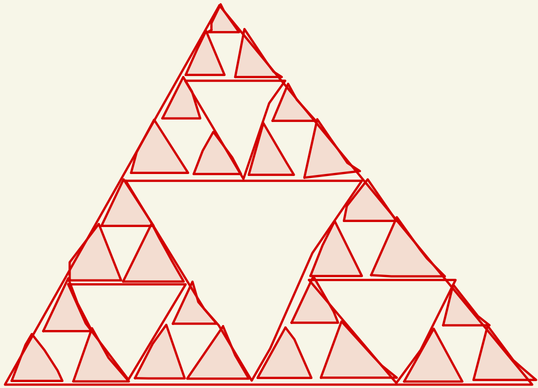
Sobolev Spaces, Integral Equations, and Scattering on non-Lipschitz and Fractal Sets

Simon Chandler-Wilde,

University of Reading



Part I: Intro

Time-harmonic acoustics

$U(x,t)$ = air pressure at x at time t

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Assume t-dependence sinusoidal. Then

$$U(x,t) = A(x)\cos(\phi(x) - \omega t)$$

where $\omega = 2\pi f$ ← frequency

Time-harmonic acoustics

$U(x,t)$ = air pressure at x at time t

$$\Delta U = \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2}$$

Assume t-dependence sinusoidal. Then

$$U(x,t) = \mathrm{Re}\left(u(x)e^{-i\omega t}\right)$$

$$u(x) = A(x)e^{i\phi(x)}$$

Time-harmonic acoustics

$U(x,t)$ = air pressure at x at time t

$$\Delta U = \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} \qquad k = \frac{\omega}{c}$$

$$\Updownarrow$$

$$\Delta u + k^2 u = 0$$

$$U(x,t) = \mathrm{Re}\left(u(x)\,e^{-i\omega t}\right)$$

$$u(x) = A(x)\,e^{i\phi(x)}$$

Eg

$$u(x) = e^{ikx_1} \Rightarrow \Delta u = -k^2 e^{ikx_1},$$

$$U(x,t) = \cos(kx_1 - \omega t)$$

$$\Delta U = \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} \qquad k = \frac{\omega}{c}$$

$$\Delta u + k^2 u = 0$$

$$U(x,t) = \mathrm{Re}\left(u(x) e^{-i\omega t}\right)$$

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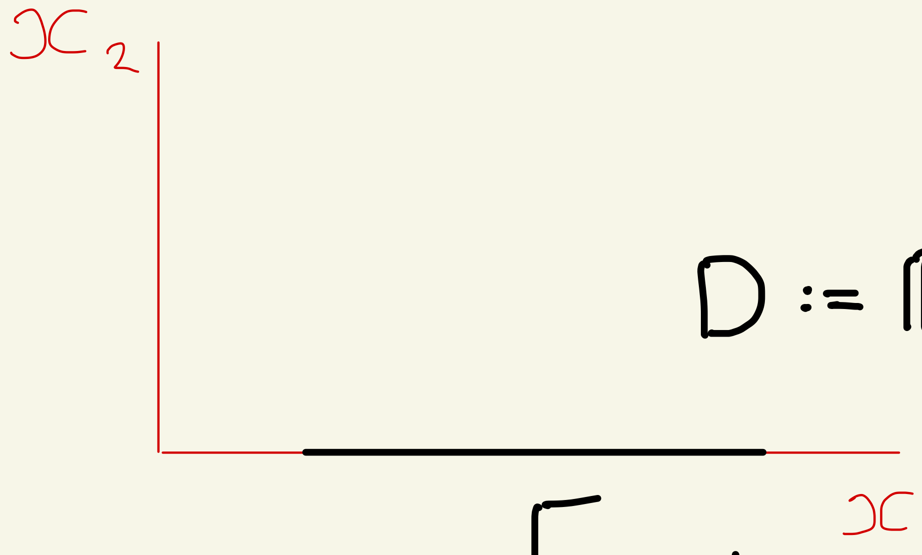
Our simple geometry

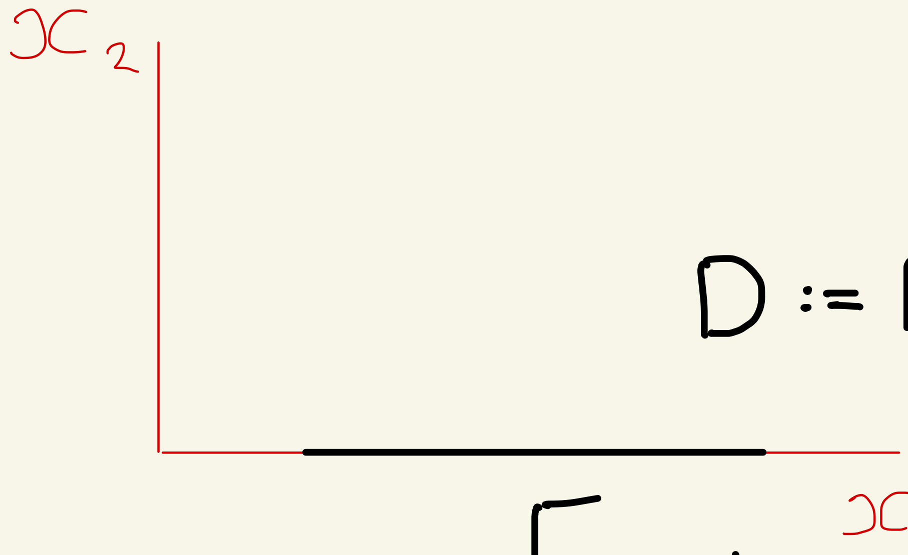


$D := \mathbb{R}^2 \setminus \Gamma$

Γ, closed

Given $g \in L^2(D)$, $\mathrm{supp}(g) \subset D$ & compact, find $u \in \tilde{H}^1(D)$ s.t.

$$\Delta u + k^2 u = g \quad \text{in } D$$



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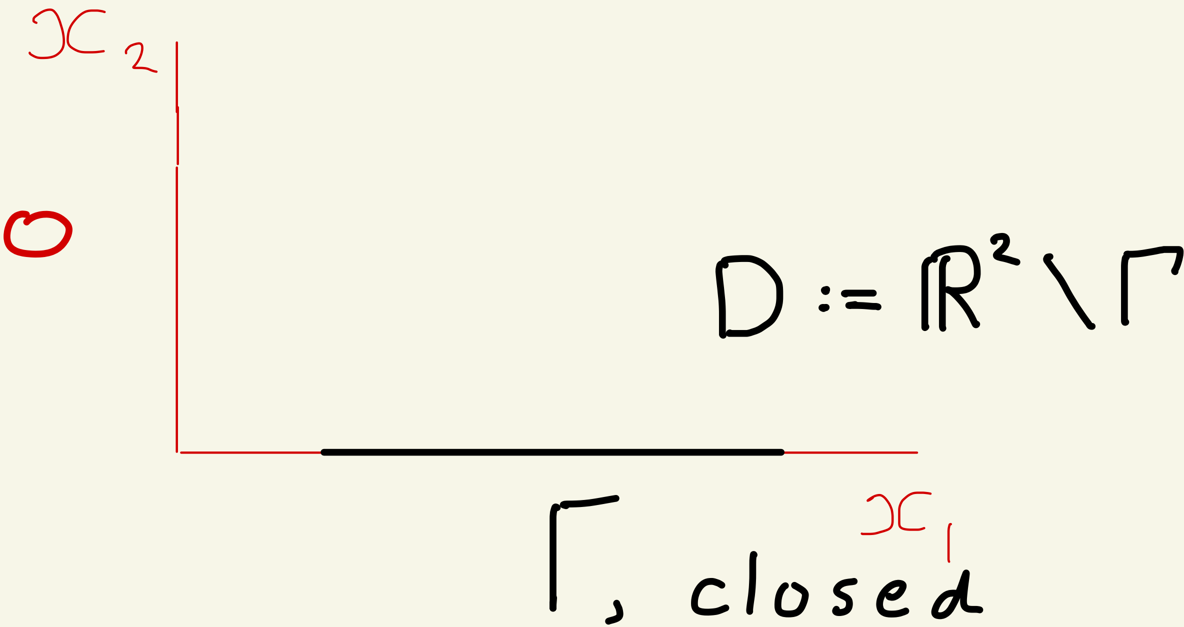
$k \in \mathbb{C}$,

$k_i := \operatorname{Im}(k) > 0$



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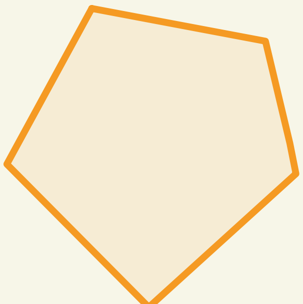
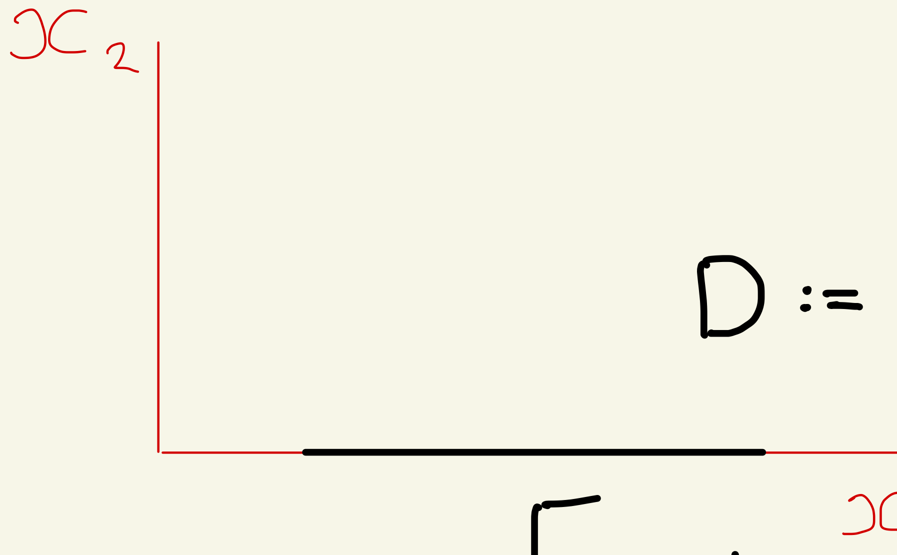


Supp(g)



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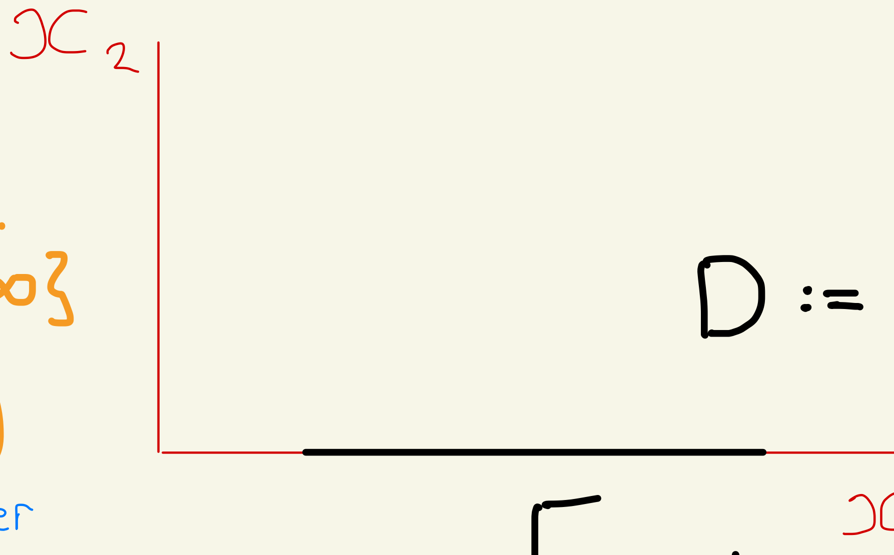
$$= \overline{C_0^\infty(D)}^{H^1(\mathbb{R}^2)}$$

$$H^1(\mathbb{R}^2) = \{ v \in L^2(\mathbb{R}^2) . \ \|v\| < \infty \}$$

$$\|v\|^2 = \int_D \left(|v|^2 + |\nabla v|^2 \right)$$

dist der



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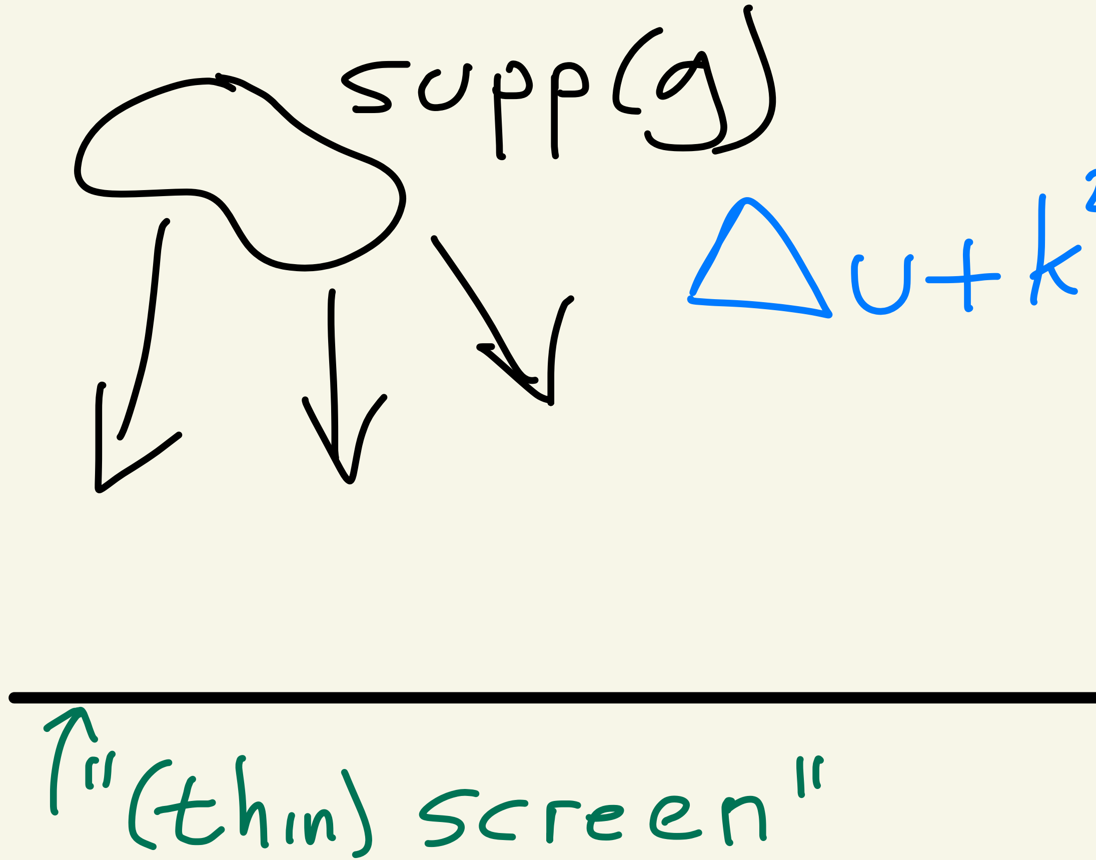
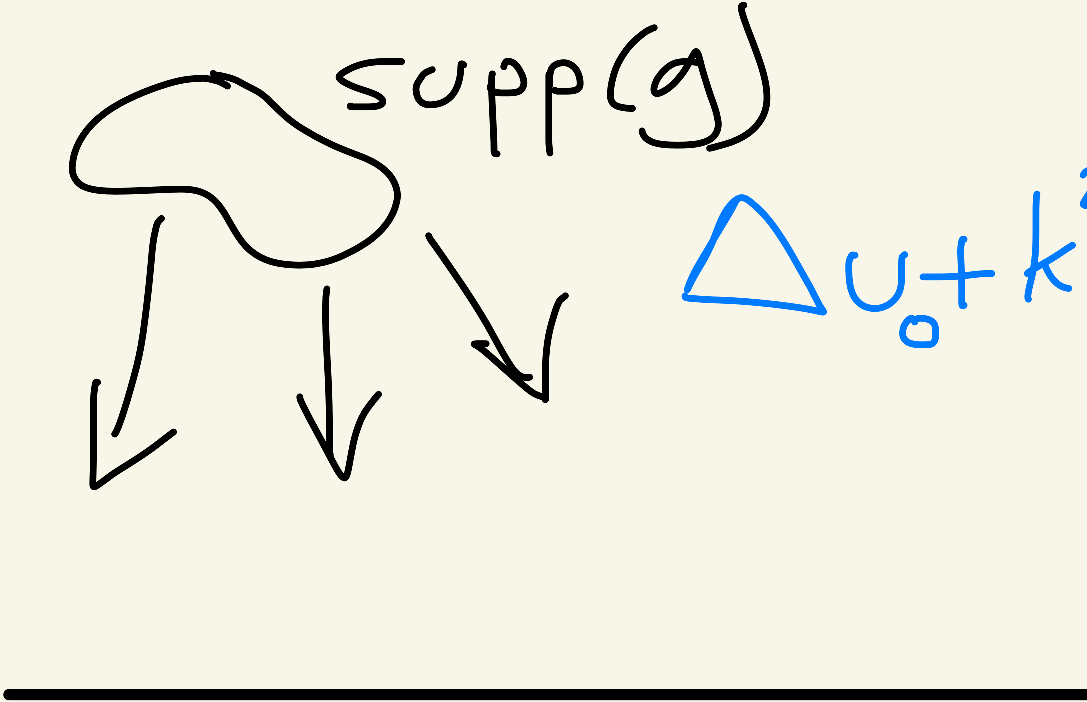
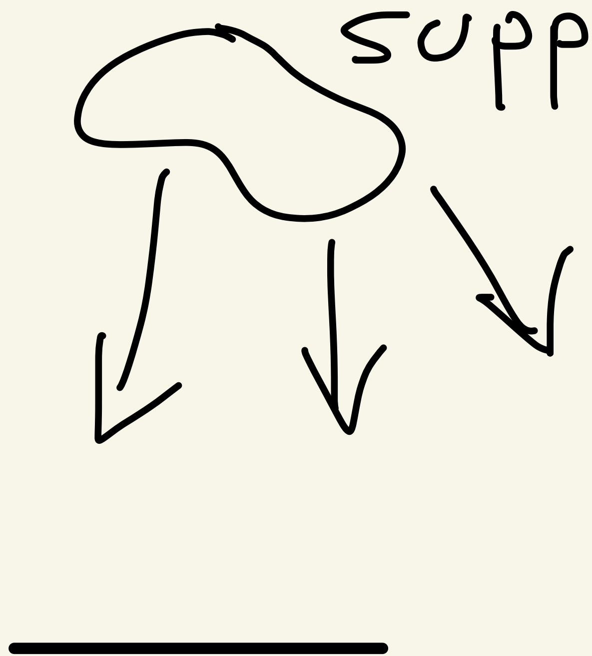




$$\Delta u_0 + k^2 u_0 = g$$

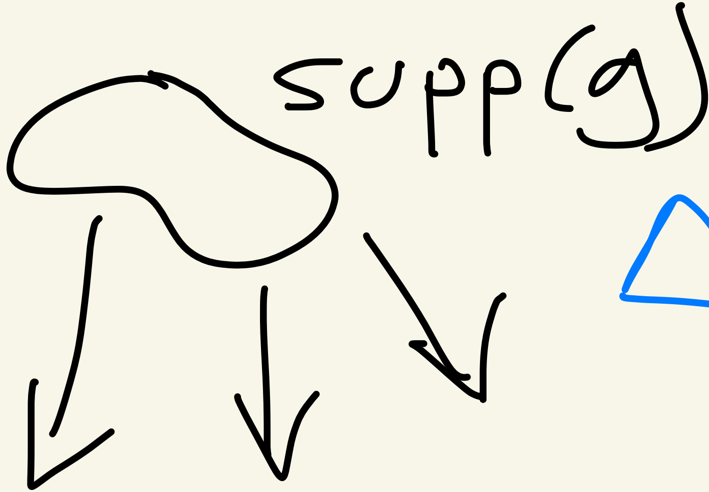
$$j = 0$$

$\mathrm{supp}(g)$



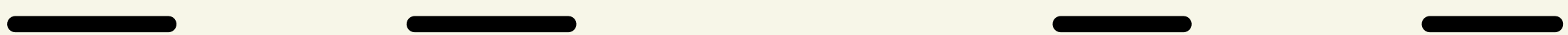
$$\Delta u_1 + k^2 u_1 = g$$

$$j = 1$$



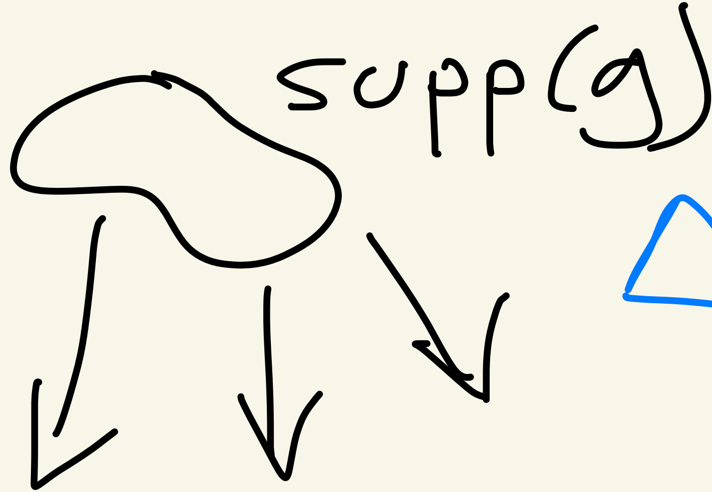
$$\Delta u_2 + k^2 u_2 = g$$

$j = 2$



What is the Cantor set limit,

$$\lim_{j\to\infty} u_j \,?$$



$$\Delta u_2 + k^2 u_2 = g$$

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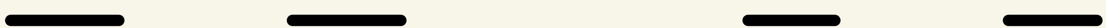
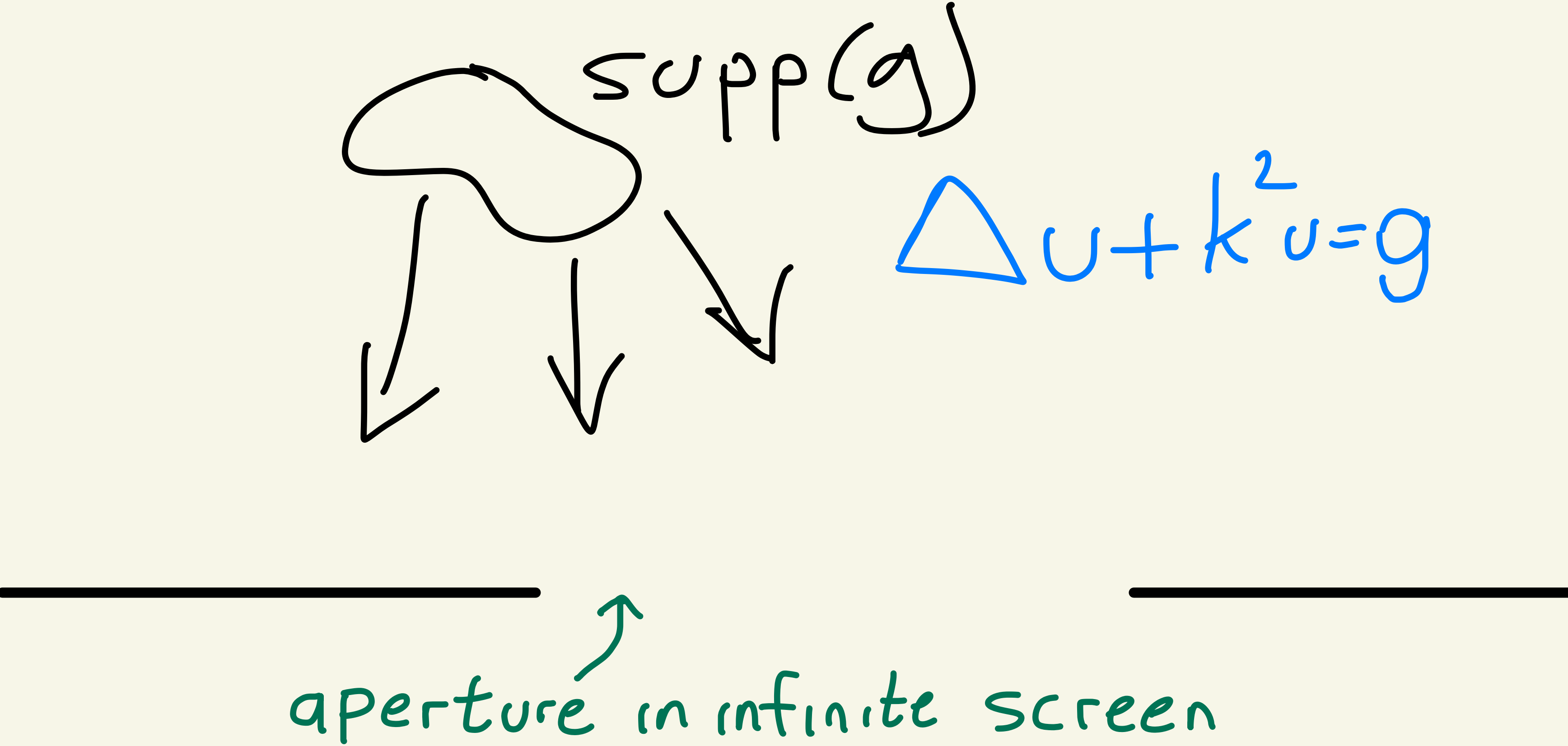


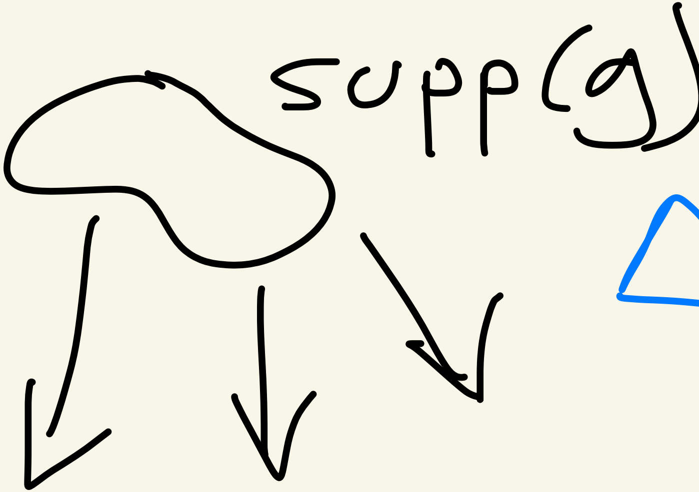
What is the Cantor set limit,

$$\lim_{j \to \infty} u_j \text{ ?}$$

Surely just $u = 0$!



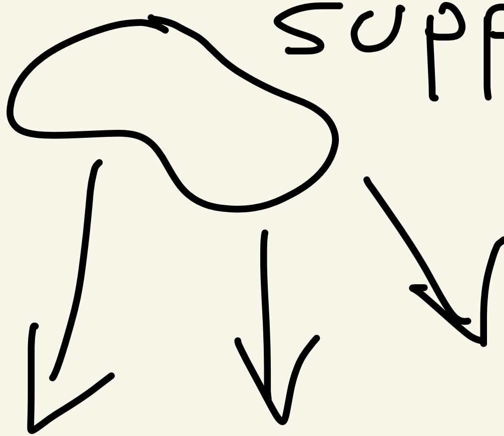
$$\Delta u + k^2 u = g$$



$$\Delta u_0 + k^2 u_0 = g$$

$$j = 0$$

supp(g)



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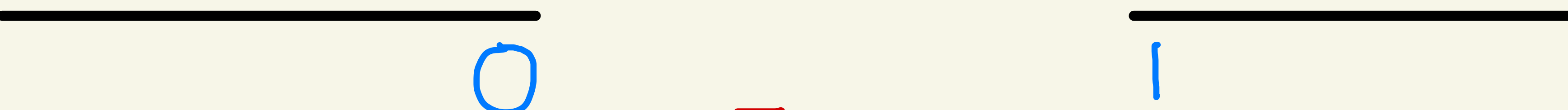
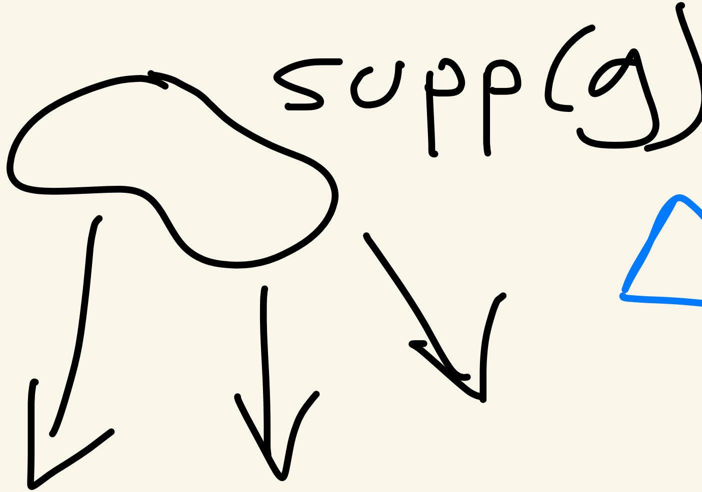
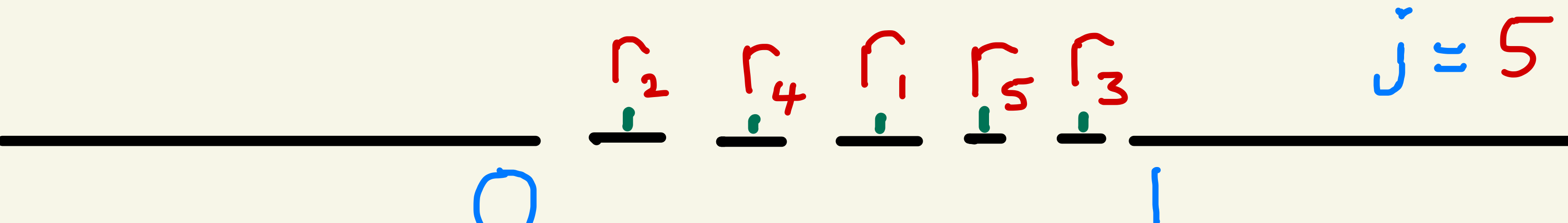


At step j add $[r_j - \epsilon_j, r_j + \epsilon_j]$, centred on jth rational $r_j \in (0,1)$

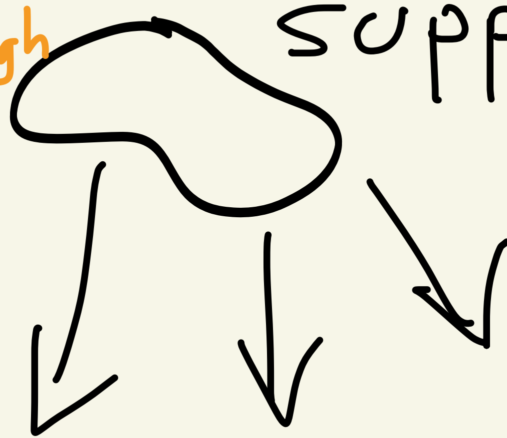


$$\Delta u_5 + k^2 u_5 = g$$



At step j add $[r_j - \epsilon_j, r_j + \epsilon_j]$, centred on jth rational $r_j \in (0,1)$

Does any sound get through in limit $j \to \infty$? Surely not!



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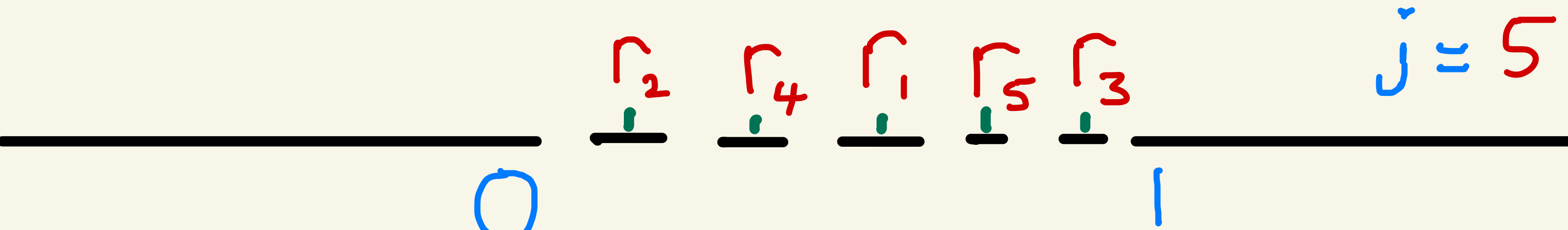

At step j add $[r_j - \epsilon_j, r_j + \epsilon_j]$, centred on jth rational $r_j \in (0,1)$

- Formulations as variational problems in D (BVP) and on Γ (integral eqn)

- VPs in Hilbert spaces, Lax-Milgram, Mosco convergence of solns of sequences of VPs

- Application to sequences of scattering problems - and answering our Qs!

Part 2: Variational Problems in Hilbert Spaces and Mosco Convergence*

*and application to the weak BVP VP formulation in D

$\boxed{\text{V.P.}}$ Given $f \in H^*$ find $u \in H$ s.t.

$$a(u,v) = \langle f, \bar{v} \rangle,^{*} \quad \forall\, v \in H$$

*See note regarding correction on next page

* Note regarding correction on previous page

I've changed $\langle f, v \rangle$ to $\langle f, \bar{v} \rangle$ on the previous because my notation is that H^* = cont linear functionals and the mapping $v \mapsto a(u,v)$ is anti-linear but $v \mapsto \langle f, v \rangle$ is linear

But what is $\bar{v}$? $\bar{v} = Jv$ where J is any fixed "conjugate map" on H, i e any fixed anti-linear isometric involution, e.g. concretely

$$v = \sum_{j \in I} a_j v_j \longmapsto \sum_{j \in I} \bar{a}_j v_j ,$$

where $\{ v_j \ j \in I \}$ is an orthonormal basis for H and $a_j \in \mathbb{C}$

If H is a function space $v \mapsto \bar{v}$ will just be complex conjugation

$\boxed{\text{V.P.}}$ Given $f \in H^*$ find $u \in H$ st

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H = Hilbert space

H^* = dual space

$a(\cdot,\cdot)$ = cont., sesquilinear form

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H^* = dual space

$a(\cdot,\cdot)$ = cont., sesquilinear form

$a(\ ,v)$ linear, $a(u,\cdot)$ antilinear,

$$|a(u,v)| \leq C\|u\|\,\|v\|, \quad \forall u,v \in H$$

Given $g \in L^2(D)$, $\text{supp}(g) \subset D$ & compact, find $u \in \tilde{H}^1(D)$ s.t.

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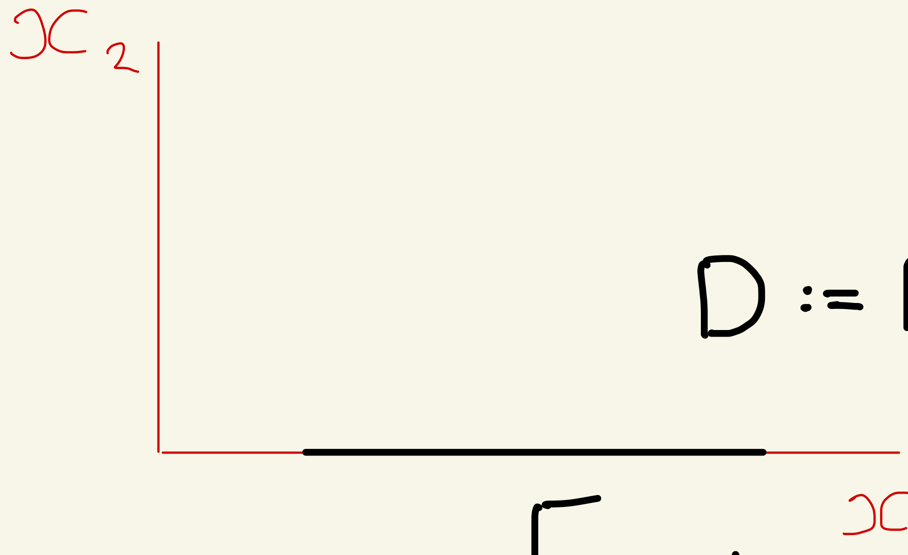

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Understood in sense of distributions

ie as $\langle \Delta u + k^2 u, v \rangle = \langle g, v \rangle$, $\forall v \in C_0^\infty(D)$

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where $\langle w, v \rangle := \int_D w v$ if $w \in L^1_{loc}(D)$

and $\langle \Delta w, v \rangle = -\int_D \nabla w \nabla v$ if $\nabla w \in L^1_{loc}(D)$

Given $g \in L^2(D)$, $\mathrm{supp}(g) \subset D$ & compact, find $u \in \tilde{H}^1(D)$ s.t.

$$\Delta u + k^2 u = g \quad \text{in } D$$

$$\Longleftrightarrow$$

$$a(u,v) = \langle g, \bar{v} \rangle, \quad \forall v \in C_0^\infty(D)$$

where $a(u,v) = \int_D (k^2 u \bar{v} - \nabla u \nabla \bar{v})$, $\forall u, v \in \tilde{H}^1(D)$

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where $a(u,v) = \int_D (k^2 u \bar{v} - \nabla u \nabla \bar{v})$

Cont as $|a(u,v)| \leq \max(1, |k|^2) \int_D (|uv| + |\nabla u||\nabla v|)$

$\leq C(\|u\|_2 \|v\|_2 + \|\nabla u\|_2 \|\nabla v\|_2) \leq 2C \, \|u\| \, \|v\|$ H^1 norm

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H^1 norm

$\boxed{\text{Lax-Milgram Lemma}}$ Suppose $a(\cdot,\cdot)$

coercive, i.e. $|a(u,u)| \geq c\|u\|^2, \forall u \in H$

Lax-Milgram Lemma Suppose $a(\cdot,\cdot)$ coercive, i.e. $|a(u,u)| \geq c\|u\|^2, \forall u \in H$

Then

$$\boxed{\text{find } u \in H \text{ s.t. } a(u,v) = \langle f, \bar{v} \rangle, \quad \forall v \in H}$$

has exactly one soln and $\|u\| \leq \frac{1}{c}\|f\|$

$\forall f \in H^*$

Given $g \in L^2(D)$, $\operatorname{supp}(g) \subset D$ & compact, find $u \in \tilde{H}^1(D)$ s.t.

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$\iff$

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Coercive as

$$|a(u,u)| = \frac{1}{|k|} \left| \int_D \Big(\underset{k|k|^2}{\cancel{\bar{k} k^2}} |u|^2 - \bar{k} |\nabla u|^2 \Big) \right| \geq \operatorname{Im}(\ ) = \frac{1}{|k|} \int_D k_i \left(|k|^2 |u|^2 + |\nabla u|^2 \right)$$

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$k|k|^2$

$$\geq \frac{k_1}{|k|} \min\left(1, |k|^2\right) \boxed{\int_D \left(|u|^2 + |\nabla u|^2\right)} = \|u\|^2$$

Given $g \in L^2(D)$, $\text{supp}(g) \subset D$ & compact, find $u \in \tilde{H}^1(D)$ st

$$\Delta u + k^2 u = g \quad \text{in } D$$

$$\Longleftrightarrow \quad \boxed{a(u,v) = \langle g, \bar{v} \rangle, \ \forall v \in \tilde{H}^1(D)}$$

Applying L–M these have exactly one soln and

$$\|u\|_{H^1} \leq c(k) \, \|g\|_{L^2}$$

Sequences of VPs

H = Hilbert space, V, V_j $(j \in \mathbb{N})$ closed subspaces, $g \in H^*$

(1) $\boxed{\text{Find } u \in V \text{ st } a(u,v) = \langle g, \bar{v} \rangle, \ \forall v \in V}$

(2) $\boxed{\text{Find } u_j \in V_j \text{ st } a(u_j,v_j) = \langle g, \bar{v}_j \rangle, \ \forall v_j \in V_j}$

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Do (1) and (2) have solutions? Does $u_j \to u$?

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Do (1) and (2) have solutions? Does $u_j \to u$?

WELL KNOWN WHEN $V=H$ (Céa's Lemma)

Weak convergence

For $v \in H$, $v_j \in H$,

$$v_j \rightharpoonup v \iff (v_j, w) \to (v, w), \ \forall\, w \in H$$

$\boxed{\text{Weak convergence}}$ For $v \in H$, $v_j \in H$,

$$v_j \rightharpoonup v \iff (v_j, w) \to (v, w),\ \forall\, w \in H$$

$$\iff \langle g, v_j \rangle \to \langle g, v \rangle,\ \forall\, g \in H^*$$

by Riesz rep thm

Weak convergence

For $v \in H$, $v_j \in H$,

$$v_j \rightharpoonup v \iff (v_j, w) \to (v, w), \ \forall \, w \in H$$

- $v_j \rightharpoonup v \implies \sup_j \|v_j\| < \infty$
- $\sup_j \|v_j\| < \infty \implies v_{j_m} \rightharpoonup v$, for some $v \in H$, subsequence j_m

Weak convergence

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- $\sup_j \|v_j\| < \infty \implies v_{j_m} \rightharpoonup v$, for some $v \in H$, subsequence j_m
- $v_j \rightharpoonup v_j$ $a(\cdot,\cdot)$ cont $\Rightarrow a(u, v_j) \to a(u, v)$, $\forall u \in V$

Weak convergence

For $v \in H$, $v_j \in H$,

$$v_j \rightharpoonup v \iff (v_j, w) \to (v, w), \ \forall\, w \in H$$

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- $\sup_j \|v_j\| < \infty \implies v_{j_m} \rightharpoonup v$, for some $v \in H$, subsequence j_m
- $v_j \rightharpoonup v$, $a(\cdot,\cdot)$ cont $\Rightarrow a(u, v_j) \to a(u, v)$, $\forall u \in V$

If also $u_j \to u$ then $|a(u_j - u, v_j)| \leq C \|u_j - u\| \|v_j\| \to 0$

so $a(u_j, v_j) = a(u_j - u, v_j) + a(u, v_j) \to a(u, v)$

Lemma Suppose (1) and (2) have unique solutions and $u_j \to u$ $\forall g \in H^*$ Then

$V_j \xrightarrow{M} V$ (Mosco convergence)

$\Leftrightarrow$

(A) $\forall v \in V \ \exists v_j \in V_j$ st $v_j \to v$

Lemma Suppose (1) and (2) have unique solutions and $u_j \to u$ $\forall g \in H^*$. Then

$$V_j \xrightarrow{M} V \text{ (Mosco convergence)}$$

$$\overset{\text{def}}{=} \begin{cases} \text{(A) } \forall v \in V \ \exists v_j \in V_j \text{ st } v_j \to v \\ \text{(B) If } v_{j_m} \in V_{j_m} \text{ and } v_{j_m} \to v \text{ then } v \in V \end{cases}$$

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$V_j \xrightarrow{M} V$ (Mosco convergence)

$\Leftrightarrow$

(A) $\forall v \in V$ $\exists v_j \in V_j$ st $v_j \to v$ ✓

if $V \subset V_j$ - take $v_j := v$

(B) If $v_j \in V_j$ and $v_j \rightharpoonup v$ then $v \in V$

Lemma Suppose (1) and (2) have unique solutions and $u_j \to u$ $\forall g \in H^*$ Then

$V_j \xrightarrow{M} V$ (Mosco convergence)

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✓ if $V_j \subset V$ - as $v_j \in V$ so $v \in V$

Lemma Suppose (1) and (2) have unique solutions and $u_j \to u$ $\forall g \in H^*$ Then

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Lemma Suppose (1) and (2) have unique solutions and $u_j \to u$ $\forall g \in H^*$. Then

(A) $\forall v \in V \; \exists v_j \in V_j$ st $v_j \to v$

Proof

(1) Find $u \in V$ st $a(u,v) = \langle g, \bar{v} \rangle$, $\forall v \in V$

(2) Find $u_j \in V_j$ st $a(u_j, v_j) = \langle g, \bar{v}_j \rangle$, $\forall v_j \in V_j$

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Then soln of (1) is $u = v$

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Proof Define $g \in H^*$ by $\langle g, w \rangle = a(v, \bar{w})$, $\forall w \in H$

Then soln of (1) is $u = v$

Define $v_j := u_j$, where u_j is soln of (2)

Then $u_j \to u$ so $v_j \to v$

Lemma Suppose (1) and (2) have unique solutions and $u_j \to u$ $\forall g \in H^*$ Then

(B) If $v_{j_m} \in V_{j_m}$ and $v_{j_m} \to v$ then $v \in V$

Proof

Lemma Suppose (1) and (2) have unique solutions and $u_j \to u$ $\forall g \in H^*$. Then

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Proof Suppose $v \notin V$. Define $g \in H^*$ by $\langle g, av+w \rangle = a$, $\forall a \in \mathbb{C}$, $w \in V$, and extend g to H by Hahn-Banach

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$\to a(u, v) = \langle g, \bar{v} \rangle = 1$

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Then $a(u_{j_m}, v_{j_m}) = \langle g, \bar{v}_{j_m} \rangle \to \langle g, \bar{v} \rangle = 0$

CONTRADICTION!

Exs of Mosco Convergence

N.B V, V_j closed

$$V_j \xrightarrow{M} V$$

$$\Leftrightarrow \begin{cases} \text{(A) } \forall v \in V \; \exists v_j \in V_j \text{ st } v_j \to v \\ \text{(B) If } v_{j_m} \in V_{j_m} \text{ and } v_{j_m} \to v \text{ then } v \in V \end{cases}$$

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1 $V_1 \subset V_2 \subset \quad , \quad V := \overline{\bigcup_{j \in \mathbb{N}} V_j}$

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1 $V_1 \subset V_2 \subset \quad , \; V := \overline{\bigcup_{j \in \mathbb{N}} V_j}$

Choose $v \in V$ Choose integers $1 \leq k_1 < k_2$ st

$\exists\, w_n \in V_{k_n}$ with $\|v - w_n\| < n^{-1}$, $n = 1, 2,$

$$V_J \xrightarrow{M} V$$

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J	1	k_1	3	4	k_2	6	k_3	k_4	9	10	k_5	12	...
V_J													...

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J	1	k_1	3	4	k_2	6	k_3	k_4	9	10	k_5	12	...
v_J	v_1	w_1	w_1	w_1	w_2	w_2	w_3	w_4	w_4	w_4	w_5	w_5	...

Exs of Mosco Convergence

$$V_j \xrightarrow{M} V$$

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② $V_1 \supset V_2 \supset \cdots$, $V := \bigcap_{j \in \mathbb{N}} V_j$

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② $V_1 \supset V_2 \supset$, $V := \bigcap_{j \in \mathbb{N}} V_j$

(A) Take $v_j := v$, $j = 1, 2,$

$$V_j \xrightarrow{M} V$$

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② $V_1 \supset V_2 \supset \cdots$, $V := \bigcap_{j \in \mathbb{N}} V_j$

(B) $\forall n \in \mathbb{N}$, $v_{j_m} \in V_{j_m} \subset V_n$ if $j_m \geq n$, so $v \in V_n$

Thus $v \in \bigcap_{n \in \mathbb{N}} V_n = V$

Exs of Mosco Convergence

$V_j \xrightarrow{M} V$

$\Leftrightarrow \begin{cases} \text{(A) } \forall v \in V \ \exists v_j \in V_j \text{ st } v_j \to v \\ \text{(B) If } v_{j_m} \in V_{j_m} \text{ and } v_{j_m} \rightharpoonup v \text{ then } v \in V \end{cases}$

③ $V = H$, V_j finite dim, $\inf_{v_j \in V_j} \|v - v_j\| \to 0$ as $j \to \infty$, $\forall v \in H$

Exs of Mosco Convergence

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③ $V = H$, V_j finite dim, $\inf_{v_j \in V_j} \|v - v_j\| \to 0$ as $j \to \infty$, $\forall v \in H$

Classical Numerical Analysis setting.

③ is the Galerkin method

$\boxed{\text{Lemma}}$ If $a(\cdot,\cdot)$ is coercive* then (1) and (2) have unique solns and

$$u_j \to u, \ \forall g \in H^* \iff V_j \xrightarrow{M} V$$

* and continuous

Lemma If $a(\cdot,\cdot)$ is coercive* then (1) and (2) have unique solns and

$$u_j \to u, \ \forall\, g \in H^* \iff V_j \xrightarrow{M} V$$

(1) $\boxed{\text{Find } u \in V \text{ st } a(u,v) = \langle g, \bar{v} \rangle, \ \forall v \in V}$

(2) $\boxed{\text{Find } u_j \in V_j \text{ st } a(u_j,v_j) = \langle g, \bar{v}_j \rangle, \ \forall v_j \in V_j}$

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Lemma If $a(\cdot,\cdot)$ is coercive* then (1) and (2) have unique solns and

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Proof That (1) and (2) have unique solns is immediate from Lax-Milgram. Moreover L-M $\implies \|u_j\| \leq c\|g\|, \ \forall j$

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$\implies$ was last lemma

So suppose $V_j \xrightarrow{M} V$

* and continuous

Lemma If $a(\cdot,\cdot)$ is coercive then (1) and (2) have unique solns and

$$u_j \to u, \ \forall g \in H^* \iff V_j \xrightarrow{M} V$$

Proof That (1) and (2) have unique solns is immediate from Lax-Milgram. Moreover L-M $\Rightarrow \|u_j\| \leq c\|g\|, \ \forall j$

$\Rightarrow$ was last lemma

So suppose $V_j \xrightarrow{M} V$. Since u_j bounded $\exists u_{j_m} \rightharpoonup u^*$, and $u^* \in V$ by (B)

Lemma If $a(\cdot,\cdot)$ is coercive then (1) and (2) have unique solns and

$$u_j \to u, \ \forall g \in H^* \Longleftrightarrow V_j \xrightarrow{M} V$$

Proof

So suppose $V_j \xrightarrow{M} V$ Since u_j bounded $\exists u_{j_m} \rightharpoonup u^*$, and $u^* \in V$ by (B)

Step 1 Show that $\boxed{a(u,w) = a(u^*,w), \forall w \in V} \Rightarrow u^* = u$

Lemma If $a(\cdot,\cdot)$ is coercive then (1) and (2) have unique solns and

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So suppose $V_j \xrightarrow{M} V$. Since u_j bounded $\exists u_{j_m} \rightharpoonup u^*$, and $u^* \in V$ by (B)

Step 1 Show that $\boxed{a(u,w) = a(u^*,w), \forall w \in V} \Rightarrow u^* = u$

Step 2 Show $u_j \rightharpoonup u$ and then $u_j \to u$ (coercivity)

Proof

So suppose $V_j \xrightarrow{M} V$. Since u_j bounded

$\exists\, u_{j_m} \rightharpoonup u^*_j$ and $u^* \in V$ by (B)

Step 1 Show that $\boxed{a(u,w) = a(u^*,w),\ \forall w \in V} \Rightarrow u^* = u$

Suppose $w \in V$. By (A) $\exists\, w_j \in V_j$ s t $w_j \to w$

$a(u,w) = \langle g, \bar{w} \rangle = \langle g, \bar{w} - \bar{w}_j \rangle + \langle g, \bar{w}_j \rangle$

u satisfies (1)

Proof

So suppose $V_j \xrightarrow{M} V$ Since u_j bounded $\exists\, u_{j_m} \rightharpoonup u^*$, and $u^* \in V$ by (B)

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u satisfies (1)

$$= \langle g, \bar{w} - \bar{w}_j \rangle + a(u_j, w_j)$$

u_j satisfies (2)

Proof

So suppose $V_j \xrightarrow{M} V$ Since u_j bounded

$\exists\, u_{j_m} \rightharpoonup u^*,$ and $u^* \in V$ by (B)

Step 1 Show that $\boxed{a(u,w) = a(u^*,w),\ \forall w \in V} \Rightarrow u^* = u$

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u satisfies (1)

$$= \langle g, \underbrace{\bar{w} - \bar{w}_{j_m}}_{\to 0} \rangle + a(\underbrace{u_{j_m}}_{\to u^*}, \underbrace{w_{j_m}}_{\to w})$$

Proof

So suppose $V_j \xrightarrow{M} V$. Since u_j bounded

$\exists\, u_{j_m} \rightharpoonup u^*$, and $u^* \in V$ by (B)

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u satisfies (1)

$$= \underbrace{\langle g, \bar{w} - \bar{w}_{j_m} \rangle}_{\to 0} + a(\underset{\to u^*}{u_{j_m}}, \underset{\to w}{w_{j_m}})$$

$$\to 0 + a(u^*, w)$$

Proof So suppose $V_j \xrightarrow{M} V$. By Step 1 $\exists$ subsequance j_m with $u_{j_m} \longrightarrow u$

Step 2 Show $u_j \rightarrow u$ and then $u_j \rightarrow u$

By same argument every subsequence has a subsequence $\longrightarrow u$, so $u_j \longrightarrow u$.

Proof So suppose $v_j \xrightarrow{M} v$. By Step 1 $\exists$ subsequence j_m with $u_{j_m} \longrightarrow u$

Step 2 Show $u_j \to u$ and then $u_j \to u$

By same argument every subsequence has a subsequence $\longrightarrow u$, so $u_j \longrightarrow u$.

Finally, since $a(\cdot,\cdot)$ coercive,

$$\underbrace{|a(u-u_j, u-u_j)|}_{e_j} \geq c \, \|u-u_j\|^2$$

Proof So suppose $v_j \xrightarrow{M} v$ By Step 1

$\exists$ subsequnce j_m with $u_{j_m} \longrightarrow u$

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$$|\underbrace{a(u-u_j, u-u_j)}_{e_j}| \geq c \, \|u-u_j\|^2$$

and

$$e_j = a(u_j, u_j - u) - a(u_j, u - u_j) = a(u_j, u_j) - a(u_j - u)$$

$$- a(u, u_j - u)$$

Proof So suppose $V_j \xrightarrow{M} V$. By Step 1

$\exists$ subsequance j_m with $u_{j_m} \longrightarrow u$

Step 2 Show $u_j \rightharpoonup u$ and then $u_j \to u$

By same argument every subsequence has a subsequence $\longrightarrow u$, so $u_j \longrightarrow u$.

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and

$$e_j = a(u_j, u_j - u) - a(u_j, u-u_j) = \underbrace{a(u_j, u_j)}_{\langle g, \bar{u}_j\rangle \to \langle g, \bar{u}\rangle} - \underbrace{a(u_j, u)}_{\to a(u,u)}$$

$$- a(u, u_j - u)$$

Proof So suppose $V_j \xrightarrow{M} V$ By Step 1 $\exists$ subsequance j_m with $u_{j_m} \rightharpoonup u$

Step 2 Show $u_j \rightharpoonup u$ and then $u_j \to u$

By same argument every subsequence has a subsequence $\rightharpoonup u$, so $u_j \rightharpoonup u$.

Finally, since $a(\cdot,\cdot)$ coercive,

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Since $a(u,u) = \langle g, \bar{u}\rangle$,
$e_j \to 0$

$\boxed{\text{Proof}}$ So suppose $v_j \xrightarrow{M} v$. By Step 1 $\exists$ subsequence j_m with $u_{j_m} \longrightarrow u$

Step 2 Show $u_j \longrightarrow u$ and then $u_j \to u$

By same argument every subsequence has a subsequence $\longrightarrow u$, so $u_j \longrightarrow u$.

Finally, since $a(\cdot,\cdot)$ coercive,

$$|\underbrace{a(u-u_j, u-u_j)}_{e_j}| \geq c\,\|u-u_j\|^2$$

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$\boxed{\text{Lemma}}$ If $a(\cdot,\cdot)$ is coercive* then (1) and (2) have unique solns and

$$u_j \to u, \ \forall g \in H^* \iff V_j \xrightarrow{M} V$$

* and continuous

Recap Part 2

- VPs $a(u,v) = \langle g, v \rangle$, $a(u_J, v_J) = \langle g, v_J \rangle$, $u, v \in V$, $u_J, v_J \in V_J$
- $a(\cdot,\cdot)$ cont., coercive $\left(|a(u,v)| \leq C \|u\| \|v\|, \right.$
  $\left. |a(u,u)| \geq c \|u\|^2 \right)$

Recap Part 2

- VPs $a(u,v) = \langle g, v \rangle$, $a(u_j, v_j) = \langle g, v_j \rangle$, $u, v \in V$, $u_j, v_j \in V_j$
- $a(\cdot,\cdot)$ cont., coercive $\left(|a(u,v)| \leq C \|u\| \|v\|, \; |a(u,u)| \geq c \|u\|^2 \right)$
- Defn $V_j \xrightarrow{M} V$
- If $a(\cdot,\cdot)$ cont, coercive, then

$$u_j \to u, \; \forall g \in H^* \iff V_j \xrightarrow{M} V$$