Mathematical Analysis of Problems in Complex Media Electromagnetics

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1. Some remarks on the history of the subject
2. Modelling: bianisotropic media
3. Time-harmonic problems
4. Time-domain problems
5. Homogenisation
6. Nonlinear media
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Optical Activity is the ability of a material to rotate the plane of polarisation of a beam of light passing through it.

Original experimental studies by D. F. J. Arago (1811), J.-B. Biot (1812), and A.-J. Fresnel (1822).

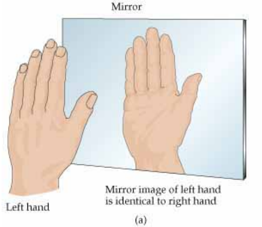
Cauchy (1842 – first mathematical work on the laws of circular polarisation).

Explanation given by Louis Pasteur (1848): introduction of Geometry into Chemistry (origin of the branch nowadays called Stereochemistry).

Handedness is a characteristic of natural and manufactured objects (e.g., DNA, certain bacteria, shells, winding plants, spiral galaxies / cork-screws, doors, cookers, computer mice, keyboards, guitars, a variety of construction tools / Möbius strips, irregular tetrahedra).

The mirror image of a right-handed object is otherwise the same as the original, but it is left-handed (the original object cannot be superposed upon its mirror image).

A handed object is called chiral, a term arising from the Greek word $\chi\varepsilon\iota\varrho$ meaning hand.



This term was introduced by Lord Kelvin in 1888 (first time in print in his famous 1904 Baltimore Lectures).

It was the relation between the chiral (micro)structure and the (macroscopic) optical rotation that was discovered by Pasteur: he noticed that that two substances which where chemically identical (in the classification scheme of that time), but which had molecules being mirror images of each other, exhibited different physical properties[1].

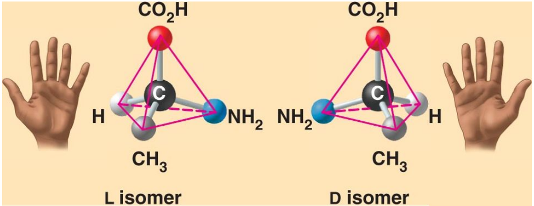


Figure: Enantiomers: L (Lævus = Left) and D (Dexter = Right) isomers of (the α-aminoacid) alanine

[1] E.g., one enantiomer of *thalidomide* may be used to cure morning sickness in pregnant women, but its mirror image induces fetal malformation - a severe problem in the U.K. in the late 1950s.

In the last part of the 19th century, after Maxwell's unification of optics with electricity and magnetism, it became possible to establish the connection between optical activity and the electromagnetic parameters of materials.

In 1914, Lindman was the first to demonstrate the effect of a chiral medium on electromagnetic waves (his work in this field was about 40 years ahead of that of other scientists); he devised a macroscopic model for the phenomenon of "optical" activity that used microwaves instead of light and wire spirals instead of chiral molecules. His related work was published in 1920 and 1922.

The revival in the interest of complex media in Electromagnetics emerged in the mid 1980s, motivated and assisted by vast technological progress, especially at microwave frequencies.

In the beginning of the 21st century, the related publications within the Applied Physics and Electrical Engineering communities were already calculated in more than 3500 papers!

Related books:

- A. Lakhtakia, V. K. Varadan, V. V. Varadan, 1989.
- A. Lakhtakia, 1994.
- I. Lindell, A. H. Sihvola, S. A. Tretyakov, A. J. Viitanen, 1994.
- A. Serdyukov, I. Semchenko, S. Tretyakov, A. Sihvola, 2001.
- S. Zouhdi, A. H. Sihvola, A. P. Vinogradov (eds.), 2009.
- G. Kristensson, 2016.

Mathematical Work – 1: Frequency domain problems (time-harmonic fields)

- (As far as I know) the first publication is by Petri Ola (1994).
- Simultaneous/independent work from the mid 1990s by the groups at
  - CMAP, École Polytechnique (Palaiseau): Jean-Claude Nédélec, Habib Ammari, and their collaborators.
  - the Department of Mathematics of the National and Kapodistrian University of Athens: Christos E. Athanasiadis, $\mathbb{S}$, and later on collaborators in various places.
- From the late 1990s, in addition to the above, many researchers enter the field; indicatively (in alphabetical, non-chronological, order) some names: A. Boutet de Monvel, G. Costakis, P. Courilleau, S. Dimitroula, T. Gerlach, S. Heumann, T. Horsin, H. Kiili, E. Kikeri, V. Kravchenko, S. Li, P. A. Martin, S. R. McDowall, M. Mitrea, D.-L. Nguyen, T.-N. Nguyen, R. Potthast, V. Sevroglou, D. Shepelsky, C. Skourogiannis, A. Spyropoulos, M.-P. Tran, N. Tsitsas, S. Vänskä, ...

Mathematical Work – 2: Time domain problems

From the early 2000s attention is focussed on the time domain, as well. Problems on the solvability, the homogenisation, and the controllability of IBVPs for the Maxwell equations, supplemented with nonlocal in time, linear constitutive relations (describing the so-called *bianisotropic* media), are studied.

Again a representative (yet incomplete) list of researchers:
E. Argyropoulou, C. E. Athanasiadis, G. Barbatis, A. Bossavit, P. Ciarlet jr., P. Courilleau, H. Freymond, D. Frantzeskakis, G. Griso, S. Halkos, T. Horsin, A. Ioannidis, A. Karlsson, G. Kristensson, G. Legendre, K. Liaskos, B. Miara, S. Nicaise, R. Picard, G. F. Roach, D. Sjöberg, $\mathbb{S}$., N. Wellander, A. N. Yannacopoulos, ...

- The biggest part of this work deals with deterministic bianisotropic media.
- But problems regarding stochastic bianisotropic media are also studied.

A big part of the mathematical theory can be found in the book

- G. F. Roach, $\mathbb{S}$, A. N. Yannacopoulos, *"Mathematical Analysis of Deterministic and Stochastic Problems in Complex Media Electromagnetics"*, Princeton University Press, 2012.

The Maxwell system

Electromagnetic phenomena are specified by 4 (vector) quantities: the *electric field* E, the *magnetic field* H, the *electric flux density* D and the *magnetic flux density* B. The inter-dependence between these quantities is given by the celebrated Maxwell system,

$$\begin{aligned} \mathrm{curl} H(t,x) &= \partial_t D(t,x) + J(t,x), \\ \mathrm{curl} E(t,x) &= -\partial_t B(t,x), \end{aligned} \tag{1}$$

where J is the electric current density. All fields are considered for $x \in \mathcal{O} \subset \mathbb{R}^3$ and $t \in \mathbb{R}$, $\mathcal{O}$ being a domain with appropriately smooth boundary. These equations are the so called Ampère's law and Faraday's law, respectively. In addition to the above, we have the two laws of Gauss

$$\begin{aligned} \mathrm{div} D(t,x) &= \rho(t,x), \\ \mathrm{div} B(t,x) &= 0, \end{aligned} \tag{2}$$

where ρ is the density of the (externally impressed) electric charge.

Initial and Boundary conditions

The initial conditions are considered to be of the form

$$E(0,x) = E_0(x)\,,\ H(0,x) = H_0(x)\,,\ x \in \mathcal{O}\,. \tag{3}$$

We consider the "perfect conductor" boundary condition

$$n(x) \times E(t,x) = 0\,,\ x \in \partial\mathcal{O}\,,\ t \in I\,, \tag{4}$$

where I is a time interval, and $n(x)$ denotes the outward normal on $\partial\mathcal{O}$.

From (1) and (2) we wish to determine the quadruplet (B, D, E, H), assuming that the vector J and the scalar ρ are known.
Need to calculate 12 scalar functions from a system of 8 scalar equations.
So, constitutive relations[2] must be introduced

$$D = D(E, H)\,,\;\; B = B(E, H)\,. \tag{5}$$

It can be seen that we may consider as "the Maxwell system" the set of equations (1) ($\mathrm{curl} H = \partial_t D + J$ and $\mathrm{curl} E = -\partial_t B$), plus the constitutive relations (5), plus the equation of continuity

$$\partial_t \rho + \mathrm{div} J = 0. \tag{6}$$

[2]The considered here form of constitutive relations $(D, B) = F(E, H)$ is compatible with energy considerations, since the Poynting vector is defined as $E \times H$. Other forms of constitutive relations are also used; e.g. in special relativity the form $(D, H) = F(E, B)$ is preferable.

The six vector notation

To express the system in more compact form, we use the six-vector notation:

- the electromagnetic flux density $\mathrm{d} := (D, B)^{tr}$,
- the electromagnetic field $u := (u_1, u_2)^{tr} := (E, H)^{tr}$,
- the current $j := (-J, 0)^{tr}$,
- the initial state $u_0 := (E_0, H_0)^{tr}$,

where the superscript tr denotes transposition.

A linear operator acting on u is written as a 2×2 (block) matrix with linear operators as its entries.

An important case is the Maxwell operator

$$\mathrm{M} := \begin{pmatrix} 0 & \mathrm{curl} \\ -\mathrm{curl} & 0 \end{pmatrix}. \tag{7}$$

The Maxwell system as an IVP

The constitutive relations are now modelled by an operator $\mathcal{L}$ and are understood as the functional equation

$$\mathrm{d} = \mathcal{L}u.$$

The properties of this operator reflect the physical properties of the medium in question.
So the Maxwell system can be written as an IVP for an abstract evolution equation

$$\begin{aligned} &(\mathcal{L}u)'(t) = \mathrm{M}u(t) + j(t) \quad , \quad \text{for } t > 0\,, \\ &u(0) = u_0\,. \end{aligned} \tag{8}$$

The prime stands for the time derivative.
The equation in the IVP (31) is an inhomogeneous *neutral functional* differential equation.

Postulates

To state the postulates that govern the evolution of the e/m field in a complex medium we follow a system-theoretic approach (Ioannidis (PhD; 2006) / Ioannidis, Kristensson, $\mathbb{S}$), in the sense that we consider the e/m field u as the cause, and the e/m flux density d as the effect.

Postulates (plausible physical hypotheses)

- Determinism: For every cause there exists exactly one effect.
- Linearity: The effect is produced linearly by its cause. Let us mention that most materials exhibit linear behaviour at a sufficiently weak field strength.
- Causality: The effect cannot precede its cause.
- Locality in space: A cause at any particular spatial point produces an effect only at this point and not elsewhere.
- Time–translation invariance: If the cause is advanced (or delayed) by some time interval, the same time-shift occurs for the effect.

Compliance with these postulates dictates the form of the operator $\mathcal{L}$.

Mathematical interpretation in terms of $\mathcal{L}$

- Determinism: $\mathcal{L}$ exists and is a single-valued nontrivial operator.
- Linearity: $\mathcal{L}$ is a linear operator.
- Causality: If $u(t,x) = 0$ for $t \leq \tau$, then $(\mathcal{L}u)(t,x) = 0$, for $t \leq \tau$.
- Locality in space: $\mathcal{L}$ is a local operator with respect to the spatial variables, i.e., $\mathcal{L}(u(\cdot,x))(\cdot,x) = \mathfrak{s}(\cdot,x)$ where $\mathfrak{s}$ is a local functional, allowing spatial derivatives of the electromagnetic fields, but not integrals with respect to the spatial variables.
  Locality with respect to temporal variables is not assumed, on the contrary memory effects are allowed.
- Time–translation invariance: For all $\varkappa \geq 0$, $\mathcal{L}$ commutes with the right $\varkappa$-shift operator $\tau_\varkappa$. Therefore, the time instant at which the observation starts does not play any significant rôle; the "present" can be chosen arbitrarily.

We do not assume continuity: it follows by linearity and time–translation invariance. Note that continuity is not ascertained in the case where the left shift replaces the right shift.

The constitutive relations for bianisotropic media

The general form of $\mathcal{L}$, consistent with the above physical postulates, turns to be a continuous operator having the convolution form

$$\mathrm{d}(t,x) = (\mathcal{L}u)(t,x) = \mathrm{A}_{\mathrm{or}}(x)u(t,x) + \int_0^t \mathrm{G}_{\mathrm{d}}(t-s,x)u(s,x)\,ds \tag{9}$$

$$\mathrm{A}_{\mathrm{or}}(x) := \begin{pmatrix} \varepsilon(x) & \xi(x) \\ \zeta(x) & \mu(x) \end{pmatrix} \;,\; \mathrm{G}_{\mathrm{d}}(t,x) := \begin{pmatrix} \varepsilon_{\mathrm{d}}(t,x) & \xi_{\mathrm{d}}(t,x) \\ \zeta_{\mathrm{d}}(t,x) & \mu_{\mathrm{d}}(t,x) \end{pmatrix}. \tag{10}$$

Each $\mathrm{A}_{\mathrm{or}}(\cdot)$, $\mathrm{G}_{\mathrm{d}}(t,\cdot)$ defines a multiplication operator in the state space. The above constitutive equation is abbreviated as

$$\mathrm{d} = \mathrm{A}_{\mathrm{or}}u + \mathrm{G}_{\mathrm{d}} \star u. \tag{11}$$

The local in space part A_{or} (optical response operator) of $\mathcal{L}$ models the instantaneous response of the medium. The nonlocal in space part $\mathrm{G}_{\mathrm{d}}\star$ of $\mathcal{L}$ models the dispersion phenomena; G_{d} is called the susceptibility kernel.

Media Classification

A material is called

- **Isotropic:** if ε, μ, ε_d, μ_d are scalar multiples of $I_{3\times 3}$ and $\xi = \zeta = \xi_d = \zeta_d = 0$.
  Recall that isotropy means that the material has identical properties in all directions; it is a microscopic property of materials.
- **Anisotropic:** if the members of at least one of the pairs $\varepsilon, \varepsilon_d$ or μ, μ_d are not scalar multiples of $I_{3\times 3}$ and $\xi = \zeta = \xi_d = \zeta_d = 0$.
- **Biisotropic:** if all the blocks of the matrices $\mathrm{A_{or}}$, $\mathrm{G_d}$ are scalar multiples of $I_{3\times 3}$.
- **Bianisotropic:** in all other cases.

The time-harmonic case

Taking the Fourier transform with respect to t, we obtain

$$\widetilde{\mathrm{d}} = \mathrm{A}_{\mathrm{or}}\widetilde{u} + \widetilde{\mathrm{G}_{\mathrm{d}}}\widetilde{u}\,, \tag{12}$$

and letting

$$\widetilde{\mathrm{A}_{\mathrm{or}}} := \mathrm{A}_{\mathrm{or}} + \widetilde{\mathrm{G}_{\mathrm{d}}} = \begin{pmatrix} \varepsilon + \widetilde{\varepsilon_d} & \xi + \widetilde{\xi_d} \\ \zeta + \widetilde{\zeta_d} & \mu + \widetilde{\mu_d} \end{pmatrix} =: \begin{pmatrix} \varepsilon_{\mathfrak{F}} & \xi_{\mathfrak{F}} \\ \zeta_{\mathfrak{F}} & \mu_{\mathfrak{F}} \end{pmatrix}\,, \tag{13}$$

we get the frequency domain constitutive relations

$$\widetilde{\mathrm{d}} = \widetilde{\mathrm{A}_{\mathrm{or}}}\widetilde{u}\,. \tag{14}$$

Media classification

We elaborate further on the classification of biisotropic media in the frequency domain. Let

$$\xi_{\mathfrak{F}} = \kappa + i\chi\,,\ \zeta_{\mathfrak{F}} = \kappa - i\chi\,. \tag{15}$$

The *chirality parameter* χ measures the degree of handedness of the material; a change in the sign of χ corresponds to the consideration of the mirror image of the material. The other parameter κ describes the magnetoelectric effect; materials with $\kappa \neq 0$ are nonreciprocal.

In the time-harmonic case a medium is called:

- **Isotropic**, if $\kappa = 0$ and $\chi = 0$, i.e., when $\xi_{\mathfrak{F}} = \zeta_{\mathfrak{F}} = 0$.
- **Nonreciprocal Nonchiral**, or **Tellegen**, if $\kappa \neq 0$ and $\chi = 0$, i.e., when $\xi_{\mathfrak{F}} = \zeta_{\mathfrak{F}}$.
- **Reciprocal Chiral**, or **Pasteur**, if $\kappa = 0$ and $\chi \neq 0$, i.e., when $\xi_{\mathfrak{F}} = -\zeta_{\mathfrak{F}}$.
- **Nonreciprocal Chiral** or **General Biisotropic**, if $\kappa \neq 0$ and $\chi \neq 0$, i.e., when $\xi_{\mathfrak{F}} \neq \zeta_{\mathfrak{F}}, -\zeta_{\mathfrak{F}}$.

The DBF constitutive relations

Reciprocal chiral media will be simply referred to as **chiral** media.
In the case of chiral media, the constitutive relations for time-harmonic fields are usually written as

$$\widetilde{D} = \varepsilon_{\mathsf{T}} \widetilde{E} + \beta_{\mathsf{T}} \widetilde{H}, \quad \widetilde{B} = \mu_{\mathsf{T}} \widetilde{H} - \beta_{\mathsf{T}} \widetilde{E}, \tag{16}$$

(where $\varepsilon_{\mathsf{T}} := \varepsilon_{\mathfrak{F}}$, $\mu_{\mathsf{T}} := \mu_{\mathfrak{F}}$, $\beta_{\mathsf{T}} := \xi_{\mathfrak{F}} = -\zeta_{\mathfrak{F}}$).

The *Drude-Born-Fedorov* (DBF) constitutive relations were introduced in 1959 by F. I. Fedorov, as a modification of constitutive relations used in 1900 by P. K. L. Drude, and in 1915 by M. Born.

The *Drude-Born-Fedorov* (DBF) constitutive relations are

$$\widetilde{D} = \varepsilon_{\mathsf{DBF}} (\widetilde{E} + \beta_{\mathsf{DBF}} \, \mathrm{curl} \widetilde{E}), \quad \widetilde{B} = \mu_{\mathsf{DBF}} (\widetilde{H} + \beta_{\mathsf{DBF}} \, \mathrm{curl} \widetilde{H}). \tag{17}$$

The medium is characterised by three (in general, complex) parameters, the electric permittivity $\varepsilon_{\mathsf{DBF}}$, the magnetic permeability μ_{DBF}, and the chirality measure β_{DBF}.

For source-free regions the constitutive parameters ε_{T}, μ_{T} and β_{T} of (16) are connected to the constitutive parameters $\varepsilon_{\mathsf{DBF}}$, μ_{DBF} and β_{DBF} of (17) via

$$\varepsilon_{\mathsf{T}} = \frac{\varepsilon_{\mathsf{DBF}}}{1 - \varpi^2 \varepsilon_{\mathsf{DBF}} \mu_{\mathsf{DBF}} \beta_{\mathsf{DBF}}^2},$$

$$\mu_{\mathsf{T}} = \frac{\mu_{\mathsf{DBF}}}{1 - \varpi^2 \varepsilon_{\mathsf{DBF}} \mu_{\mathsf{DBF}} \beta_{\mathsf{DBF}}^2},$$

$$\beta_{\mathsf{T}} = i \varpi \varepsilon_{\mathsf{DBF}} \mu_{\mathsf{DBF}} \frac{\beta_{\mathsf{DBF}}}{1 - \varpi^2 \varepsilon_{\mathsf{DBF}} \mu_{\mathsf{DBF}} \beta_{\mathsf{DBF}}^2}.$$

The Interior Problem

The study of time-harmonic problems for the Maxwell equations supplemented with the DBF constitutive relations is a well-developed area. The solvability of interior and exterior BVPs is established by variational techniques.

The interior problem reads

$$\left.\begin{aligned} \operatorname{curl} E &= \beta\gamma^2 E + i\varpi\mu \left(\tfrac{\gamma}{k}\right)^2 H, \\ \operatorname{curl} H &= \beta\gamma^2 H - i\varpi\varepsilon \left(\tfrac{\gamma}{k}\right)^2 E, \end{aligned}\right. \quad \text{in } \mathcal{O}, \tag{18}$$

where $\varpi > 0$ is the angular frequency and

$$k^2 := \varpi^2 \varepsilon\mu \quad , \quad \gamma^2 := k^2(1-\beta^2 k^2)^{-1}. \tag{19}$$

These equations are complemented with the boundary condition

$$n \times E = f, \quad \text{on } \partial\mathcal{O}, \tag{20}$$

where $f \in H^{-1/2}(\operatorname{div}, \partial\mathcal{O})$ is a prescribed electric field on $\partial\mathcal{O}$.

Assumptions

A typical assumption on the data is

Assumption

(i) $\mathcal{O}$ is a bounded domain and $\partial\mathcal{O}$ is of class $C^{1,1}$.

(ii) The coefficients ε, μ and β are real valued and positive $C^2(\overline{\mathcal{O}})$ functions.

(iii) The function $\mu^{-1}(1 - \varpi^2\varepsilon\mu\beta^2)$ is positive in $\overline{\mathcal{O}}$.

The solvability of the interior problem ((18), (20)) will be the subject of my mini-course.

The Exterior Problem – Radiation Conditions

As for the exterior problem, it consists of the equations

$$\left.\begin{aligned} \operatorname{curl} E_e &= \beta_e \gamma_e^2 E_e + i\varpi\mu_e \left(\frac{\gamma_e}{k_e}\right)^2 H, \\ \operatorname{curl} H_e &= \beta_e \gamma_e^2 H_e - i\varpi\varepsilon_e \left(\frac{\gamma_e}{k_e}\right)^2 E, \end{aligned}\right. \qquad \text{in } \mathcal{O}_e, \tag{21}$$

with boundary condition

$$n \times E_e = f_e, \quad \text{on } \partial\mathcal{O}_e$$

and with one of the two Silver-Müller radiation conditions

$$\lim_{|x|\to\infty} |x| \left(\sqrt{\mu} H_e \times \widehat{x} - \sqrt{\varepsilon} E_e\right) = 0\,, \tag{22}$$

or

$$\lim_{|x|\to\infty} |x| \left(\sqrt{\varepsilon} E_e \times \widehat{x} + \sqrt{\mu} H_e\right) = 0\,, \tag{23}$$

uniformly over all directions $\widehat{x}$.

- It is known (Ammari and Nédélec) that the "standard" (achiral) Silver-Müller radiation conditions (written above) are adequate to cover the chiral case, too.
- Another important property is a "continuity result" in terms of the chirality measure β. It is known (Ammari and Nédélec) that if β is assumed to be a non-negative constant, then the limit of the solution of the chiral problem as $\beta \to 0$ coincides with the solution of the corresponding achiral ($\beta = 0$) problem.

Scattering problems

Another very well developed area of research regarding chiral media in the time-harmonic regime deals with scattering problems.
Consider that an electromagnetic wave propagating in a chiral (or achiral) homogeneous[3] environment is incident upon a chiral (or achiral) obstacle. Depending on the materials with which the surrounding space and the obstacle are filled, and on the boundary condition(s) on the obstacle's surface, a variety of scattering problems (exterior BVPs, transmission problems) is treated. For example,

- Using classical potentials, BIEs are employed to study the solvability, the determination of uniquely solvable equations, and the "low-chirality" approximation:
  Ammari, Nédélec / Athanasiadis, $\mathbb{S}$ / Athanasiadis, Costakis, $\mathbb{S}$ Athanasiadis, Martin, $\mathbb{S}$ / Mitrea / Ola.

[3] There are no variations on a macroscopic length scale. Note that isotropy does not imply homogeneity.

- Typical scattering results (e.g., Reciprocity Principle, Optical Theorem, General Scattering Theorem) are extended to the chiral case for plane and spherical incident waves:
  Athanasiadis, Martin, $\mathbb{S}$ / Athanasiadis, Giotopoulos
  Athanasiadis, Skourogiannis / Athanasiadis, Tsitsas.
- Low-Frequency theory:
  Ammari, Laouadi, Nédélec / Athanasiadis, Costakis, $\mathbb{S}$.
- Herglotz functions and pairs:
  Athanasiadis, Kardasi, Kikeri.
- Infinite Fréchet differentiability of the mapping from the boundary of the scatterer onto the far-field patterns, and a characterisation of the Fréchet derivative as a solution to a BVP:
  Potthast, $\mathbb{S}$.

- Periodic structures, Gratings:
  Ammari, Bao / Zhang, Ma.
- Quaternionic methods:
  Kravchenko in various combinations with Grudsky, Khmelnytskaya, Oviedo-Galdeano, and V. Rabinovich.
- Inverse scattering problems:
  Athanasiadis, Berketis / Athanasiadis, Dimitroula
  Athanasiadis, Sevroglou, Skourogiannis / Athanasiadis, $\mathbb{S}$
  Boutet de Monvel, Shepelsky / Gao, Ma, Zhang / Gerlach / Heumann
  Kusunoki / Li / Mc Dowall / Rikte, Sauviac, Kristensson, Mariotte.
- Galerkin approximation for the forward problem:
  D.-L. Nguyen, T.-N. Nguyen, Tran.

An introduction to BIEs – The Helmholtz equation

Consider the Helmholtz equation

$$\Delta u + k^2 u = 0 \tag{24}$$

in either a bounded, simply connected domain $\mathcal{O} \subset \mathbb{R}^3$, with boundary $\partial\mathcal{O} \in C^2$, or in its complement $\mathcal{O}_e$ in $\mathbb{R}^3$.

It is well known that a classical solution $u \in C^2(\mathcal{O}) \cap C^1(\overline{\mathcal{O}})$ (respectively, $u \in C^2(\mathcal{O}_e) \cap C^1(\overline{\mathcal{O}_e})$) can be represented in terms of surface potentials via the first Green identity for the Laplacian and the fundamental solution

$$\Phi(x, y; k) = \frac{e^{\mathrm{i}\, k|x-y|}}{4\pi\, |x-y|}, \quad x \neq y, \tag{25}$$

of the Helmholtz equation. As usual, $k \in \mathbb{C}$ with $\mathcal{I}\mathrm{m}\, k \geq 0$.

In the case of exterior problems the behaviour at infinity is required to satisfy the Sommerfeld radiation conditions

$$u(x) = O(|x|^{-1})\,, \ \frac{\partial u}{\partial |x|}(x) - iku(x) = o(|x|^{-1})\,, |x| \to \infty\,.$$

Single- and Double-layer potentials

The aforementioned representation reads

$$u(x) = \pm \left(\int_{\partial \mathcal{O}} \Phi(x, y; k) \frac{\partial u}{\partial n}(y) ds(y) - \int_{\partial \mathcal{O}} u(y) \frac{\partial \Phi(x, y; k)}{\partial n_y} ds(y) \right) , \quad (26)$$

with "+" for $x \in \mathcal{O}$ and "–" for $x \in \mathcal{O}_e$, where n_y denotes the exterior normal to $\partial \mathcal{O}$ at $y \in \partial \mathcal{O}$.

For given Cauchy data $u|_{\partial \mathcal{O}}$ and $\frac{\partial u}{\partial n}|_{\partial \mathcal{O}}$, the above representation formula defines the solution of the Helmholtz equation everywhere in $\mathcal{O}$, or in $\mathcal{O}_e$, respectively.

The surface potentials appearing in the representation formula are the single-layer potential

$$\mathbb{V}_k \phi(x) := \int_{\partial \mathcal{O}} \Phi(x, y; k) \phi(y) ds(y) , \; x \in \mathcal{O} \cup \mathcal{O}_e ,$$

and the double-layer potential

$$\mathbb{W}_k \varphi(x) := \int_{\partial \mathcal{O}} \frac{\partial \Phi(x, y; k)}{\partial n_y} \varphi(y) ds(y) , \; x \in \mathcal{O} \cup \mathcal{O}_e ,$$

where ϕ and φ are the corresponding densities.

Boundary integral operators

Provided the corresponding limits exist, the boundary integral operators we are going to employ are related to the limits of the boundary potentials from either $\mathcal{O}$, or $\mathcal{O}_e$, on $\partial\mathcal{O}$.
Let $\partial\mathcal{O} \in C^2$ and ϕ and ψ be continuous. Then the required limits exist uniformly with respect to all $x \in \partial\mathcal{O}$ and all ϕ and ψ with $\sup_{x\in\partial\mathcal{O}} |\phi(x)| \leq 1\,, \sup_{x\in\partial\mathcal{O}} |\psi(x)| \leq 1$. The BIOs needed are

$$(\mathbb{V}_{\mathrm{k}}\phi)(x) = \int_{y\in\partial\mathcal{O}\setminus\{x\}} \Phi(x,y;k)\phi(y)ds(y)\,,\ x \in \partial\mathcal{O}\,,$$

$$(\mathbb{K}_{\mathrm{k}}\psi)(x) = \int_{y\in\partial\mathcal{O}\setminus\{x\}} \frac{\partial\Phi(x,y;k)}{\partial n_y}\psi(y)ds(y)\,,\ x \in \partial\mathcal{O}\,,$$

$\mathbb{K}_{\mathrm{k}}^*$ is the adjoint of $\mathbb{K}_{\mathrm{k}}$, i.e.,

$$(\mathbb{K}_{\mathrm{k}}^*\phi)(x) = \int_{y\in\partial\mathcal{O}\setminus\{x\}} \frac{\partial\Phi(x,y;k)}{\partial n_x}\phi(y)ds(y)\,,\ x \in \partial\mathcal{O}\,,$$

while, for ϕ Hölder continuously differentiable with $||\phi||_{C^{1,\theta}} \leq 1\,, \theta \in (0,1)$,

$$(\mathbb{D}_{\mathrm{k}}\phi)(x) = \mathrm{pv}\int_{\partial\mathcal{O}} \frac{\partial^2\Phi(x,y;k)}{\partial n_x\partial n_y}(\phi(y)-\phi(x))ds(y)\,,$$

where "pv" denotes the Cauchy principal value integral.

Cauchy PV integral

Because of the strong singularity of the kernel the integral in the definition of $\mathbb{D}_\mathsf{k}$ has to be interpreted as a Cauchy principal value integral.
Let us briefly recall the definition in a simple case: let $f \in C^{0,\theta}([a,b])$ and $x \in (a,b)$. The Cauchy principal value integral is defined as

$$\mathrm{pv}\int_a^b \frac{f(t)}{x-t}dt := \lim_{\epsilon\to 0}\left(\int_a^{x-\epsilon}\frac{f(t)}{x-t}dt + \int_{x+\epsilon}^b \frac{f(t)}{x-t}dt\right).$$

If $f \in C^1([a,b])$, we have that

$$\mathrm{pv}\int_a^b \frac{f(t)}{x-t}dt = f(a)\ln(x-a) - f(b)\ln(b-x) + \int_a^b f'(t)\ln|x-t|dt.$$

Further, if $f \in C^{1,\theta}([a,b])$, we have

$$\frac{d}{dx}\left(\mathrm{pv}\int_a^b \frac{f(t)}{x-t}dt\right) = \frac{f(a)}{x-a} + \frac{f(b)}{b-x} + \mathrm{pv}\int_a^b \frac{f'(t)}{x-t}dt.$$

Weakly singular kernels

It is not hard to see that the kernels of $\mathbb{V}_\mathsf{k}, \mathbb{K}_\mathsf{k}$ and $\mathbb{K}_\mathsf{k}^*$ are weakly singular (with $\alpha = 1$, see below).

Consider the Banach space $C(\partial\mathcal{O})$ of complex-valued functions on $\partial\mathcal{O}$ equipped with the maximum norm, and the boundary integral operator $\mathfrak{C} : C(\partial\mathcal{O}) \to C(\partial\mathcal{O})$ defined by

$$(\mathfrak{C}\phi)(x) := \int_{\partial\mathcal{O}} \mathsf{k}(x,y)\,\phi(y)\,ds(y)\,, \; x \in \partial\mathcal{O}\,.$$

The kernel k is said to be weakly singular if it is defined and continuous for all $x, y \in \partial\mathcal{O}, x \neq y$, and there exist positive constants c and $\alpha \in (0, 2]$ such that for all $x, y \in \partial\mathcal{O}, x \neq y$, we have

$$|\mathsf{k}(x,y)| \leq c|x-y|^{\alpha-2}\,.$$

Then it is well known that, under the assumption that the kernel k is either continuous, or weakly singular, the operator $\mathfrak{C}$ is compact.

BVPs

We are now in a position to consider boundary value problems; we discuss the interior Dirichlet problem,

$$\Delta u + k^2 u = 0 \text{ in } \mathcal{O}\,, \text{ with } u = f \text{ on } \partial\mathcal{O}\,.$$

In this case only one of the two Cauchy data is given in (26), namely $u(x) = f(x)\,, x \in \partial\mathcal{O}$.

The "missing" datum is $\frac{\partial u}{\partial n}(x) = g(x)$.

The BIE for the determination of g then reads

$$(\mathbb{V}_{\mathrm{k}} g)(x) = \left(\left(\frac{1}{2}\mathbb{I} + \mathbb{K}_{\mathrm{k}}\right) f\right)(x)\,, \; x \in \partial\mathcal{O}\,,$$

which is a Fredholm integral equation of the first kind.

This equation, despite the fact that it is ill posed, has proved to be very important both from the analytic as well as from the numerical point of view.

It is known that for $k \neq 0$ and $f \in C^{1,\alpha}(\partial\mathcal{O})\,, \alpha \in (0,1)$, the above BIE is uniquely solvable with $g \in C^{\alpha}(\partial\mathcal{O})$ except for certain values of $k \in \mathbb{C}$ which are the exceptional (or irregular) frequencies of the boundary integral operator $\mathbb{V}_{\mathrm{k}}$.

The interior Dirichlet problem for the Helmholtz equation can alternatively be expressed as

$$\left(\left(\frac{1}{2}\mathbb{I} - \mathbb{K}_{\mathsf{k}}^{*}\right) g\right)(x) = \left(\mathbb{D}_{\mathsf{k}} f\right)(x)\,,\ x \in \partial\mathcal{O}\,,$$

which is a Fredholm integral equation of the second kind.
This BIE has been and still is extensively studied, both analytically and numerically.

We notice that we may use different BIEs for the same boundary value problem; further, these boundary integral equations may be uniquely, or non-uniquely, solvable.

This is an important issue: for example, the exterior Dirichlet problem for the Helmholtz equation is known to have a unique solution for all k with $\mathcal{I}\mathrm{m}\, k \geq 0$; so the complication of non-uniqueness for the boundary integral equation at the irregular frequencies arises from the selected method of solution rather than from the nature of the problem itself.

PDOs ⇝ BIOs / Pros and Cons

So we see that the Dirichlet boundary value problem for the Helmholtz equation in $\mathcal{O}$ (or in $\mathcal{O}_e$) is reduced (in view of appropriate integral representations of the solution) to a problem defined on a bounded domain of lower dimension, namely, on the boundary $\partial\mathcal{O}$.

It is not necessary for the problem to be elliptic, but the ones treated with BIE methods often are.

Such reductions shift the setting from partial differential (unbounded) operators to boundary integral (compact) operators. The very rich Riesz-Fredholm theory for compact operators is then an indispensable arsenal that, combined with potential theory, provides a powerful mathematical framework.

Boundary integral equation methods are also closely related to constructive techniques and are well suited for numerical computations.

Drawbacks:

- Only linear problems can be treated.
- Only problems with piece-wise constant coefficients can be treated.

In conclusion, BIE methods are not very versatile, but when they apply they are very powerful!

The Bohren decomposition – Beltrami fields

In view of the DBF constitutive relations, the Maxwell equations can be written as

$$\begin{pmatrix} \operatorname{curl} E \\ \operatorname{curl} H \end{pmatrix} = \frac{\gamma^2}{k^2} \begin{pmatrix} \beta k^2 & \mathrm{i}\,\varpi\mu \\ -\mathrm{i}\,\varpi\varepsilon & \beta k^2 \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} . \tag{27}$$

Diagonalising the matrix in (27), we obtain

$$\begin{pmatrix} \operatorname{curl}(\mathrm{i}\,\eta^{-1} E + H) \\ \operatorname{curl}(E + \mathrm{i}\,\eta\, H) \end{pmatrix} = \begin{pmatrix} \frac{k}{1-k\beta} & 0 \\ 0 & -\frac{k}{1+k\beta} \end{pmatrix} \begin{pmatrix} \mathrm{i}\,\eta^{-1} E + H \\ E + \mathrm{i}\,\eta\, H \end{pmatrix} , \tag{28}$$

where $\eta = \mu^{1/2}\varepsilon^{-1/2}$ is the intrinsic impedance of the medium.

The Bohren decomposition – Beltrami fields

Introducing the fields

$$Q_{\mathrm{L}} := \mathrm{i}\,\eta^{-1}\,E + H\,,\ \ Q_{\mathrm{R}} := E + \mathrm{i}\,\eta\,H\,,$$

we note that (28) is written as

$$\begin{aligned} \mathrm{curl} Q_{\mathrm{L}} &= \gamma_{\mathrm{L}}\, Q_{\mathrm{L}}, \\ \mathrm{curl} Q_{\mathrm{R}} &= -\gamma_{\mathrm{R}}\, Q_{\mathrm{R}}, \end{aligned} \tag{29}$$

where

$$\gamma_{\mathrm{L}} := k(1 - k\beta)^{-1}\,,\ \ \gamma_{\mathrm{R}} := k(1 + k\beta)^{-1}\,.$$

Note that $\gamma^2 = \gamma_{\mathrm{L}}\gamma_{\mathrm{R}}$. Q_{L} and Q_{R} satisfy the vector Helmholtz equation

$$\Delta Q_\lambda + \gamma_\lambda^2 Q_\lambda = 0,\ \ \lambda = L, R\,,$$

and $\gamma_{\mathrm{L}}, \gamma_{\mathrm{R}}$ are the wave numbers of the Beltrami fields Q_{L}, Q_{R}, respectively. If E, H are divergence free, the same holds for Q_{L}, Q_{R}, as well.

The Bohren decomposition – Beltrami fields

Thus we obtain the *Bohren decomposition* of E, H into Q_{L}, Q_{R}

$$\begin{aligned} E &= Q_{\mathsf{L}} - \mathrm{i}\,\eta\, Q_{\mathsf{R}}, \\ H &= Q_{\mathsf{R}} - \mathrm{i}\,\eta^{-1}\, Q_{\mathsf{L}}. \end{aligned} \tag{30}$$

From (29), and if the two complex-valued wave numbers γ_{L} and γ_{R} have positive real parts, we note that while Q_{L} is a left-handed Beltrami field, Q_{R} is a right-handed one.

This decomposition is very useful in the study of chiral media, since representation formulae for the fields in chiral media - that constitute the first basic step in developing BIE methods - can easily be deduced from the corresponding ones of metaharmonic fields.

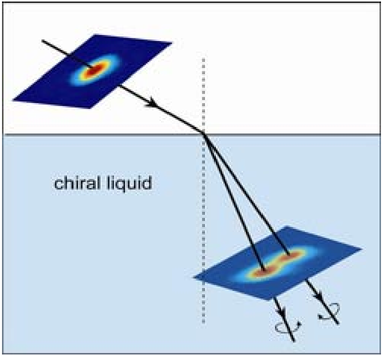


Figure: Reflection by a chiral medium

Time-domain problems: Well-posedness

Several alternative approaches to the solvability of the IVP for the Maxwell system

$$\begin{aligned} &(\mathcal{L}u)'(t) = \mathrm{M}u(t) + j(t) \quad , \quad \text{for } t > 0\,, \\ &u(0) = u_0\,, \end{aligned} \tag{31}$$

supplemented with the constitutive relations for dissipative bianisotropic media

$$(\mathcal{L}u)(t,x) = \mathrm{A}_{\mathrm{or}}(x)u(t,x) + \int_0^t \mathrm{G}_{\mathrm{d}}(t-s,x)u(s,x)\,ds\,, \tag{32}$$

can be considered, e.g., semigroups, evolution families, the Faedo–Galerkin method.
We adopt the former, based on the semigroup generated by the Maxwell operator. Then the convolution terms are treated as perturbations of this semigroup.

The choice of the semigroup approach is plausible since

- the semigroup (group actually) generated by the Maxwell operator is very well studied,
- the kernels in the convolution terms are known to be physically small, therefore, it is reasonable to consider them as perturbations.

These approaches have been used in different variations by Bossavit, Griso and Miara / Ciarlet jr. and Legendre / Ioannidis, Kristensson and $\mathbb{S}$ / Liaskos, $\mathbb{S}$ and Yannacopoulos.

For the integrodifferential equation (31) a variety of different types of solutions can be defined, regarding spatial - or temporal - regularity.

Other constitutive relations in the time-domain

Instead of using the (general) non-local in time constitutive relations $\mathsf{d} = \mathsf{A}_{\mathsf{or}} u + \mathsf{G}_{\mathsf{d}} \star u$, employed so far, a variety of problems on the class of complex media described by adopting the "DBF-like" (local in time) constitutive relations in the time-domain

$$\mathsf{d} = \mathsf{A}_0 u + \beta\, \mathsf{C}\, u$$

where

$$\mathsf{A}_0 := \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \;,\; \mathsf{C} := \begin{pmatrix} \varepsilon\, \mathrm{curl} & 0 \\ 0 & \mu\, \mathrm{curl} \end{pmatrix},$$

has been studied by a number of authors, e.g., Argyropoulou / Ciarlet jr., Legendre / Ciarlet jr., Legendre, Nicaise / Courilleau, Horsin / Courilleau, Horsin, $\mathbb{S}$ / Kravchenko , Oviedo / Liaskos, $\mathbb{S}$, Yannacopoulos.

- In these studies there are subtleties involved, regarding the spectrum of the operator curl.
- These constitutive relations may lead to nonphysical solutions.

Homogenisation

Modelling of physical processes in composite media with heterogeneous microstructure is a common problem in Material Science and Engineering. The analysis of such problems requires solving differential equations with rapidly varying coefficients.

Homogenisation is a mathematical theory for deriving the form of the effective homogeneous medium as the length scale of the microstructure approaches zero.

It is a mathematical theory for studying

- differential operators with rapidly oscillating coefficients
- BVPs with rapidly changing boundary conditions
- equations in perforated domains

The first results in homogenisation theory were obtained by E. De Giorgi and S. Spagnolo in the late 1960s.

Some *sine qua non* names / alphabetical order

- (mostly) Periodic Homogenisation
  G. Allaire, I. Babuška, A. Bensoussan, D. Cioranescu, A. Damlamian, P. Donato, E. De Giorgi, G. Francfort, G. Griso, J.-L. Lions, B. Miara, F. Murat, G. Nguetseng, O. A. Oleinik, G. Papanicolaou, É. Sanchez-Palencia, S. Spagnolo, L. Tartar

- (mostly) Stochastic Homogenisation
  S. N. Armstrong, L. A. Caffarelli, P. Cardaliaguet, G. A. Chechkin, G. Dal Maso, Y. Efendiev, A. Gloria, V. V. Jikov, S. Kozlov, C. Le Bris, P.-L. Lions, L. Modica, F. Otto, A. Pankov, A. L. Piatnitski, A. S. Shamaev, C. K. Smart, P. E. Souganidis, S. R. S. Varadhan, G. A. Yosifian, V. Yurinskii

Homogenisation techniques:

- multiscale convergence
- asymptotic expansion method
- compensated compactness
- Γ-convergence
- G-convergence
- H-convergence
- Young measures
- periodic unfolding method
- stochastic homogenisation

Self-averaging environments

- Periodic, quasi-periodic (linear combination of periodic of incommensurate periods), almost periodic (closure of quasi-periodic).
- Random: for each experiment we encounter a different material (lack of information of the exact composition, or true randomness); the frequency of appearance of each particular material configuration is described by a probability measure on the space of all possible configurations .

  For a random environment to be self-averaging, statistical self repetition is required (from a sufficiently large block of a particular realisation of the material, one should be able to reconstruct all possible realisations of the material at a particular point): stationarity and ergodicity.

  This is a kind of a generalisation of (classical) periodicity, in which from a single cell the whole material can be reproduced by translations.

1–dim periodic example

Consider the problem of steady-state heat diffusion in a rod whose conductivity profile is given by the function $a^\epsilon(x) = a(\epsilon^{-1}\, x)$, where a is a bounded periodic function defined on the physical domain $\mathcal{O} := (0, 1)$, with the temperature being fixed equal to zero at the end points of $\mathcal{O}$.

$$-\frac{d}{dx}\left(a^\epsilon\,\frac{du^\epsilon}{dx}\right) = f\ ,\ \ x \in \mathcal{O} = (0,1)\,,$$

$$u^\epsilon = 0\ ,\ \ x \in \partial\mathcal{O} = \{0,1\}\,.$$

The right-hand side function f describes a source-term.
Since a is periodic, considering a^ϵ with successively smaller values of ϵ implies working with rapidly oscillating conductivity functions.
Speaking in terms of material properties, considering successively smaller values of ϵ entails considering conductors with successively finer microstructure.

The basic question of homogenisation is what happens as $\epsilon \to 0$?

Is there a limiting homogenised material ?
In mathematical terminology:

- Do u^ϵ converge to a limit ?
- If so, in what topology does the convergence take place ?
- Can we describe/characterise the limit ?

The above questions are answered in the classical homogenisation result:

Theorem

The functions u^ϵ converge in $L^2(\mathcal{O})$-norm to u^o, which is the solution of the problem:

$$-\frac{d}{dx}\left(a^o\,\frac{du^o}{dx}\right) = f\ ,\ x \in \mathcal{O} = (0,1)\,,$$

$$u^o = 0\ ,\ x \in \partial\mathcal{O} = \{0,1\}\,,$$

where a^o is the harmonic mean of a over the interval $(0,1)$:

$$a^o := \left(\int_0^1 \frac{1}{a(x)}\,dx\right)^{-1}\,.$$

The coefficient a^o is called the homogenised coefficient, or the effective conductivity.

Homogenisation in Electromagnetics

Within the electromagnetic (applied physics/electrical engineering) community, homogenisation of composites has a huge literature, the major part of which is devoted to dielectrics.

In this community, the related literature on bianisotropic composites is much smaller. Among the recent developments are the work on Maxwell Garnett and Bruggeman formalisms for different classes of bianisotropic inclusions and the work on the Strong Property Fluctuation Theory for bianisotropic composites.

There are many important applications, e.g., in biomedical engineering and optics (optical waveguides, high-dielectric thin-film capacitors, captive video disc units, novel antennas and design of complementary split-ring resonators).

For isotropic media there is important rigorous mathematical work by many authors: see, in particular, the contribution by Artola and Cessenat / Bensoussan, J.-L. Lions and Papanicolaou / Jikov, Kozlov and Oleinik / Markowich and Poupaud / Sanchez-Hubert / Sanchez-Palencia / Visintin / Wellander.

For dissipative bianisotropic media the problem was originally studied by Barbatis and $\mathbb{S}$ (2003), then by $\mathbb{S}$ and Yannacopoulos (2012), and further developed by Barbatis, $\mathbb{S}$ and Yannacopoulos (2015 & in progress).

Related work by Bossavit, Griso and Miara (2005) / Sjöberg (2005) / J.S. Jiang, C.K. Lin and C.H. Liu (2008) / L.Cao, Y. Zhang, Allegretto and Y. Lin (2010).

The periodic unfolding method and two scale convergence

In 1990, Arbogast, Douglas and Hornung defined a “dilation” operator to study homogenisation for a periodic medium with double porosity.
In 2002, Cioranescu, Damlamian and Griso expanded this idea and presented a general and simple approach for classical or multiscale periodic homogenisation, under the name periodic unfolding method.
This method is essentially based on two ingredients: the unfolding operator (which is similar to the dilation operator and whose effect is to “zoom” the microscopic structure in a periodic manner), and the separation of the characteristic scales by decomposing every function $\phi \in W^{1,p}(\mathcal{O})$ into two parts; this scale-splitting can be either achieved by using the local average, or by a procedure inspired by the Finite Element Method.
The periodic unfolding method simplifies many of the two-scale convergence proofs.

Nonlinear problems

Regarding chiral media, although third-order nonlinear effects were predicted as early as 1967, nonlinear optical rotation experiments were not undertaken before 1993.
We consider a nonlinear complex electromagnetic medium modelled by

$$\mathrm{d} = \mathcal{L}u = \mathrm{A}_0 u + \mathrm{G}_0 \star u + \mathrm{G}_{0,\mathrm{n}\ell} \star \mathrm{N}(u)u\,,$$

where the linear part is given by

$$\mathrm{A}_0(x) = \begin{bmatrix} \varepsilon(x) & 0 \\ 0 & \mu(x) \end{bmatrix}, \mathrm{G}_0\,(t,x) = \begin{bmatrix} \varepsilon(x)\chi^{\mathrm{e}}(t) & \chi^{\mathrm{em}}(t) \\ \chi^{\mathrm{me}}(t) & \mu(x)\chi^{\mathrm{m}}(t) \end{bmatrix},$$

while the nonlinearity is given by

$$\mathrm{N}(u) := \begin{bmatrix} N_1|u_1|^q & 0 \\ 0 & N_2|u_2|^q \end{bmatrix},\;\; \mathrm{G}_{0,\mathrm{n}\ell}\,(t,x) := \begin{bmatrix} \chi^{\mathrm{e}}_{\mathrm{n}\ell}(t,x) & 0 \\ 0 & \chi^{\mathrm{m}}_{\mathrm{n}\ell}(t,x) \end{bmatrix},$$

where $q \in \mathbb{N}$, $N_1, N_2 \in \mathbb{R}^{3\times 3}$ are matrices independent of the spatial and temporal variables.

Let $\mathrm{B_A} := \mathrm{G}_0^{-1}\mathrm{G}_{0,n\ell}(0)$, $\mathrm{G}_{A,n\ell} := \mathrm{G}_0^{-1}\mathrm{G}'_{0,n\ell}$, $\mathrm{M_A} := \mathrm{G}_0^{-1}\mathrm{M}$, $\mathrm{G_A} := -\mathrm{G}_0^{-1}\mathrm{G}'_0$, $\mathrm{J_A} := \mathrm{G}_0^{-1} j$.

Then, under suitable regularity assumptions on G_0 and $\mathrm{G}_{0,n\ell}$, the Maxwell system takes the form

$$u' + \mathrm{B_A}\mathrm{N}(u)u + \mathrm{G}_{A,n\ell} \star \mathrm{N}(u)u = \mathrm{M_A}u + \mathrm{G_A} \star u + \mathrm{J_A}\,,$$

with initial condition

$$u(0,x) = u_0(x), \quad x \in \mathcal{O}\,,$$

and the perfect conductor boundary condition

$$n \times u_1 = 0, \quad (t,x) \in [0,T] \times \partial\mathcal{O}\,.$$

Presently, more general nonlinearities are being considered (Kristensson, $\mathbb{S}$, Wellander, Yannacopoulos).

Nonlinear systems present interesting types of solutions in the form of travelling waves that propagate with unchanged shape through the medium as an effect of the interplay between dispersion and nonlinearity. This type of behaviour is typical and is very well studied in integrable systems; however, solutions of similar type are often present in nonintegrable systems and find important applications in various branches of science.

A formal approach to the evolution of nonlinear waves in chiral media with weak dispersion and weak nonlinearity of the Kerr type in the low chirality case has been studied by Frantzeskakis, $\mathbb{S}$, Yannacopoulos and by Tsitsas, Frantzeskakis, Lakhtakia.

A set of modulation equations is obtained for the evolution of the slowly varying field envelopes that is in the form of 4 coupled nonlinear Schrödinger equations. This set of equations is nonintegrable; however, with the use of reductive perturbation theory, under certain conditions these equations may be reduced to an integrable system, the Melnikov system. This system is known to possess vector soliton solutions. Thus, by the above reduction, in certain (limiting) cases the existence of vector solitons in chiral media may be shown; these appear in pairs of dark and bright solitons. Depending on the chosen behaviour at infinity, the dark component can be along the right-handed component of the field and the bright component along the left-handed component of the field, or vice versa.

Stochastic problems

Recently there has been a considerable amount of work on the **stochastic** Maxwell system for random (finite variation of source term $\rightarrowtail$ partial integrodifferential equation with random coefficients) (or, even stochastic (infinite variation of source term $\rightarrowtail$ SPDE)) bianisotropic media.
Main lines:

- Solvability
  Semigroup methods, Wiener-chaos method
  (Liaskos, $\mathbb{S}$, Yannacopoulos)
- Controllability
  Difficulties related to the adaptivity of the solution to the filtration generated by the noise process / study of approximate controllability by random PDE and SPDE approaches
  ($\mathbb{S}$, Yannacopoulos / Liaskos, Pantelous, $\mathbb{S}$)
- Homogenisation
  Ergodicity + Stationarity / generalisation to systems of stochastic scalar elliptic regularity results by Blanc, Le Bris and P.-L. Lions / Laplace transform
  (Barbatis, $\mathbb{S}$, Yannacopoulos)

Obviously the topic is too wide for me to manage to exhaust it.

So all I can hope for, is that, at least, YOU are not exhausted...!

Thank you for your patience!

A domani!