_\Delta \;:\; \widetilde{H}^{\frac{1}{2}}(S_2) \to H^{-\frac{1}{2}}(S_2)$$

are positive definite in the following sense,

$$\left\langle r_{_{S_1}}\mathcal{V}_\Delta\Psi\,,\,\Psi\right\rangle_{S_1} \geq c\,\|\Psi\|^2_{H^{-\frac{1}{2}}(S)} \tag{51}$$

$$\left\langle -r_{_{S_2}}\mathcal{L}^+_\Delta\hat{\Phi}\,,\,\hat{\Phi}\right\rangle_{S_2} \geq c\,\|\hat{\Phi}\|^2_{H^{\frac{1}{2}}(S)} \tag{52}$$

for arbitrary $\Psi \in \widetilde{H}^{-\frac{1}{2}}(S_1)$ and arbitrary $\hat{\Phi} \in \widetilde{H}^{\frac{1}{2}}(S_2)$.

In addition, the operators

$$r_{_{S_1}}\mathcal{W}_\Delta : \widetilde{H}^{\frac{1}{2}}(S_2) \to H^{\frac{1}{2}}(S_1), \quad r_{_{S_2}}\mathcal{W}'_\Delta : \widetilde{H}^{-\frac{1}{2}}(S_1) \to H^{-\frac{1}{2}}(S_2)$$

are mutually adjoint, i.e., $\left\langle r_{_{S_1}}\mathcal{W}_\Delta\hat{\Phi}\,,\,\Psi\right\rangle_{S_1} = \left\langle \hat{\Phi}\,,\,r_{_{S_2}}\mathcal{W}'_\Delta\Psi\right\rangle_{S_2}$

for arbitrary $\Psi \in \widetilde{H}^{-\frac{1}{2}}(S_1)$ and arbitrary $\hat{\Phi} \in \widetilde{H}^{\frac{1}{2}}(S_2)$.

Consequently, we derive the inequality

$$\langle \mathcal{K} X , X \rangle = \left\langle r_{S_1} \mathcal{V}_\Delta \Psi , \Psi \right\rangle_{S_1} + \left\langle -r_{S_2} \mathcal{L}_\Delta^{+} \hat{\Phi} , \hat{\Phi} \right\rangle_{S_2} -$$

$$- \left\langle r_{S_1} \mathcal{W}_\Delta \hat{\Phi} , \Psi \right\rangle_{S_1} + \left\langle \hat{\Phi} , r_{S_2} \mathcal{W}'_\Delta \Psi \right\rangle_{S_2} \geq$$

$$\geq c \left(\|\Psi\|^2_{H^{-\frac{1}{2}}(S)} + \|\hat{\Phi}\|^2_{H^{\frac{1}{2}}(S)} \right),$$

implying $\Psi = 0$, $\Phi = 0$. ∎

REDUCTION OF THE MIXED BVP TO BOUNDARY-DOMAIN INTEGRAL EQUATIONS

$$A\,u = f \quad \text{in} \quad \Omega^+, \qquad f \in L_2(\Omega^+) \tag{53}$$

$$r_{S_D}\, u^+ = \varphi_0 \quad \text{on} \quad S_D, \qquad \varphi_0 \in H^{\frac{1}{2}}(S_D), \tag{54}$$

$$r_{S_N}\, T^+ u = \psi_0 \quad \text{on} \quad S_N, \qquad \psi_0 \in H^{-\frac{1}{2}}(S_N). \tag{55}$$

Let $\Phi_0 \in H^{\frac{1}{2}}(S)$ be a fixed extension of the given function φ_0 from the sub–manifold S_D to the whole of S. An arbitrary extension $\Phi \in H^{\frac{1}{2}}(S)$ preserving the function space can be then represented as $\Phi = \Phi_0 + \varphi$ with $\varphi \in \widetilde{H}^{\frac{1}{2}}(S_N)$.

Analogously, let $\Psi_0 \in H^{-\frac{1}{2}}(S)$ be a fixed extension of the given function ψ_0 from the sub–manifold S_N to the whole of S. An arbitrary extension $\Psi \in H^{-\frac{1}{2}}(S)$ preserving the function space can be then represented as $\Psi = \Psi_0 + \psi$ with $\psi \in \widetilde{H}^{-\frac{1}{2}}(S_D)$.

Consider Green's third formula in Ω^+ and its traces on S :

$$u + \mathcal{R}u - VT^+u + Wu^+ = \mathcal{P}Au \text{ in } \Omega^+, \tag{56}$$

$$2^{-1}u^+ + \mathcal{R}^+u - \mathcal{V}T^+u + \mathcal{W}u^+ = [\mathcal{P}Au]^+ \text{ on } S, \tag{57}$$

$$2^{-1}T^+u + T^+\mathcal{R}u - \mathcal{W}'T^+u + \mathcal{L}^+u^+ = T^+\mathcal{P}Au \text{ on } S. \tag{58}$$

Substitute here $u^+ = \Phi_0 + \varphi$, $T^+u = \Psi_0 + \psi$, and $Au = f$.
We arrive at the following BOUNDARY DOMAIN INTEGRAL EQUATION SYSTEM (BDIE) with respect to the unknowns u, ψ, and φ:

$$u + \mathcal{R}u - V\psi + W\varphi = F_0 \text{ in } \Omega^+, \tag{59}$$

$$r_{S_D}\mathcal{R}^+u - r_{S_D}\mathcal{V}\psi + r_{S_D}\mathcal{W}\varphi = r_{S_D}F_0^+ - \varphi_0 \text{ on } S_D, \tag{60}$$

$$r_{S_N}T^+\mathcal{R}u - r_{S_N}\mathcal{W}'\psi + r_{S_N}\mathcal{L}^+\varphi = r_{S_N}T^+F_0 - \psi_0 \text{ on } S_N, \tag{61}$$

where $F_0 := \mathcal{P}f + V\Psi_0 - W\Phi_0 \in H^{1,0}(\Omega^+, A)$.

EQUIVALENCE THEOREM.

Let $f \in L_2(\Omega^+)$ and let $\Phi_0 \in H^{\frac{1}{2}}(S)$ and $\Psi_0 \in H^{-\frac{1}{2}}(S)$ be some extensions of $\varphi_0 \in H^{\frac{1}{2}}(S_D)$ and $\psi_0 \in H^{-\frac{1}{2}}(S_N)$, respectively.

(i) If some $u \in H^{1,0}(\Omega^+, A)$ solves the mixed BVP, then the solution is unique and the triple

$$(u, \psi, \varphi) \in H^{1,0}(\Omega^+, A) \times \widetilde{H}^{-\frac{1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N), \qquad (62)$$

where

$$\psi = T^+ u - \Psi_0, \qquad \varphi = u^+ - \Phi_0 \qquad \text{on} \quad S, \qquad (63)$$

solves BDIE system (59)-(61).

EQUIVALENCE THEOREM.

Let $f \in L_2(\Omega^+)$ and let $\Phi_0 \in H^{\frac{1}{2}}(S)$ and $\Psi_0 \in H^{-\frac{1}{2}}(S)$ be some extensions of $\varphi_0 \in H^{\frac{1}{2}}(S_D)$ and $\psi_0 \in H^{-\frac{1}{2}}(S_N)$, respectively.

(i) If some $u \in H^{1,0}(\Omega^+, A)$ solves the mixed BVP, then the solution is unique and the triple

$$(u, \psi, \varphi) \in H^{1,0}(\Omega^+, A) \times \widetilde{H}^{-\frac{1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N), \tag{62}$$

where

$$\psi = T^+u - \Psi_0, \qquad \varphi = u^+ - \Phi_0 \qquad \text{on} \quad S, \tag{63}$$

solves BDIE system (59)-(61).

(ii) If a triple $(u, \psi, \varphi) \in H^{1,0}(\Omega^+, A) \times \widetilde{H}^{-\frac{1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N)$ solves BDIE system, then the solution is unique, u solves mixed BVP, and equations (63) hold.

Proof. The item (i) directly follows from uniqueness theorem and Green's third formula.
Indeed, if u solves mixed BVP, then $u \in H^{1,0}(\Omega^+)$ and Green's third formula holds,

$$u + \mathcal{R}u - VT^+u + Wu^+ = \mathcal{P}f \quad \text{in} \quad \Omega^+, \tag{64}$$

which can be rewritten as

$$u + \mathcal{R}u - V\big(T^+u - \Psi_0\big) + W\big(u^+ - \Phi_0\big) = \\ = \mathcal{P}f + V\big(\Psi_0\big) - W\big(\Phi_0\big) \equiv F_0 \quad \text{in} \quad \Omega^+, \tag{65}$$

and since $T^+u - \Psi_0 = \psi$ and $u^+ - \Phi_0 = \varphi$ we get the first equation of the BDIE system for the triplet (u, ψ, φ),

$$u + \mathcal{R}u - V\psi + W\varphi = F_0 \quad \text{in} \quad \Omega^+. \tag{66}$$

The traces of this relation and its conormal derivative coincide with the second and the third equations of the BDIE system for the triplet (u, ψ, φ).

Let now a triplet $(u,\psi,\varphi) \in H^{1,0}(\Omega^+, A) \times \widetilde{H}^{-\frac{1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N)$ solve BDIE system:

$$u + \mathcal{R}u - V\psi + W\varphi = F_0 \quad \text{in} \quad \Omega^+, \tag{67}$$

$$r_{S_D}\mathcal{R}^+u - r_{S_D}\mathcal{V}\psi + r_{S_D}\mathcal{W}\varphi = r_{S_D} F_0^+ - \varphi_0 \quad \text{on} \quad S_D, \tag{68}$$

$$r_{S_N}T^+\mathcal{R}u - r_{S_N}\mathcal{W}'\psi + r_{S_N}\mathcal{L}^+\varphi = r_{S_N}T^+F_0 - \psi_0 \quad \text{on} \quad S_N, \tag{69}$$

with $F_0 := \mathcal{P}f + V\Psi_0 - W\Phi_0 \in H^{1,0}(\Omega^+)$.
Taking trace of equation (67) on S_D and subtracting equation (68) from it, we obtain,

$$r_{S_D} u^+ = \varphi_0 \quad \text{on} \quad S_D, \tag{70}$$

i.e. u satisfies the Dirichlet condition on S_D.
Taking the co-normal derivative of equation (67) on S_N and subtracting equation (69) from it, we obtain

$$r_{S_N} T^+u = \psi_0 \quad \text{on} \quad S_N, \tag{71}$$

i.e. u satisfies the Neumann condition onS_N.

Equation (67) can be rewritten as

$$u + \mathcal{R}u - V(\Psi_0 + \psi) + W(\Phi_0 + \varphi) = \mathcal{P}f \quad \text{in} \quad \Omega^+. \tag{72}$$

By AUXILIARY LEMMA 1 we deduce that u is a solution of PDE $A(x,\partial)u = f$ in Ω^+ and

$$V\Psi^* - W\Phi^* = 0 \quad \text{in} \quad \Omega^+, \tag{73}$$

where

$$\Psi^* = \Psi_0 + \psi - T^+u \in \widetilde{H}^{-\frac{1}{2}}(S_D), \tag{74}$$

$$\Phi^* = \Phi_0 + \varphi - u^+ \in \widetilde{H}^{\frac{1}{2}}(S_N). \tag{75}$$

From (73) by AUXILIARY LEMMA 2 we deduce $\Psi^* = 0$ and $\Phi^* = 0$ on S. Thus, u is a solution to the mixed BVP and

$$\psi = T^+u - \Psi_0, \qquad \varphi = u^+ - \Phi_0 \qquad \text{on} \quad S, \tag{76}$$

which completes the proof. ■

BDIE System can be rewritten in vector-matrix form

$$\mathcal{M}\mathcal{U} = \mathcal{F}, \tag{77}$$

$$\mathcal{U} := (u, \psi, \varphi)^\top \in \mathbb{X} \equiv H^1(\Omega^+) \times \widetilde{H}^{-\frac{1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N), \tag{78}$$

$$\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)^\top \in \mathbb{Y} \equiv H^1(\Omega^+) \times H^{\frac{1}{2}}(S_D) \times H^{-\frac{1}{2}}(S_N), \tag{79}$$

$$\mathcal{M} := \begin{bmatrix} I + \mathcal{R} & -V & W \\ r_{S_D}\mathcal{R}^+ & -r_{S_D}\mathcal{V} & r_{S_D}\mathcal{W} \\ r_{S_N}T^+\mathcal{R} & -r_{S_N}\mathcal{W}' & r_{S_N}\mathcal{L}^+ \end{bmatrix} \tag{80}$$

Due to the properties of the potential operators involved in (80), the followin operator is continuous:

$$\mathcal{M} : \mathbb{X} \to \mathbb{Y}.$$

THEOREM 4. The operator

$$\mathcal{M} : H^1(\Omega^+) \times \widetilde{H}^{-\frac{1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N) \to \\ \to H^1(\Omega^+) \times H^{\frac{1}{2}}(S_D) \times H^{-\frac{1}{2}}(S_N), \quad \textbf{(81)}$$

is invertible.

THEOREM 4. The operator

$$\mathcal{M} : H^1(\Omega^+) \times \widetilde{H}^{-\frac{1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N) \to$$
$$\to H^1(\Omega^+) \times H^{\frac{1}{2}}(S_D) \times H^{-\frac{1}{2}}(S_N), \quad \textbf{(80)}$$

is invertible.

Proof. Let us consider the upper triangular matrix operator

$$\mathcal{M}_0 := \begin{bmatrix} I & -V & W \\ 0 & -r_{S_D}\mathcal{V} & 0 \\ 0 & 0 & r_{S_N}\widehat{\mathcal{L}} \end{bmatrix} \qquad \textbf{(81)}$$

where $\widehat{\mathcal{L}}g = \mathcal{L}^+_{\Delta}(ag)$ on S.

The operator $\mathcal{M}_0$ is a compact perturbation of the operator $\mathcal{M}$.

The diagonal operators are invertible

$$\begin{aligned} I \quad &: \quad H^1(\Omega^+) \to H^1(\Omega^+), \\ r_{S_D}\, \mathcal{V} \quad &: \quad \widetilde{H}^{-\frac{1}{2}}(S_D) \to H^{\frac{1}{2}}(S_D), \\ r_{S_N}\, \widehat{\mathcal{L}} \quad &: \quad \widetilde{H}^{\frac{1}{2}}(S_N) \to H^{-\frac{1}{2}}(S_N). \end{aligned}$$

Therefore the triangular operator

$$\begin{aligned} \mathcal{M}_0 \ : \ & H^1(\Omega^+) \times \widetilde{H}^{-\frac{1}{2}}(S_D) \times \widetilde{H}^{\frac{1}{2}}(S_N) \to \\ & \to H^1(\Omega^+) \times H^{\frac{1}{2}}(S_D) \times H^{-\frac{1}{2}}(S_N) \end{aligned} \tag{82}$$

is invertible. Whence it follows that operator $\mathcal{M}$ possesses the Fredholm property and its index is zero.

The Equivalence Theorem yields that the null-space of the operator (80) is trivial and consequently the operator $\mathcal{M}$ in (80) is invertible.

The invertibility of the operator $\mathcal{M}$ and the Equivalence Theorem lead to the following assertions.

THEOREM 5. **The operator**

$$\mathcal{M}\ :\ H^{1,0}(\Omega^+;L)\times\widetilde{H}^{-\frac{1}{2}}(S_D)\times\widetilde{H}^{\frac{1}{2}}(S_N)\to$$
$$\to H^{1,0}(\Omega^+;L)\times H^{\frac{1}{2}}(S_D)\times H^{-\frac{1}{2}}(S_N) \quad \textbf{(83)}$$

is invertible.

COROLLARY 6. **The mixed boundary value problem as well as the corresponding BDIE system are uniquely solvable.**

THANK YOU!