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METHOD OF COMPLEX POTENTIAL IN THE THEORY OF COMPOSITES

Vladimir Mityushev
(Pedagogical University of Kraków, Poland)

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ABSTRACT

Method of complex potential is a powerful constructive method of applied complex analysis where an analytic function plays the fundamental role. A wide class of physical two-dimensional stationary fields are described by harmonic and biharmonic functions, hence, by analytic functions. The relation between analytic and harmonic functions in simply and multiply connected domains are derived. The classic problems of mathematical physics are stated as boundary value problems. The special attention is devoted to problems of the theory of composites. The methods of integral and functional equations associated to Schwarz's alternating method are presented. The constructive homogenization procedure is explained in terms of the elliptic functions on torus.

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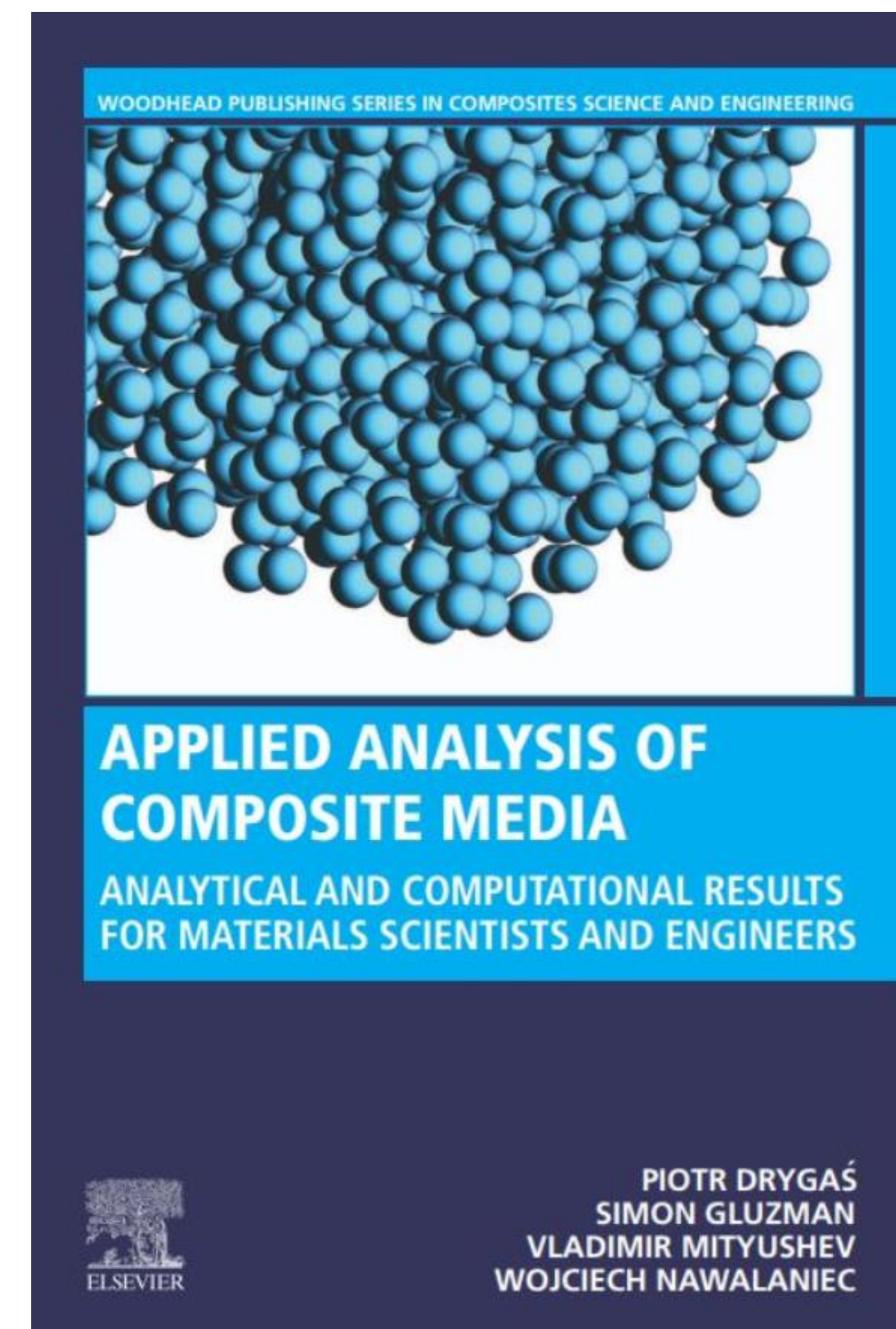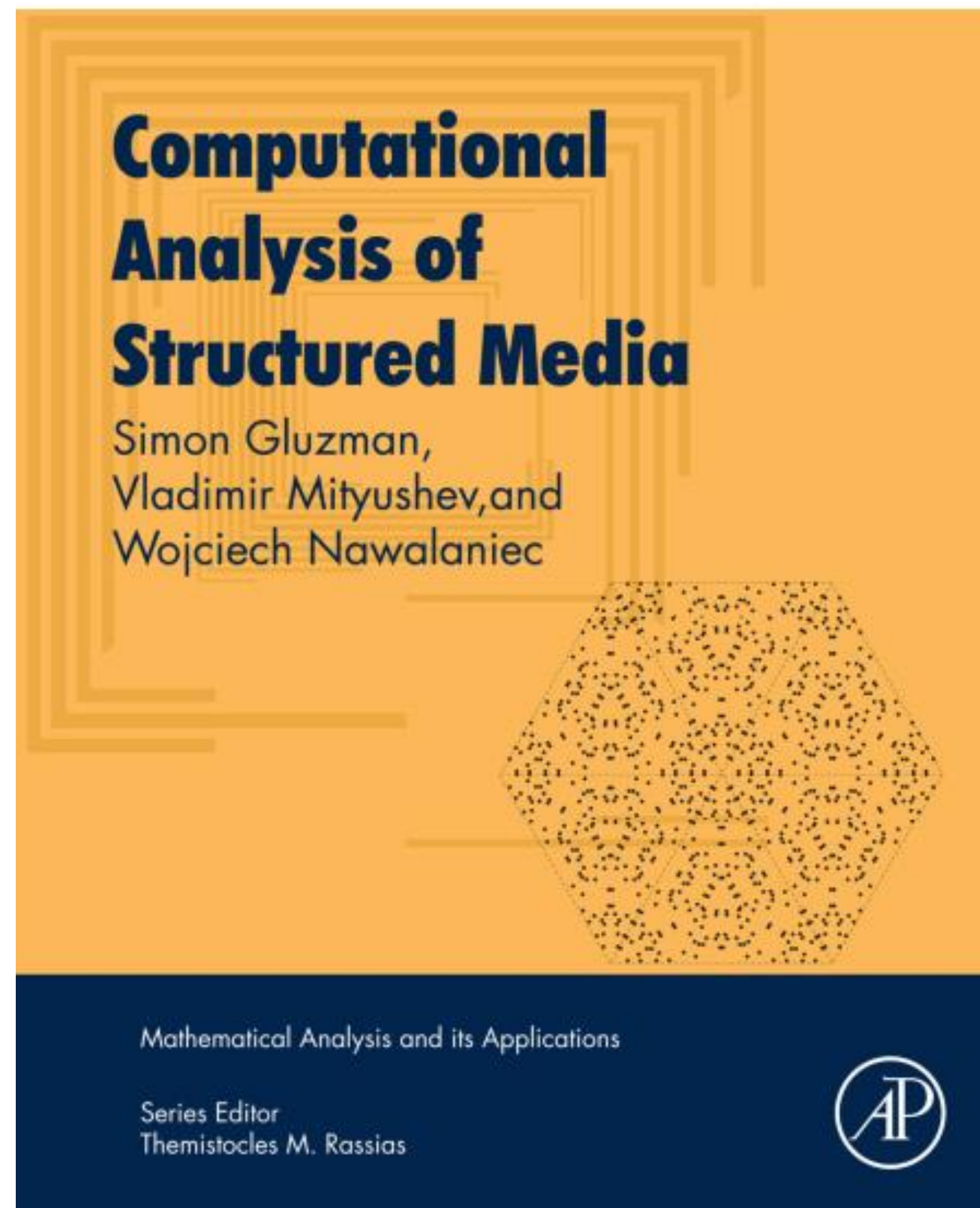
In many interesting cases, the general problems of continuum mechanics can be reduced to boundary value problems for 2D Laplace's equation

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = 0 \qquad (2.1.1)$$

in a domain $D \subset \mathbb{R}^2$. The function $u(x_1, x_2)$ is called harmonic in D. Let $z = x_1 + ix_2$ denote a complex variable. It is well known that any function $u(x_1, x_2)$ harmonic in a simply connected domain D can be presented as the real part of the function $\varphi(z)$ analytic in D

$$u(x_1, x_2) = \mathrm{Re}\ \varphi(z), \quad z \in D. \qquad (2.1.2)$$

The function $\varphi(z)$ is called *complex potential*. This equation relates the important in application class of partial differential equations and analytic functions. This is the main reason why we pay attention to boundary value problems stated and solved in terms of Complex Analysis.

```
f[z_] = z Sin[z] + 1/(z^2 + 1); u[x_, y_] = Re[f[x + i y]];

Plot3D[u[x, y], {x, 0, 1}, {y, 0, 1}]
```
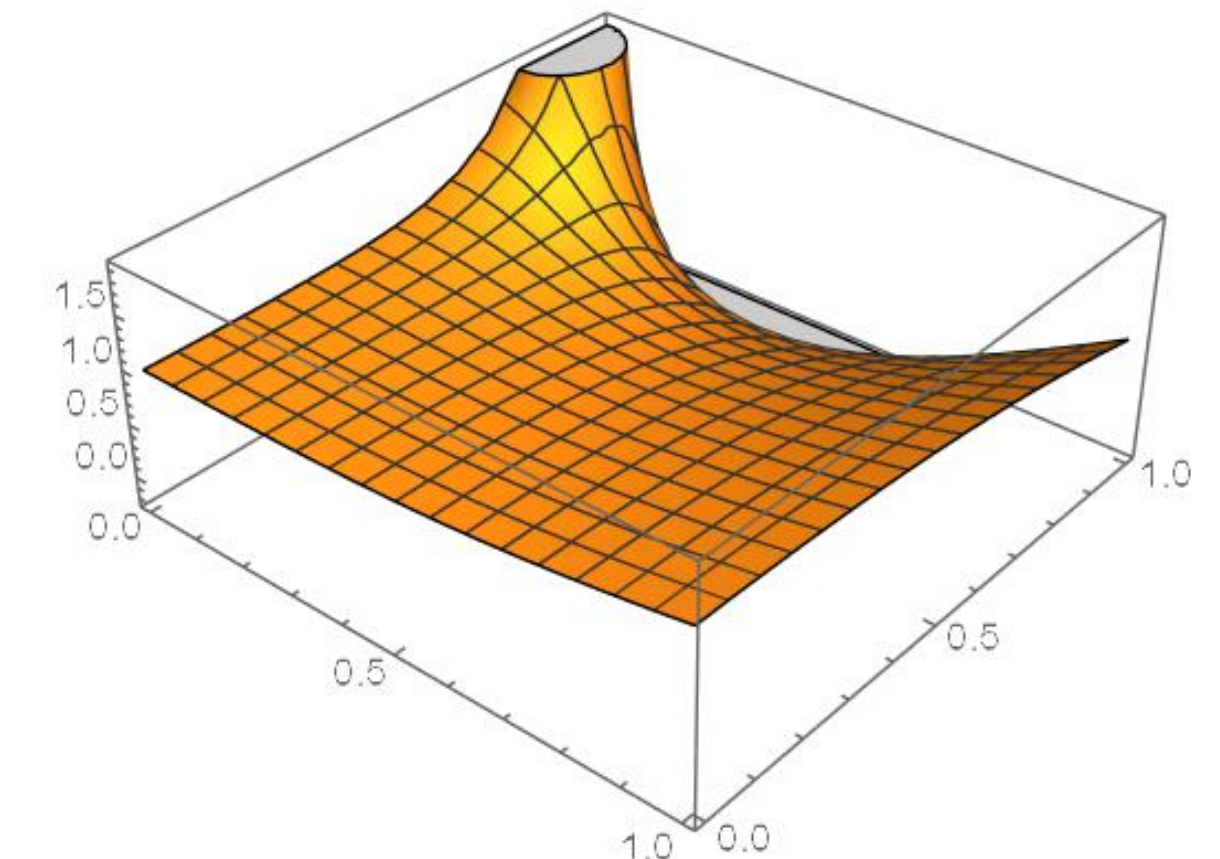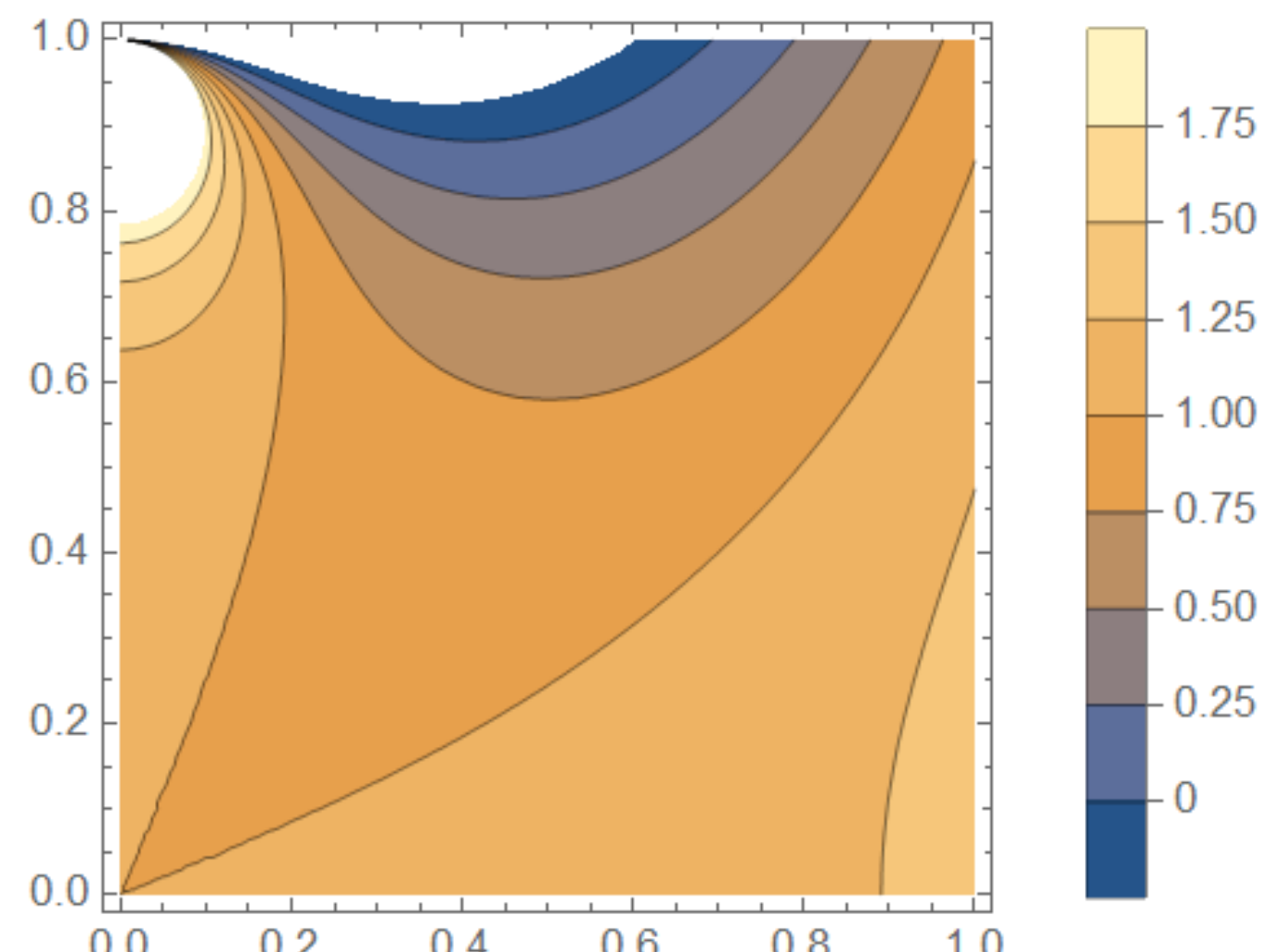
The function $\varphi(z)$ is determined by $u(z)$ up to an arbitrary purely imaginary additive constant due to the formula

$$\varphi(z) = u(z) + iv(z), \quad \text{where } v(z) = \int\limits_{w}^{z} -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy + C, \qquad (2.1.16)$$

where w is an arbitrary point in D and C is a real constant.

```
ContourPlot[u[x, y], {x, 0, 1},
  {y, 0, 1}, PlotLegends → Automatic]
```

Let L be a finite union of simple closed curves on $\widehat{\mathbb{C}}$. Functions continuous in L form the Banach space of *continuous functions* $C(L)$ endowed with the norm

$$\| \varphi \|_C := \max_{z \in L} | \varphi(z) | .$$

Analogous space is introduced for functions continuous in $D \cup \partial D$ with the norm

$$\| \varphi \|_C := \max_{z \in D \cup \partial D} | \varphi(z) | . \tag{2.1.6}$$

A function φ is called the Hölder continuous function on L (we write $\varphi \in \mathcal{H}^\alpha(L)$) with the power $0 < \alpha \leq 1$ if there exists such a constant $C > 0$ that $|\varphi(x) - \varphi(y)| < C|x - y|^\alpha$ for all $x, y \in L$. $\mathcal{H}^\alpha(L)$ is a subspace of the continuous functions. It is a Banach space with the norm

$$\| \varphi \|_\alpha := \| \varphi \|_C + \sup_{x,y \in L, x \neq y} \frac{| \varphi(x) - \varphi(y) |}{| x - y |^\alpha} . \tag{2.1.7}$$

$$\log z := \log|z| + i \arg z$$

```
ContourPlot[Re[Log[x + i y]], {x, -1, 1}, {y, -1, 1},
  PlotLegends → Automatic, ColorFunction → "TemperatureMap"]
```



```
ContourPlot[Im[Log[x + i y]], {x, -1, 1}, {y, -1, 1},
  PlotLegends → Automatic, ColorFunction → "RoseColors"]
```
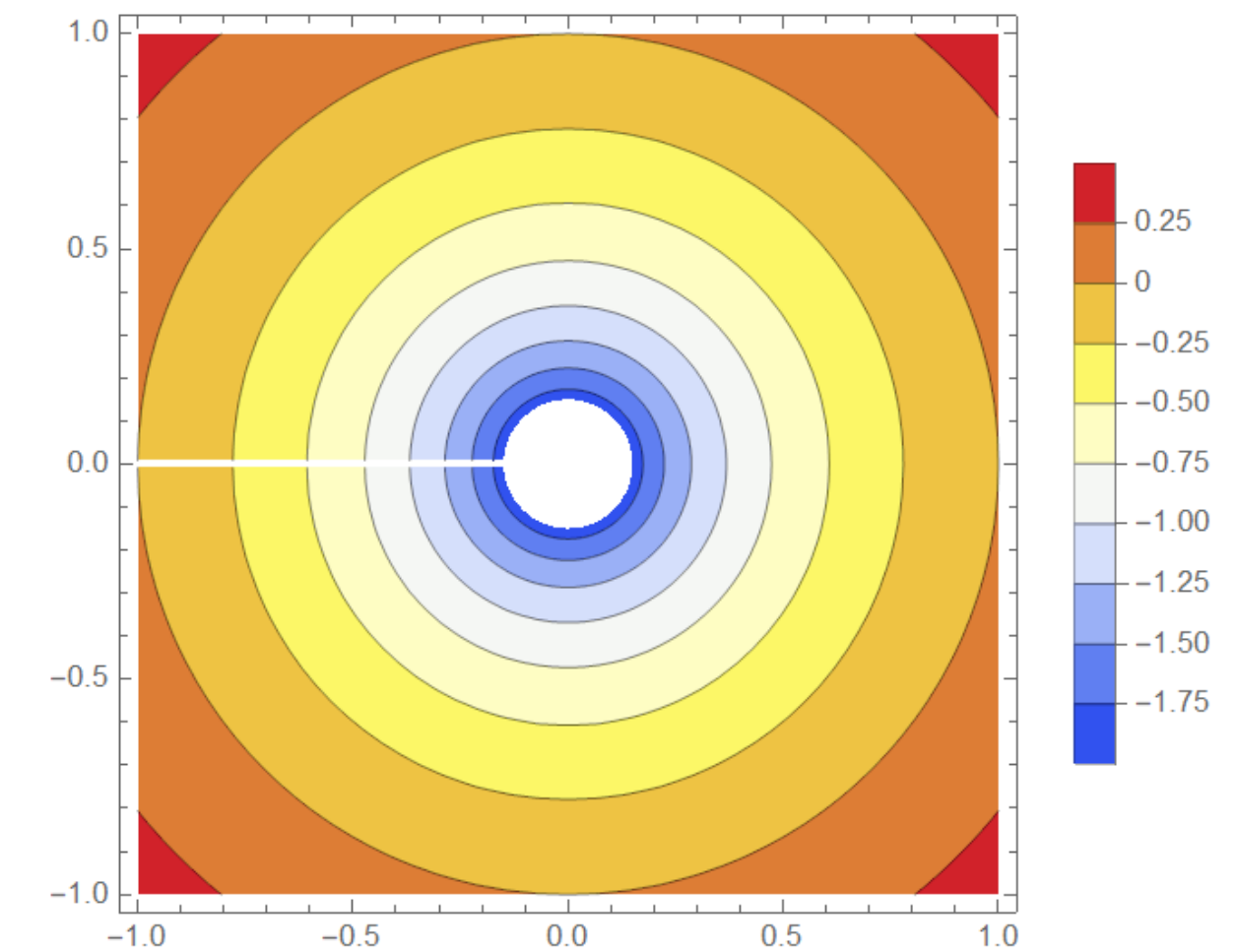

Singular points of the complex potentials are supposed to be poles and logarithms. The complex logarithm $\log z = \log|z| + i \arg z$ is introduced as a branch of the multi-valued function $\mathrm{Log}\, z = \log|z| + i \arg z + 2\pi i k$ $(k \in \mathbb{Z})$ inverse to the exponential function. The argument $\arg z$ of the complex number z can be fixed, for instance, as $0 \le \arg z < 2\pi$. Then, the function $\log z$ is uniquely defined in $\mathbb{C} \backslash [0, +\infty]$. It is assumed that the cut $[0, +\infty]$ has two sides considered as different lines. The real function $\log x$ is defined for positive x as the limit $\log x = \lim_{z \to x, Im\, z>0} \log z$. From other side, we have $\lim_{z \to x, Im\, z<0} \log z = \log x + 2\pi i$; hence, the complex function $\log z$ has the jump $2\pi i$ across the half-axis $(0, +\infty)$. The cut $(0, +\infty)$ can be replaced by another smooth simple curve Γ connecting the points $z = 0$ and $z = \infty$. Then, the logarithm is defined in $\mathbb{C} \backslash \Gamma$ by equation $\log z = \log|z| + i \arg z$ where the argument has the jump $2\pi i$ when z passes across Γ. Frequently, Γ is taken as $(-\infty, 0)$, then $-\pi < \arg z \le \pi$. It is worth noting that the function $\mathrm{Re}\, \log z = \log|z|$ is always single-valued contrary to $\mathrm{Im}\, \log z = \arg z$. This is the reason why a single-valued harmonic function does not necessarily yield by (2.1.16) a single-valued analytic function in a multiply connected domain. The function $\frac{Q}{2\pi} \log z$ for real Q determines a source or sink at zero of intensity Q. This follows from the total normal derivative calculated through the integral
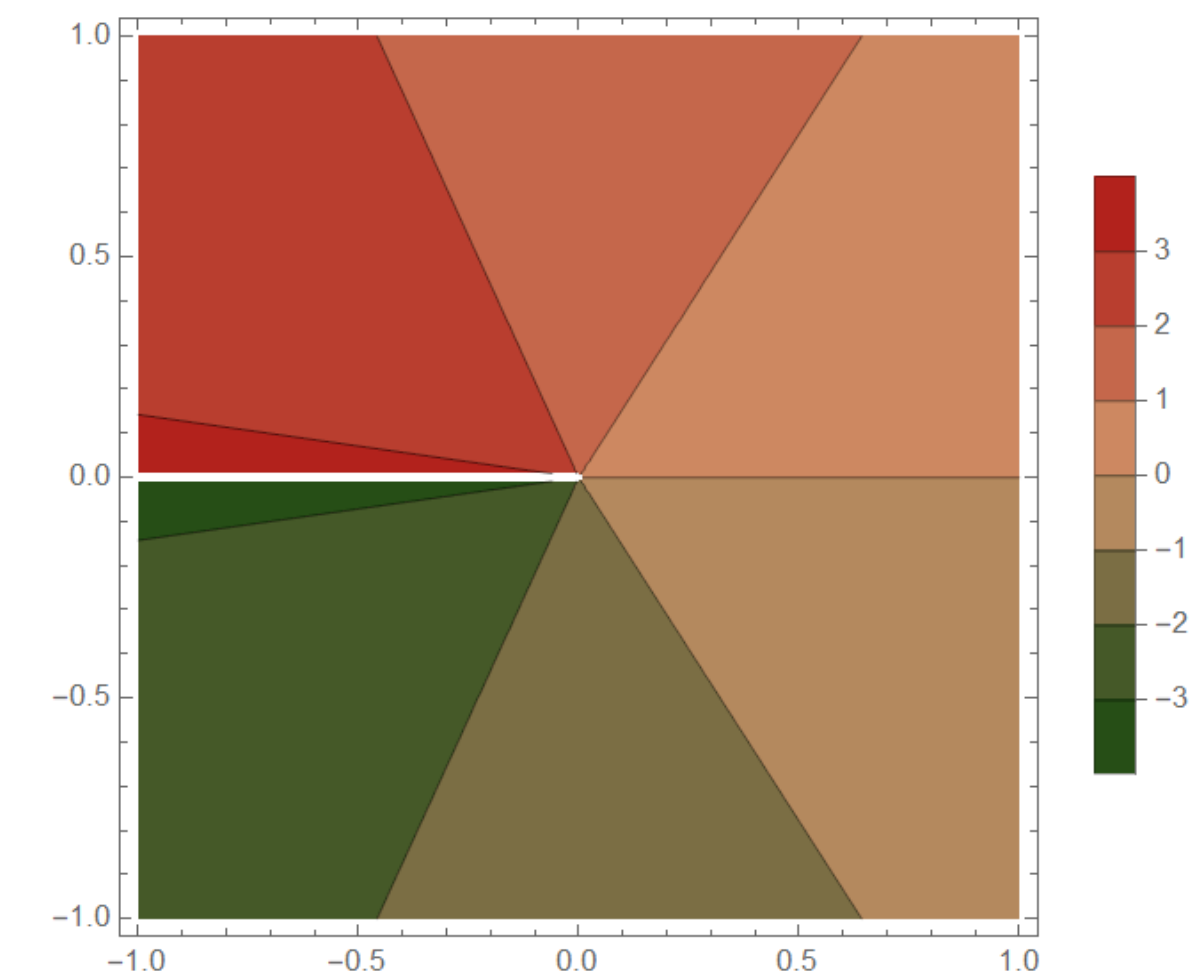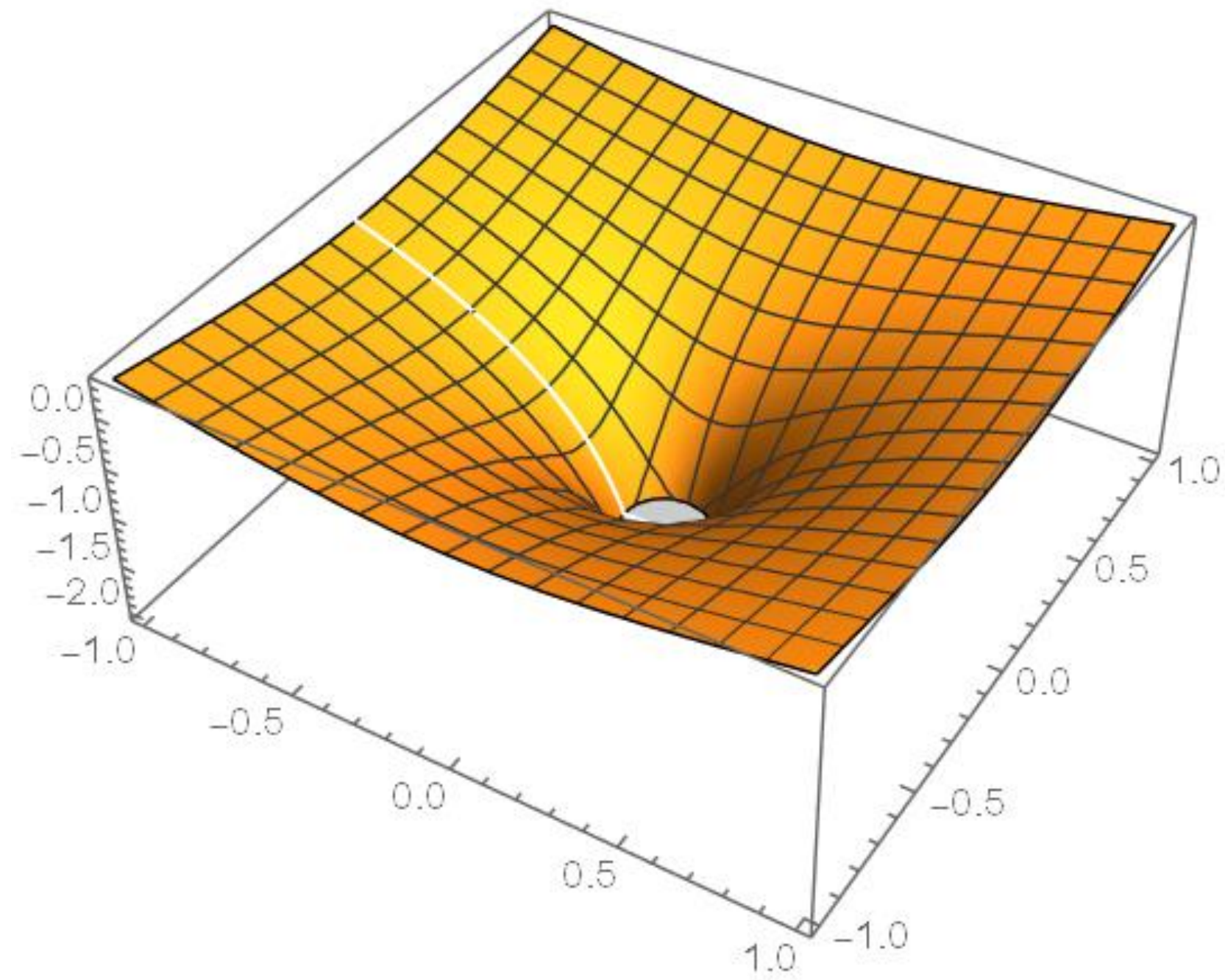
$$\frac{Q}{2\pi} \int_{|z|=r_0} \frac{\partial}{\partial \mathbf{n}} \log|z| \, ds = Q. \tag{2.2.41}$$

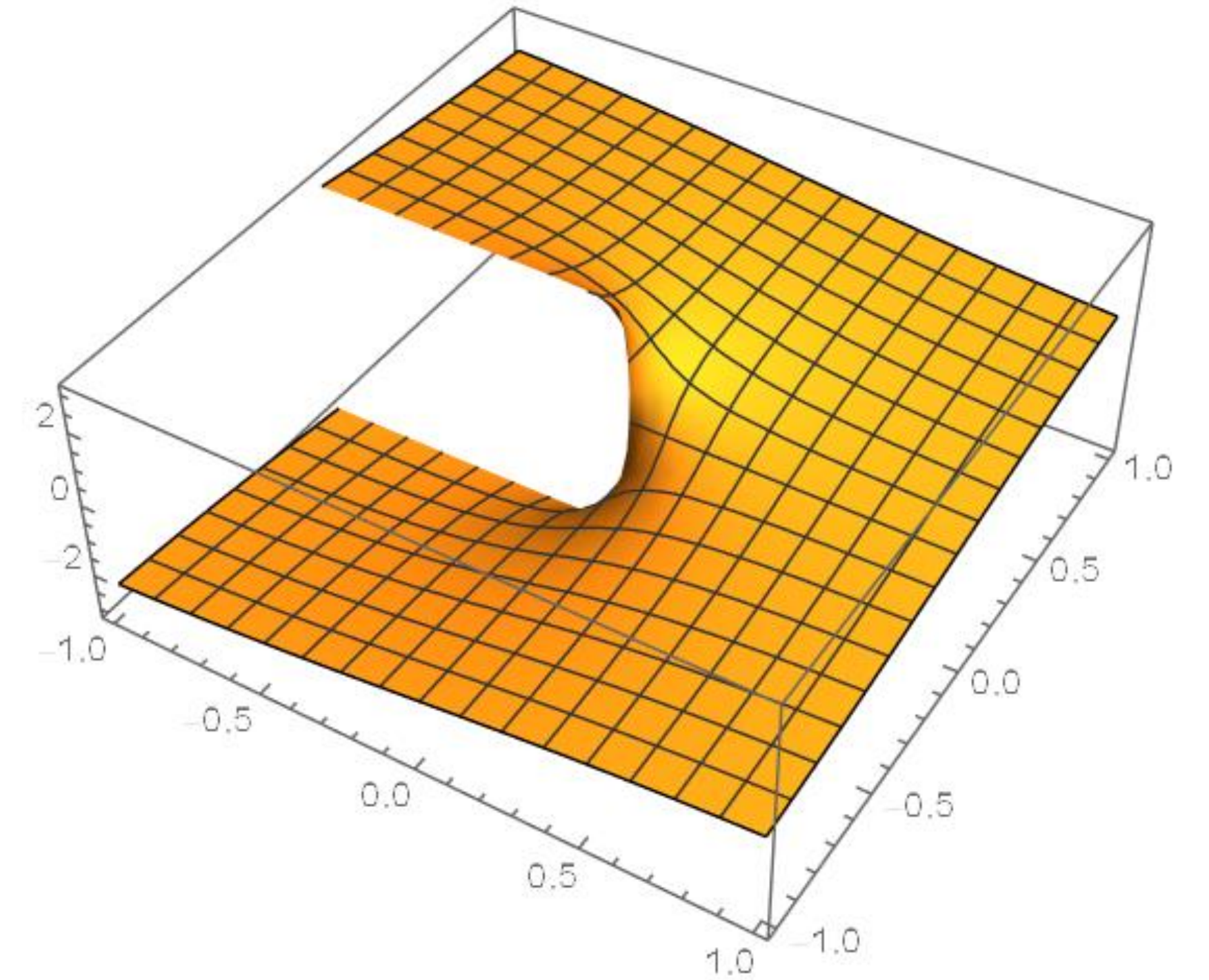The derivative $(\log z)' = \frac{1}{z}$ is a single-valued function. This function models the dipole. The functions $\frac{1}{z^n}$ for $n = 2, 3, \dots$ model multipoles.

Plot3D[Re[Log[x + i y]], {x, -1, 1}, {y, -1, 1}]

Plot3D[Im[Log[x + i y]], {x, -1, 1}, {y, -1, 1}]

$$\varphi(z) = u(x_1, x_2) + iv(x_1, x_2)$$
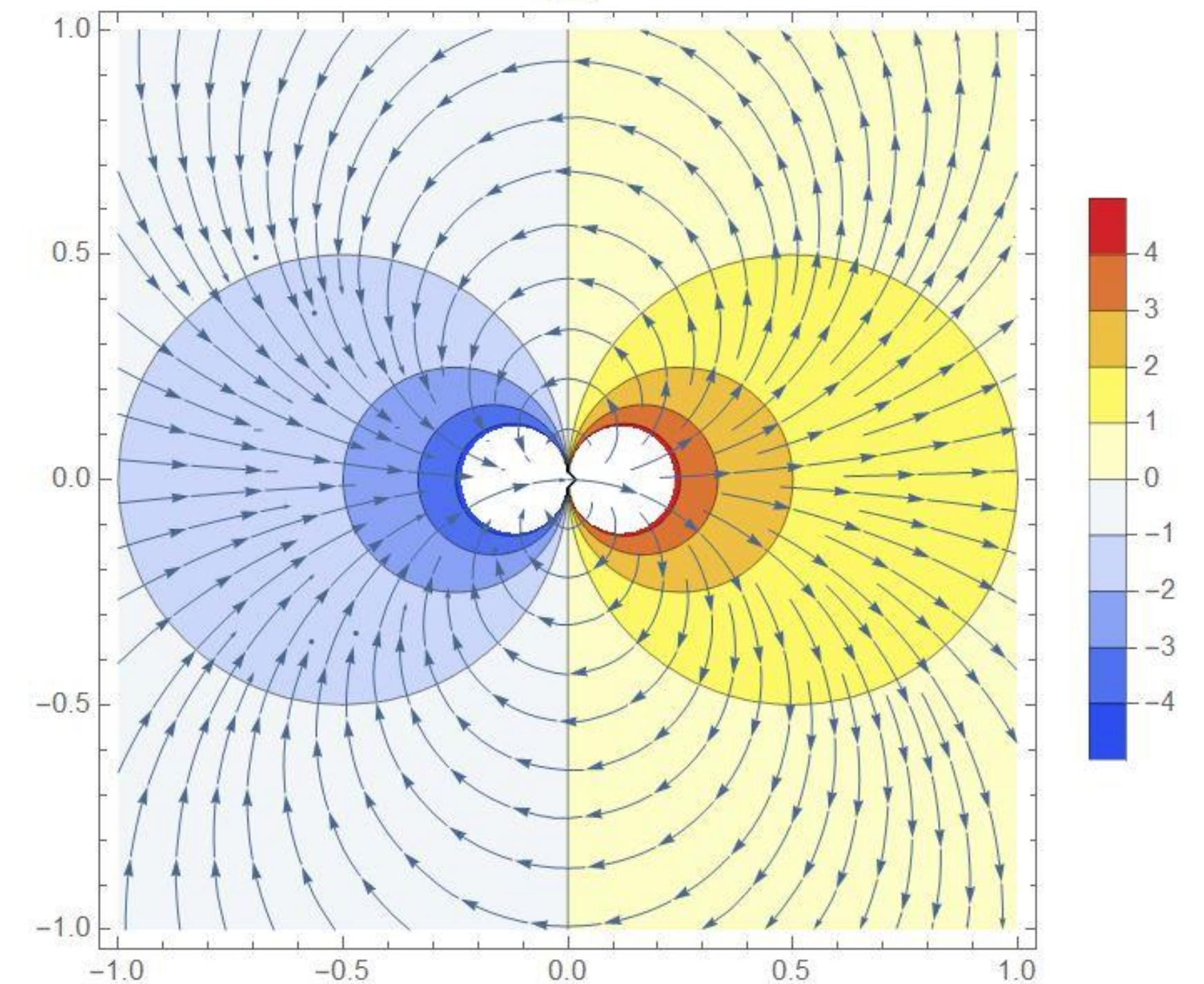
$$\varphi'(z) = \frac{\partial u}{\partial x_1} + i\frac{\partial v}{\partial x_1} = \frac{\partial u}{\partial x_1} - i\frac{\partial u}{\partial x_2}$$

Complex potential

Analytic function

Complex velocity

log z

$$\frac{1}{z}$$

Theorem 3 (Decomposition Theorem [32, p. 23]). *Let D and D_j be domains defined in the previous theorem. Any function u harmonic in D and continuous in its closure has the unique decomposition*

$$u(z) = Re \sum_{j=1}^{n} \left[\varphi_j(z) + A_j \log (z - z_j) \right] + A, \qquad (2.1.18)$$

where φ_j is analytic in $D \cup \partial D_j \cup D_j$ and $\varphi_j(\infty) = 0$ for all $j = 1, 2, \ldots, n$. The constants A_j and A are real and $\sum_{j=1}^{n} A_j = 0$.

For L being a closed curve the integral $\int_L d\varphi(z)$ is called the *period* of $\varphi(z)$ along L.

Calculating periods of the function $\omega(z) := \sum_{j=1}^{n} \left[\varphi_j(z) + A_j \log (z - z_j) \right]$ along each component L_m of $(-\partial D)$, one can see by virtue of logarithmic function properties that

$$\int_{L_m} d\omega(z) = 2\pi i A_m.$$

Therefore the constants $2\pi A_m$ which appear in (2.1.17), (2.1.18) are imaginary parts of the periods of the multi-valued analytic function $\omega(z)$ along the components of $(-\partial D)$.
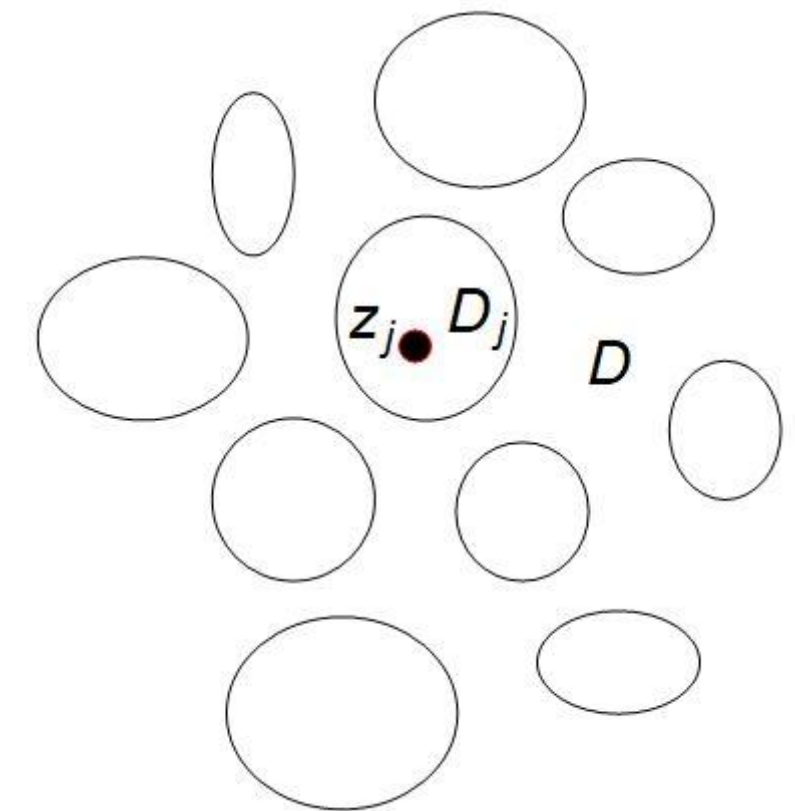
1.4. Cauchy type integral and singular integrals

The classical Cauchy integral formula [14] can be presented in the following way. Let L be a simple, closed, piece-wise smooth curve on the complex plane $\mathbb{C}$ dividing $\widehat{\mathbb{C}}$ onto two simply connected domains D^+ and $D^- \ni \infty$. If function $\Phi(z)$ is analytic in D^+ and continuous up to the boundary, it can be represented in the form of *Cauchy integral*

$$\frac{1}{2\pi i}\int_L \frac{\Phi(t)}{t-z}dt = \begin{cases} \Phi(z), & z \in D^+, \\ 0, & z \in D^-. \end{cases} \qquad (2.1.19)$$

If $\Phi(z)$ is analytic in D^- and continuous up to the boundary, we have

$$\frac{1}{2\pi i}\int_L \frac{\Phi(t)}{t-z}dt = \begin{cases} \Phi(\infty), & z \in D^+, \\ -\Phi(z) + \Phi(\infty), & z \in D^-. \end{cases} \qquad (2.1.20)$$

D^+

D^-
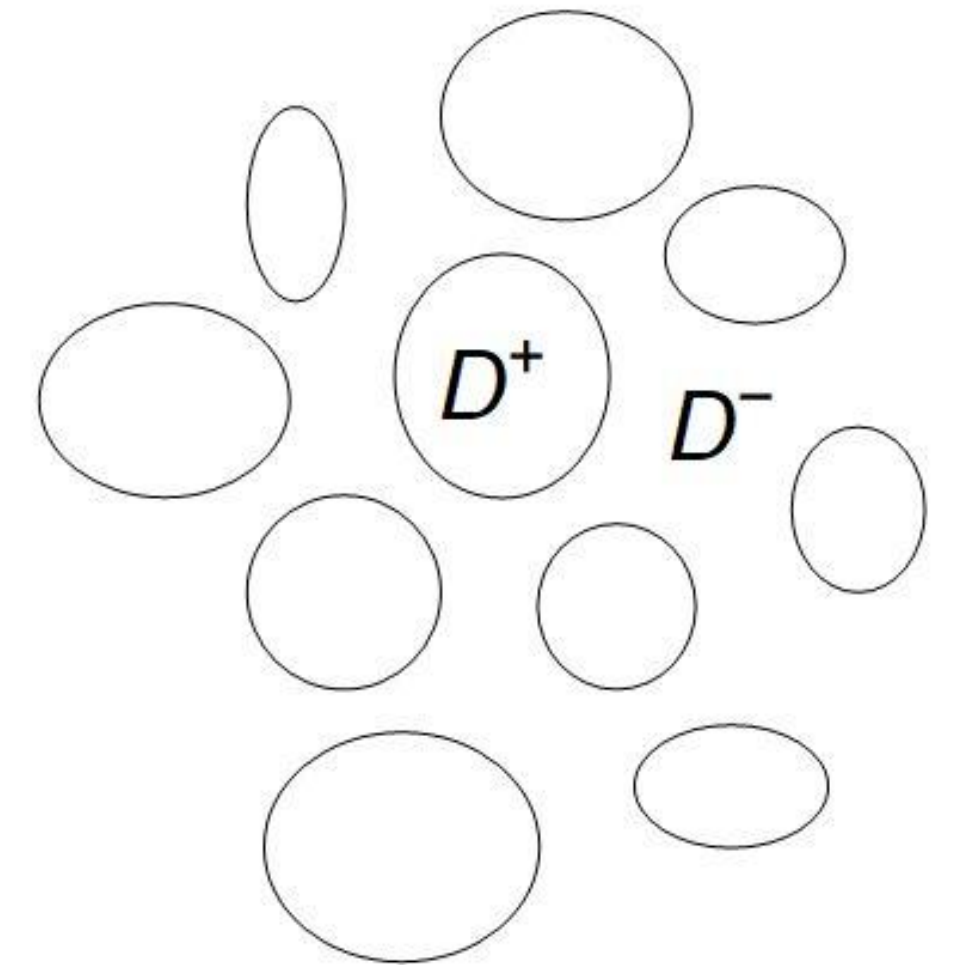
Let L be a simple, rectifiable Jordan closed curve (or open arc) on $\mathbb{C}$ and $\phi \in \mathcal{H}^{\alpha}(L)$, then

$$\frac{1}{2\pi i} \int_L \frac{\phi(t)}{t-z} dt, \quad z \in \widehat{\mathbb{C}} \setminus L$$

is called *Cauchy type integral*. It gives two analytic functions $\Phi^+(z)$ and $\Phi^-(z)$ in the domains D^+ and D^- respectively if L is closed curve (or unique function $\Phi(z)$, analytic in $\widehat{\mathbb{C}} \setminus L$, if L is open arc).

Let again L be a simple, rectifiable Jordan closed curve on $\mathbb{C}$ and $\phi \in \mathcal{H}^{\alpha}(L)$. Let $B(t, \varepsilon)$ denote the disk with the center at t of radius ε. The following limit

$$\lim_{\varepsilon \to 0} \frac{1}{\pi i} \int_{L \setminus B(t,\varepsilon)} \frac{\phi(\tau)}{\tau - t} d\tau = v.p. \frac{1}{\pi i} \int_L \frac{\phi(\tau)}{\tau - t} d\tau \qquad (2.1.21)$$

is called *Cauchy principal value of a singular integral* $\frac{1}{\pi i} \int_L \frac{\phi(\tau)}{\tau - t} d\tau$ or simply *singular integral* (with Cauchy kernel).
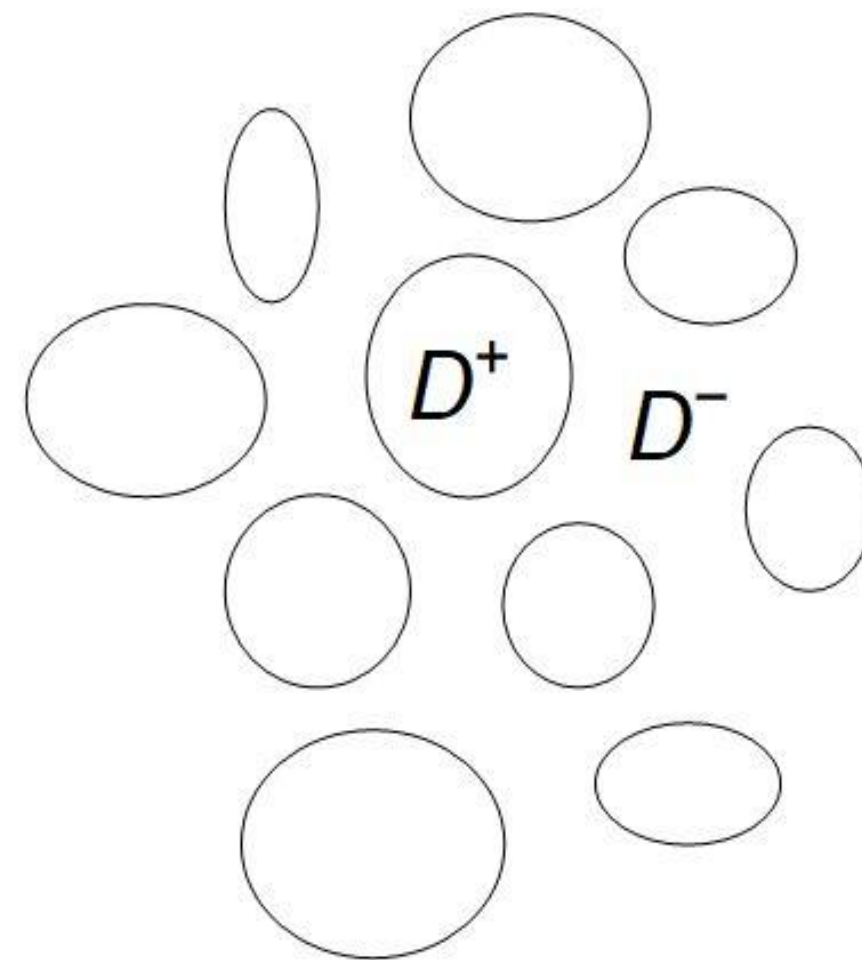
Let simple, closed, smooth curve L divide $\overline{\mathbb{C}}$ onto two domains D^+, $D^- \ni \infty$. Then the boundary functions $\Phi^+(t)$, $\Phi^-(t)$ of the Cauchy type integral

$$\Phi^{\pm}(z) = \frac{1}{2\pi i} \int\limits_{L} \frac{\phi(\tau)d\tau}{\tau - z}, \quad z \in D^{\pm},$$

do satisfy the *Sochocki–Plemelj formulae*[1]

$$\Phi^{\pm}(t) = \pm\frac{1}{2}\phi(t) + \frac{1}{2\pi i} \int\limits_{L} \frac{\phi(\tau)d\tau}{\tau - t}, \quad t \in L. \qquad (2.1.22)$$

Any function from h_p ($\mathbb{U}$) $(1 < p \leq \infty)$ can be represented in the form of *Poisson integral*, i.e., if $u \in h_p$ ($\mathbb{U}$) $(1 < p \leq \infty)$, there exists a function $\phi \in L_p$ $(0, 2\pi)$ such that

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - t)} \phi(t)dt = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t)\phi(t)dt, \qquad (2.1.23)$$

$0 \leq r < 1, \theta \in (-\pi, \pi]$, where $P_r(\tau) = \frac{1-r^2}{1+r^2-2r\cos\tau}$ is called *Poisson kernel*. It should be noted that Poisson kernel P_r is in fact the real part of the so called *Schwarz kernel* :

$$P_r(\theta - t) = \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - t)} = \mathrm{Re}\,\frac{e^{it} + re^{i\theta}}{e^{it} - re^{i\theta}} = \mathrm{Re}\,\frac{e^{it} + z}{e^{it} - z}. \qquad (2.1.24)$$

It is not always hold for an arbitrary harmonic function in $\mathbb{U}$, but it is true for bounded harmonic functions in $\mathbb{U}$, hence for any one continuous up to the boundary.

Let a harmonic function u be continuous in the closure of the unit disk. Then,

$$u(e^{it}) = \lim_{re^{i\theta} \to e^{it}} u(re^{i\theta}).$$

Moreover, if $\phi(t)$ is an arbitrary continuous function on $[-\pi, \pi]$ and $\phi(-\pi) = \phi(\pi)$, the Poisson integral solves the Dirichlet problem (see Section 2.5 of the book [32])

$$u(e^{it}) = \phi(t), \quad |t| = 1.$$

The problem of determination of an analytic function in a domain via boundary values of its real part is often called *Schwarz problem* (see, e.g., [14, 32]). In the case of simply connected domain it is simply reduced to the Dirichlet problem for harmonic functions. An operator that stays in correspondence to a given on ∂D real-valued function u and analytic in D function φ, such that Re $\varphi_{|\partial D} = u$, is called *Schwarz operator* $\mathbb{T}: u \mapsto \varphi$. In the case of unit disc $\mathbb{U}$, the Schwarz operator has an explicit form

$$(\mathbb{T}u)(z) := \frac{1}{2\pi i} \int_{\partial \mathbb{U}} u(\zeta) \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta}. \qquad (2.1.25)$$

For the upper half-plane, we have

$$(\mathbb{T}u)(z) := \frac{1}{\pi} \int_{-\infty}^{\infty} u(\xi) \frac{x_2}{(\xi - x_1)^2 + x_2^2} d\xi, \quad x_2 > 0.$$

The Riemann-Hilbert problem $\boxed{\operatorname{Re} \overline{\lambda(t)}\varphi(t) = g(t),\ t \in \partial D.}$

generalizes the Dirichlet and Neumann problems

$$\boxed{D^- = D \qquad D^+ = \bigcup_{k=1}^{n} D_k}$$

Theorem 4 ([32, 34]). *The problem*

$$\operatorname{Re} \varphi(t) = g(t), \qquad t \in \partial D \qquad (2.1.27)$$

is equivalent to the problem

$$\varphi^-(t) = \varphi^+(t) - \overline{\varphi^+(t)} + g(t), \qquad t \in \partial D, \qquad (2.1.28)$$

i.e., the problem (2.1.27) is solvable if and only if (2.1.28) is solvable. If (2.1.27) has a solution $\varphi(z)$, it is a solution of (2.1.28) in D and solution of (2.1.28) in D_k can be found from the following simple problem for the simply connected domain D_k with respect to function $2\operatorname{Im} \varphi^+(z)$ harmonic in D_k

$$2\operatorname{Im} \varphi^+(t) = \operatorname{Im} \varphi^-(t) - g(t), \qquad t \in \partial D_k, \qquad (k = 1, 2, \ldots, n). \qquad (2.1.29)$$

The problem (2.1.29) has a unique solution up to an arbitrary additive real constant.
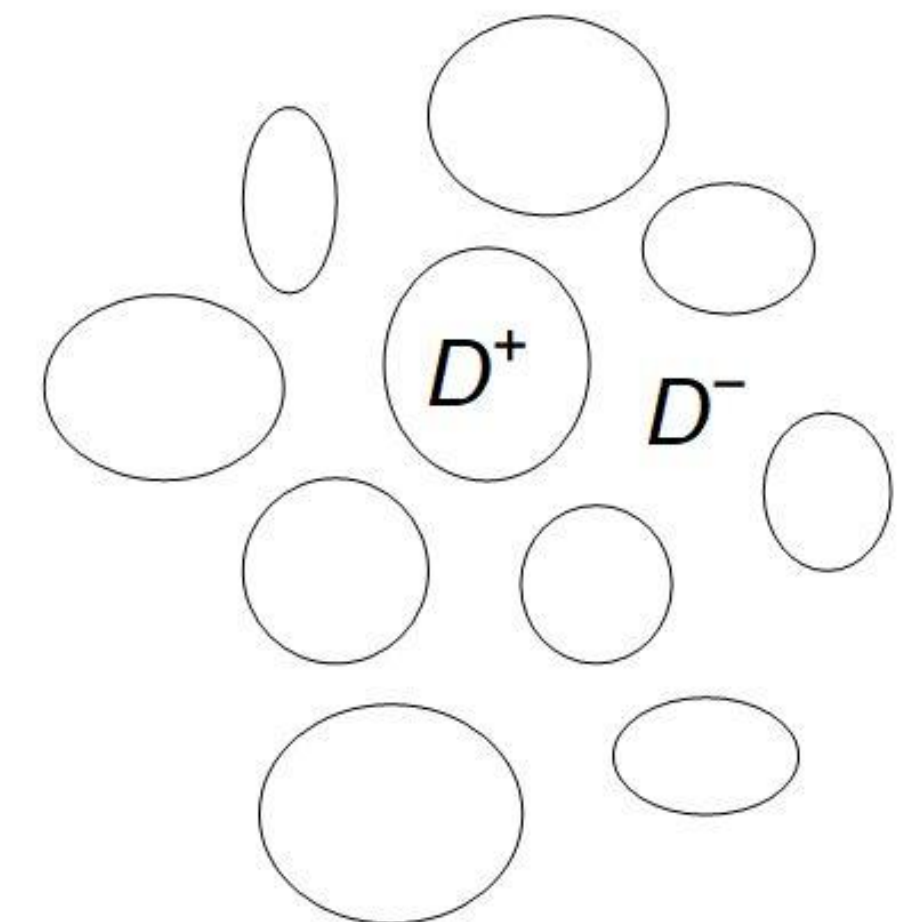
LOCAL CONDUCTIVITY LAWS

Consider locally the 2D steady heat conduction in a domain U occupied by a conducting material with a constant conductivity $\sigma > 0$. The temperature distribution $u(x_1, x_2)$ and the heat flux $\mathbf{q}(x_1, x_2) = (q_1(x_1, x_2), q_2(x_1, x_2))$ satisfy in U the Fourier law

$$\mathbf{q} = -\sigma \nabla u \qquad (2.2.34)$$

and the conservation energy law

$$\nabla \cdot \mathbf{q} = 0 \iff \frac{\partial q_1}{\partial x_1} + \frac{\partial q_2}{\partial x_2} = 0. \qquad (2.2.35)$$

Here, $\nabla := (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$. Substitution of (2.2.35) into (2.2.34) yields Laplace's equation in the domain U
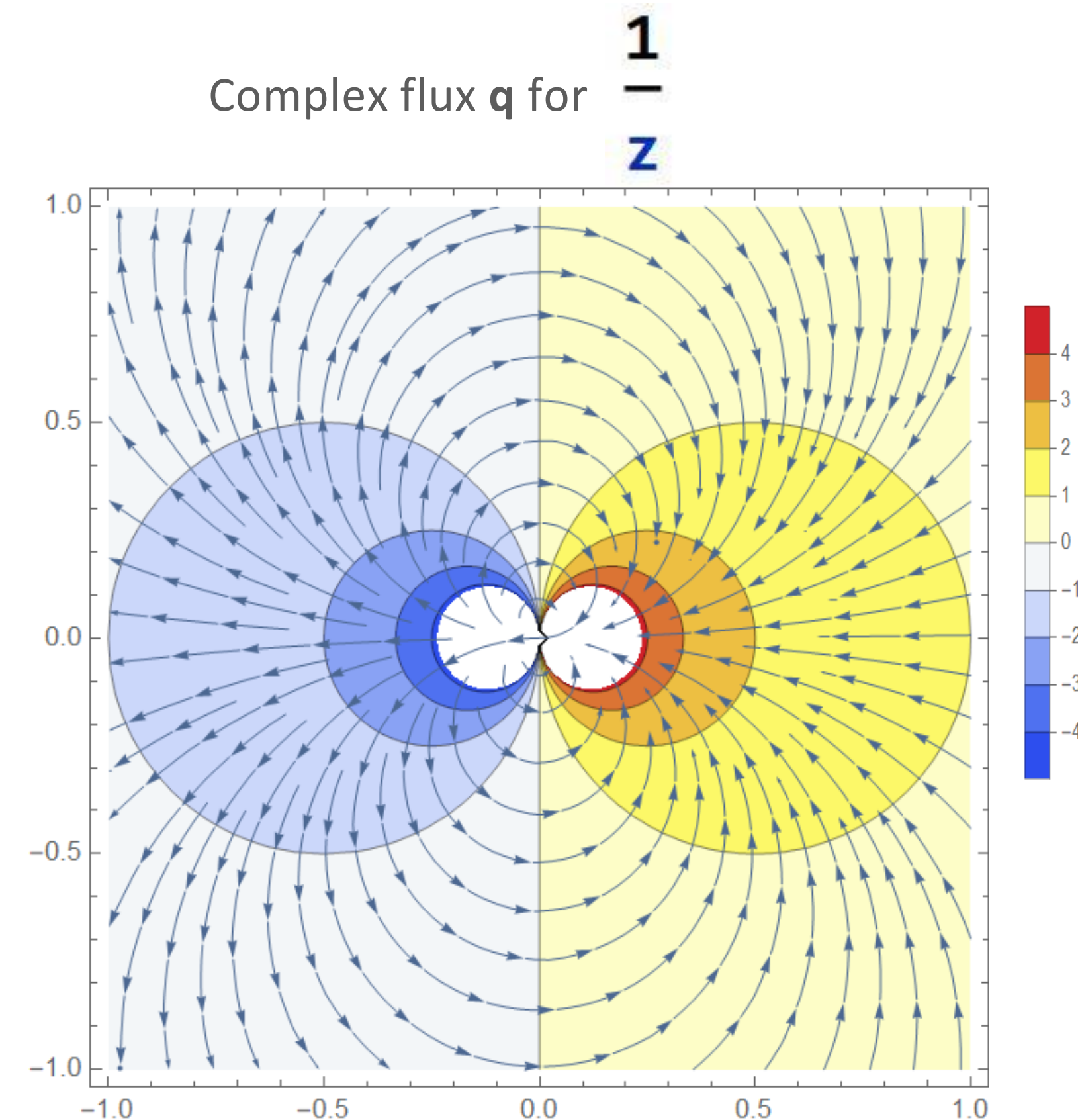
$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = 0. \qquad (2.2.36)$$

Let a smooth oriented curve $L \subset U$ be fixed. The normal heat flux across a point of L is given by formula

$$q_{\mathbf{n}} = \mathbf{q} \cdot \mathbf{n} = -\sigma \frac{\partial u}{\partial \mathbf{n}},$$

where $\mathbf{n}$ denotes the unit normal vector to L.

Complex flux $\mathbf{q}$ for $\dfrac{1}{\mathbf{z}}$

Let the inclusions D_k has the conductivity $\sigma^+ = \sigma_k$ $(k = 1, 2, \ldots, n)$ and the conductivity of D be normalized to unity as $\sigma^- = 1$. Such a normalization does not limit the generality of the problem. The coefficients σ_k then become dimensionless and are considered as the ratios of the conductivities of the kth inclusion to the conductivity of matrix. In these designations, (2.2.33) becomes

$$u(t) = u_k(t), \quad \frac{\partial u}{\partial \mathbf{n}}(t) = \sigma_k \frac{\partial u_k}{\partial \mathbf{n}}(t), \quad t \in L_k \quad (k = 1, 2, \ldots, n). \tag{2.2.37}$$

The subscript k pertains to the inclusions.

We now reduce (2.2.37) to an $\mathbb{R}$-linear problem. To this end, introduce the complex potentials $\varphi(z)$ and $\varphi_k(z)$ analytic (meromorphic) in D and D_k, respectively. The harmonic and analytic functions are related by the equalities

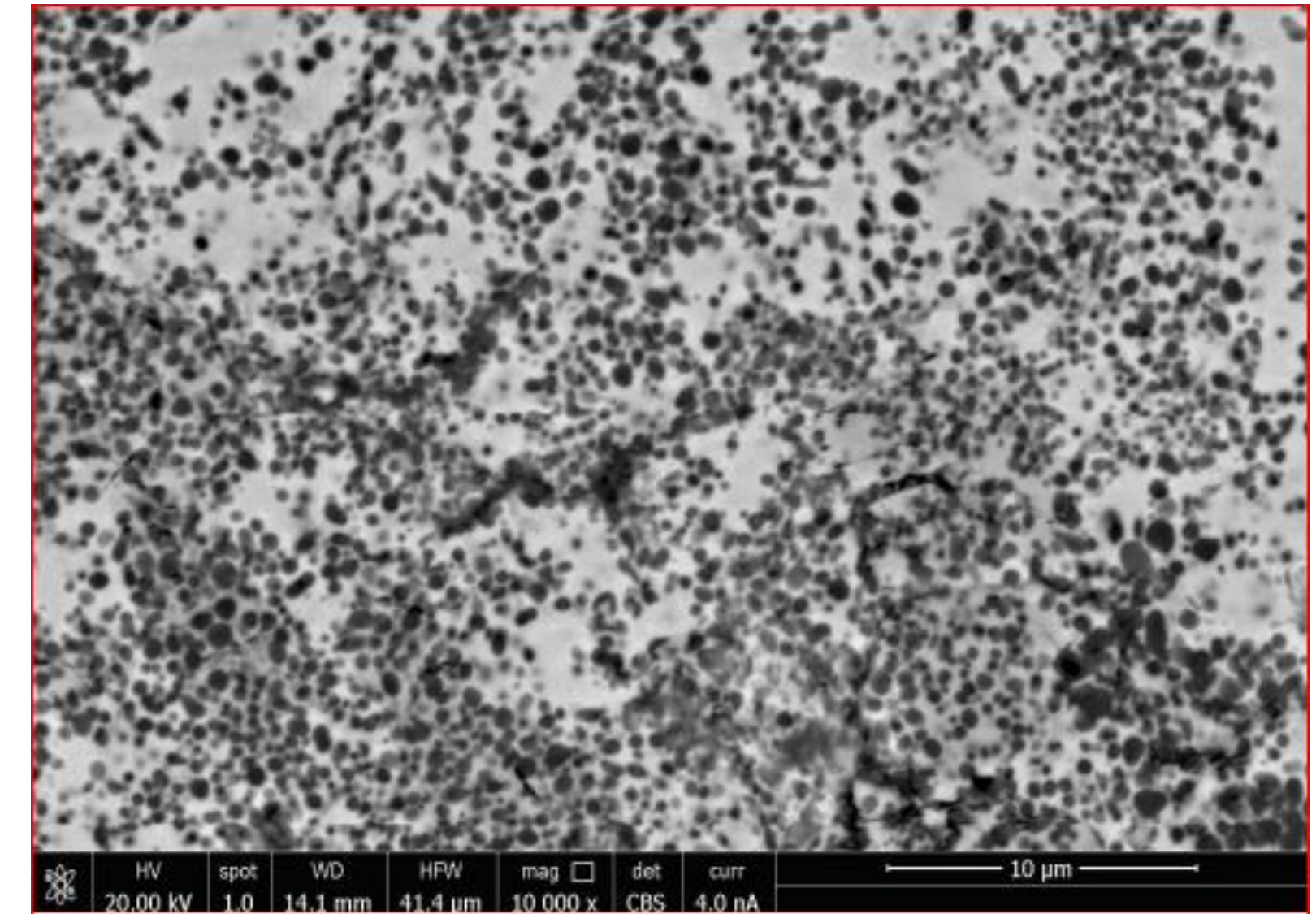$$u(z) + iv(z) = \varphi(z), \quad z \in D, \tag{2.2.38}$$

$$u_k(z) + iv_k(z) = \frac{2}{\sigma_k + 1} \varphi_k(z), \quad z \in D_k \ (k = 1, 2, \ldots, n), \tag{2.2.39}$$

Microstructure of the TiC–FeCr composite

R-linear problem

$$\varphi(t) = \varphi_k(t) - \rho_k \overline{\varphi_k(t)}$$

Contrast parameter

$$\rho_k = \frac{\sigma_k - 1}{\sigma_k + 1}$$

BOX A.2.5 Cauchy–Riemann equations on a curve
Let functions $u(x_1, x_2)$ and $v(x_1, x_2)$ be continuously differentiable in the closure of the open simply connected domain V with the Hölder continuous boundary ∂V and satisfy the Cauchy–Riemann equations (2.1.3) in V. Consider the unit tangent $\mathbf{s} = (-n_2, n_1)$ and normal $\mathbf{n} = (n_1, n_2)$ vectors to ∂V. Continuous differentiability implies that the Cauchy–Riemann equations (2.1.3) are fulfilled in $V \cup \partial V$. Consider the directional derivatives

$$\frac{\partial u}{\partial \mathbf{n}} = n_1 \frac{\partial u}{\partial x_1} + n_2 \frac{\partial u}{\partial x_2}, \quad \frac{\partial v}{\partial \mathbf{s}} = -n_2 \frac{\partial v}{\partial x_1} + n_1 \frac{\partial v}{\partial x_2} \tag{2.2.43}$$

and $\frac{\partial u}{\partial \mathbf{s}}$, $\frac{\partial v}{\partial \mathbf{n}}$. Using (2.1.3) we obtain the Cauchy–Riemann equations on ∂V

$$\frac{\partial u}{\partial \mathbf{n}} = \frac{\partial v}{\partial \mathbf{s}}, \quad \frac{\partial u}{\partial \mathbf{s}} = -\frac{\partial v}{\partial \mathbf{n}}. \tag{2.2.44}$$

Complex flux in domains

$$\psi(z) \equiv \varphi'(z) = \frac{\partial u}{\partial x_1}(z) - i\,\frac{\partial u}{\partial x_2}(z), \quad z \in D,$$

$$\psi_k(z) \equiv \varphi'_k(z) = \frac{\sigma_k + 1}{2}\left(\frac{\partial u_k}{\partial x_1}(z) - i\,\frac{\partial u_k}{\partial x_2}(z)\right),\ z \in D_k \quad (k = 1, 2, \ldots, n).$$

R-linear problem for a circular domain in terms of complex flux

$$\psi(t) = \psi_k(t) + \rho_k \overline{n(t)^2 \overline{\psi_k(t)}}, \quad t \in L_k\ (k = 1, 2, \ldots, n)$$

In the present section, we apply Schwarz's method to the $\mathbb{R}$-linear problem corresponding to the perfect contact between components of the composite with an external field modeled by a function $g(t)$

$$\varphi^-(t) = \varphi_k(t) - \rho_k\overline{\varphi_k(t)} - g(t), \quad t \in L_k \ (k = 1, 2, \ldots, n), \qquad (2.2.72)$$

where the contrast parameter ρ_k has the form (2.2.48).

For fixed m introduce the operator

$$A_m g(z) = \frac{1}{2\pi i} \int_{L_m} \frac{g(t)dt}{t - z}, \quad z \in G_m. \qquad (2.2.73)$$

In accordance with the Sochocki formulae,

$$A_m g(\zeta) = \lim_{z \to \zeta} A_m g(z) = \frac{1}{2}g(\zeta) + \frac{1}{2\pi i} \int_{L_m} \frac{g(t)dt}{t - \zeta}, \quad \zeta \in L_m. \qquad (2.2.74)$$

Equations (2.2.73)–(2.2.74) determine the operator A_m in the space $\mathcal{H}(D_m)$.

Lemma 1. *The linear operator A_m is bounded in the space $\mathcal{H}(D_m)$.*

The conjugation condition (2.2.72) can be written in the form

$$\varphi_k(t) - \varphi^-(t) = \rho_k \overline{\varphi_k(t)} + g(t), \quad t \in L_k \ (k = 1, 2, \ldots, n). \tag{2.2.75}$$

The difference between functions analytic in $D^+ = \cup_{k=1}^n D_k$ and in D appears in the left-hand part of the latter relation. Then, application of the Sochocki formulae (2.1.22) yield

$$\varphi_k(z) = \sum_{m=1}^n \frac{\rho_m}{2\pi i} \int_{L_m} \frac{\overline{\varphi_m(t)}}{t - z} dt + g_k(z), \quad z \in D_k \ (k = 1, 2, \ldots, n), \tag{2.2.76}$$

where the function

$$g_k(z) = \frac{1}{2\pi i} \sum_{m=1}^n \int_{L_m} \frac{g(t)}{t - z} dt$$

One can consider equations (2.2.76), (2.2.77) as an equation with linear bounded operator in the space $\mathcal{H}(D^+)$ consisting of functions analytic in D^+ and Hölder continuous in the closure of D^+ (see page 10). The complex potential in the matrix surrounding inclusions is not directly presented in (2.2.76), (2.2.77) and the problem is reduced "only" to finding inclusions potentials.

Equations (2.2.76), (2.2.77) constitute the generalized method of Schwarz. Write, for instance, equation (2.2.76) in the form

$$\varphi_k(z) - \frac{\rho_k}{2\pi i} \int_{L_k} \frac{\overline{\varphi_k(t)}}{t-z} dt = \sum_{m \neq k} \frac{\rho_m}{2\pi i} \int_{L_m} \frac{\overline{\varphi_m(t)}}{t-z} dt + g_k(z), \ z \in D_k \ (k = 1, 2, \ldots, n).$$

$$(2.2.78)$$

At the zeroth approximation we arrive at the problem for the single inclusion D_k ($k = 1, 2, \ldots, n$)

$$\varphi_k(z) - \frac{\rho_k}{2\pi i} \int_{L_k} \frac{\overline{\varphi_k(t)}}{t-z} dt = g_k(z), \ z \in D_k. \qquad (2.2.79)$$

Let the problem (2.2.79) be solved. Further, its solution is substituted into the RHS of (2.2.78). Then, we arrive at the first-order problem and so forth. Therefore, the generalized method of Schwarz can be considered as the method of implicit iterations applied to integral equations (2.2.76), (2.2.77).
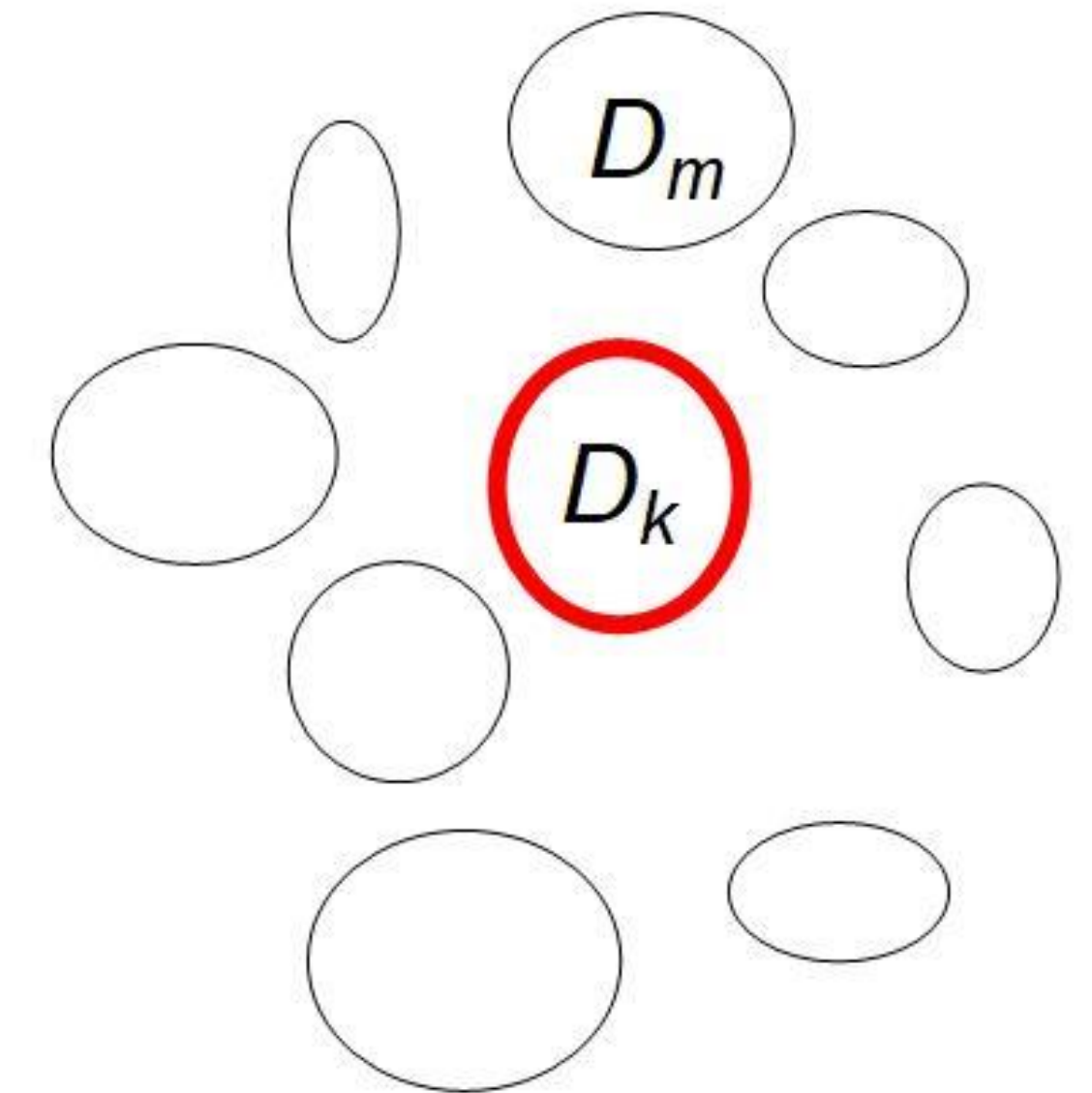
R-linear problem for a circular domain in terms of complex flux

$$\psi(t) = \psi_k(t) + \rho_k \left(\frac{r_k}{t - a_k}\right)^2 \overline{\psi_k(t)}, \quad |t - a_k| = r_k, \ k = 1, 2, \ldots, n.$$
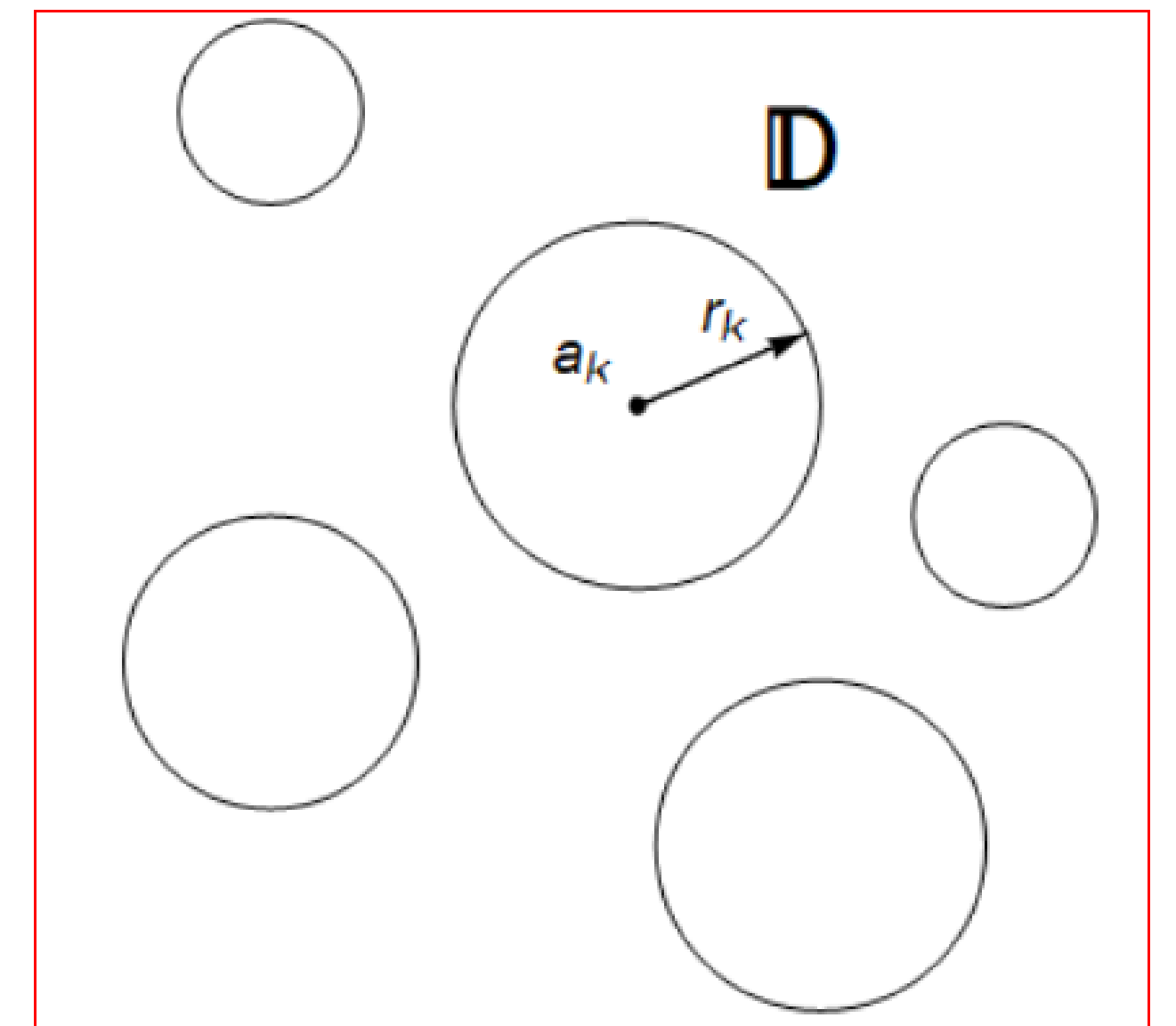
$$\rho_k = \frac{\sigma_k - 1}{\sigma_k + 1}$$

contrast

parameter

The problem is reduced to the system of functional equations

$$\psi_k(z) = \sum_{m \neq k} \rho_m \left(\frac{r_m}{z - a_m}\right)^2 \overline{\psi_m \left(z_{(m)}^*\right)} + 1, \quad |z - a_k| \leq r_k, \ k = 1, 2, \ldots, n.$$
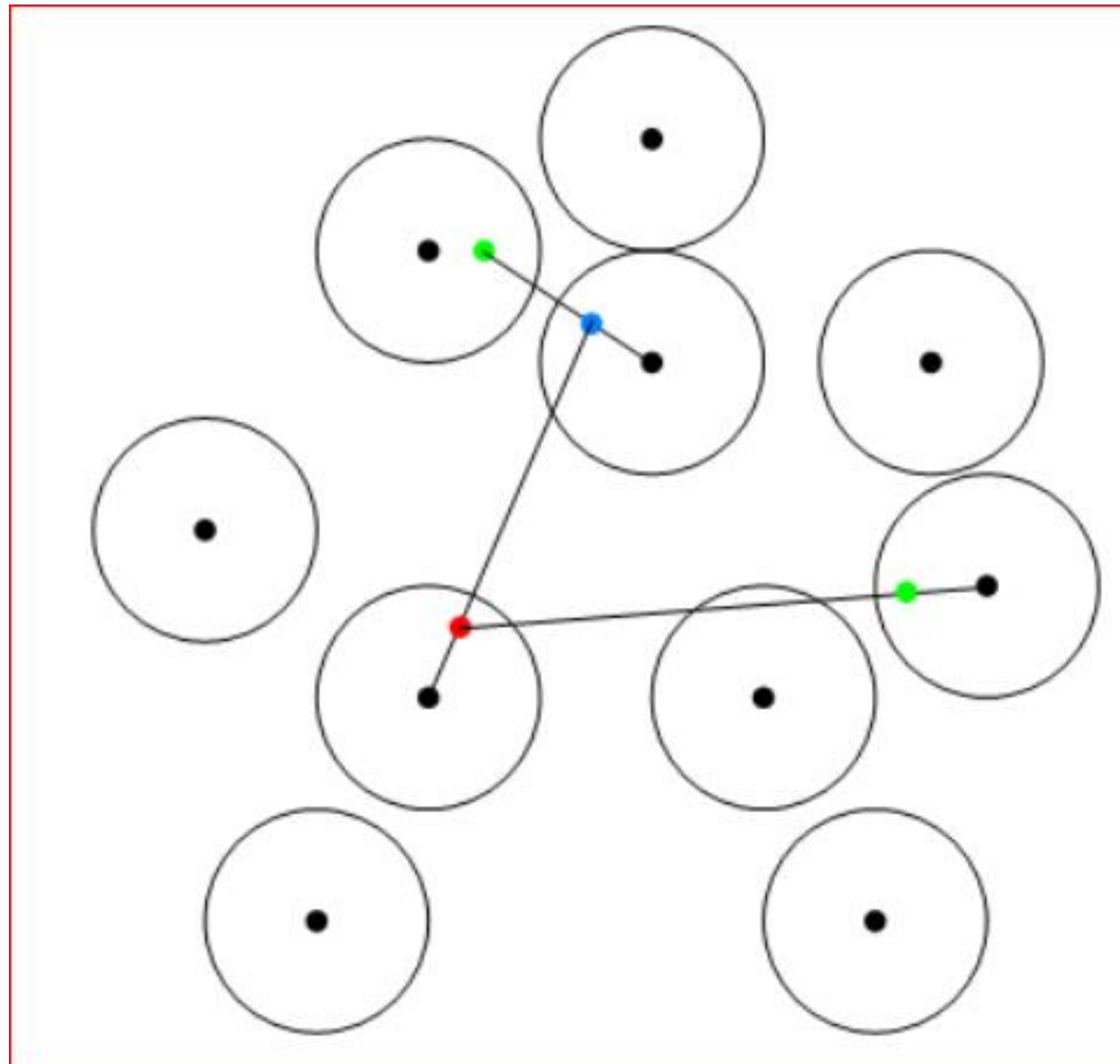

Schottky group of inversions and their compositions

$$z_{(k)}^* = \frac{r^2}{\overline{z} - \overline{a_k}} + a_k, \quad z_{(k_1 k_2 \ldots, k_m)}^* := \left(z_{(k_2 \ldots, k_{m-1})}^*\right)_{k_1}^*, \quad (k_{j+1} \neq k_j).$$
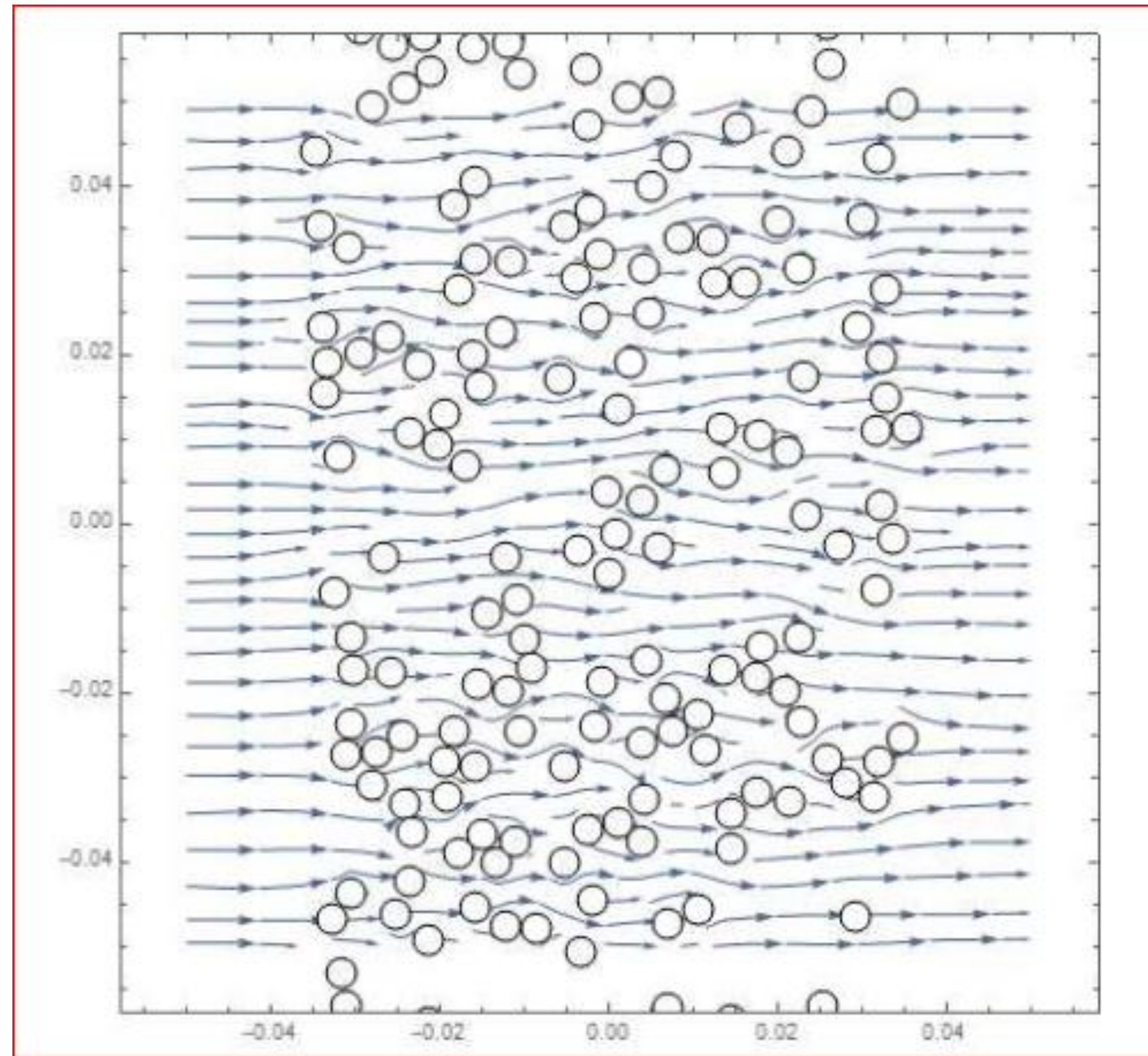
Schottky group $z^{*}_{(k_1 k_2 \ldots, k_m)}$

Poincare series for the Schottky group

$$\psi(z) = 1 + \varrho \sum_{k=1}^{n} \frac{d}{dz} \overline{z^{*}_{(k)}} + \varrho^2 \sum_{k=1}^{n} \sum_{k_1 \neq k} \frac{d}{dz} z^{*}_{(k_1 k)}$$

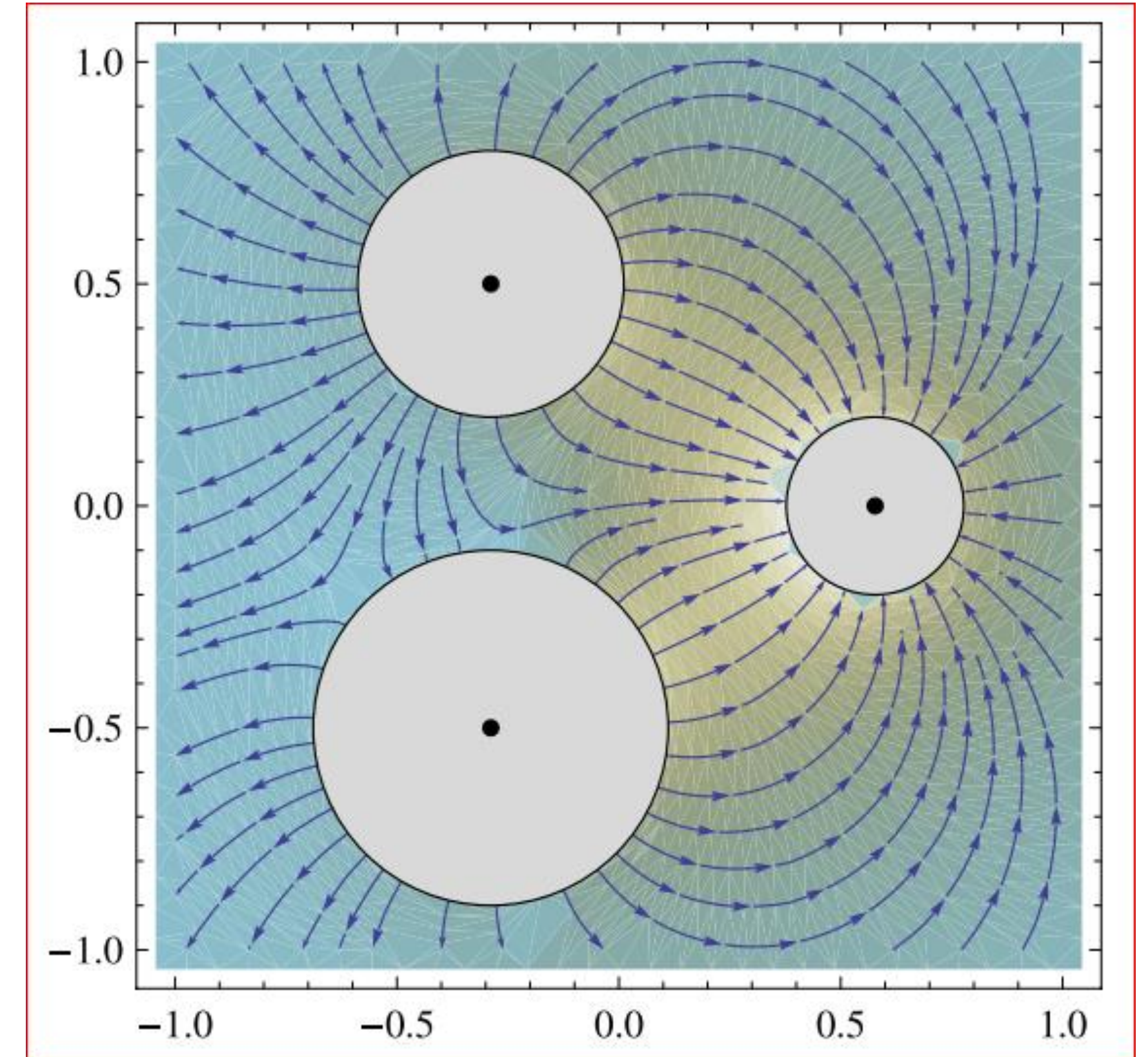$$+ \varrho^3 \sum_{k=1}^{n} \sum_{k_1 \neq k} \sum_{k_2 \neq k_1} \frac{d}{dz} \overline{z^{*}_{(k_2 k_1 k)}} + \cdots$$

Theorem. The Poincare series convrge uniformly for any multiply connected Domain D.
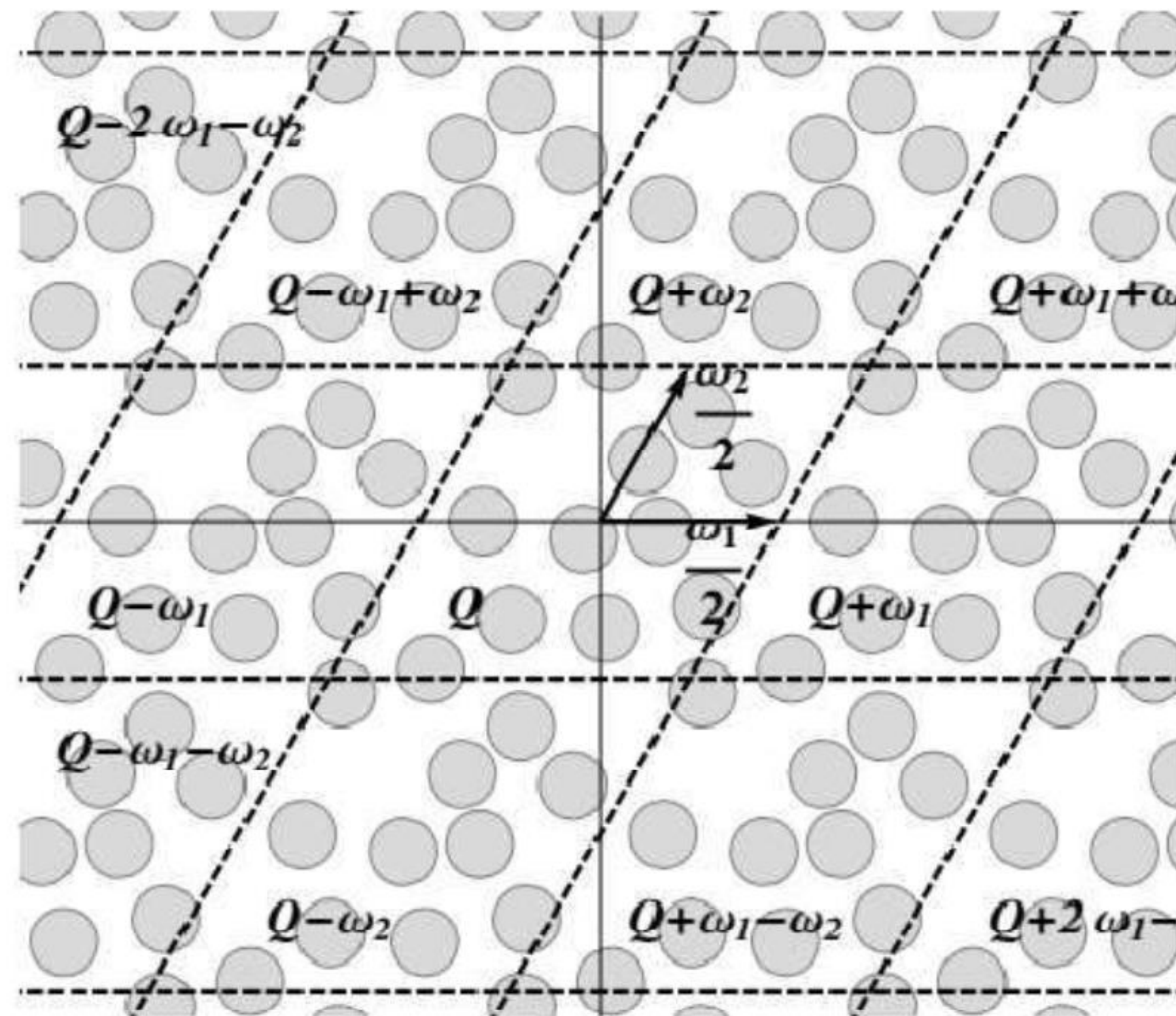
The flux for holes with

the external flux (-1,0)

The higher intensity of the flux corresponds to more lighter areas.

Potential on the circles hold 2.82, -1.12, -0.85

Two –dimensional periodic structure

Q is the fundamental domain

R-linear problem in a class

of doubly periodic functions

$$\psi(t) = \psi_k(t) + \rho_k \left(\frac{r_k}{t - a_k} \right)^2 \overline{\psi_k(t)}, \quad |t - a_k| = r_k$$

Concentration

$$f = N \pi r^2$$

The effective

conductivity tensor

$$\sigma_e = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

The main formula

$$\sigma_{11} - i\sigma_{12} = 1 + 2\rho f \frac{1}{N} \sum_{k=1}^{N} \psi_k(a_k)$$

EISENSTEIN FUNCTIONS (1847)

Eisenstein summation

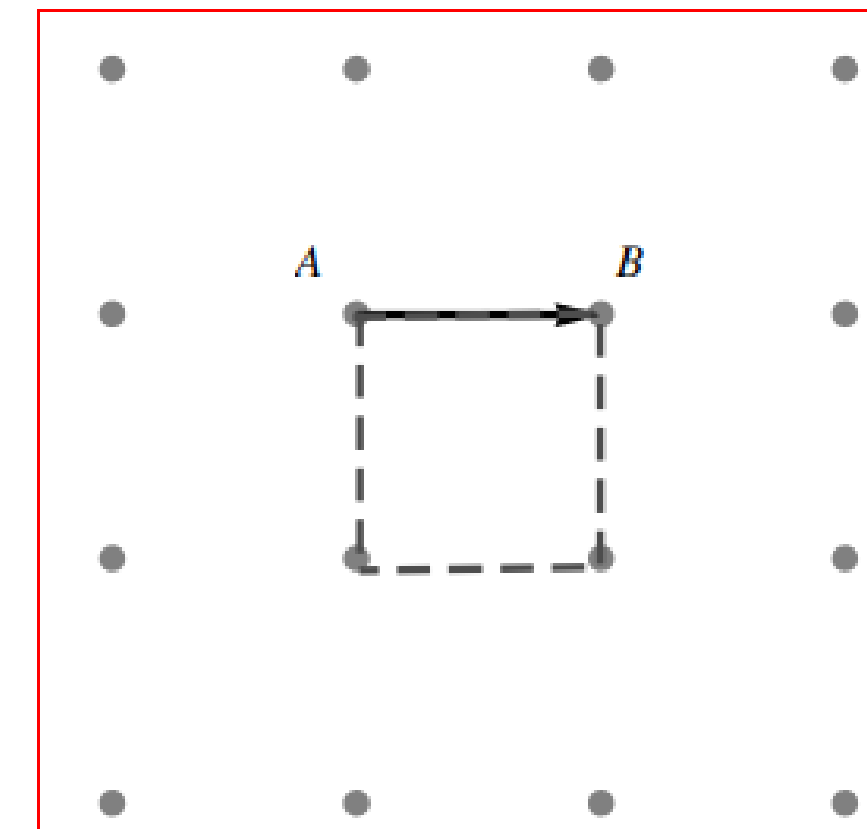$$\sum_{p,q}^{e} := \lim_{M_2 \to +\infty} \lim_{M_1 \to +\infty} \sum_{q=-M_2}^{M_2} \sum_{p=-M_1}^{M_1}$$

Eisenstein–Rayleigh lattice sums

$$S_m = \sum_{m_1,m_2} (m_1 \omega_1 + m_2 \omega_2)^{-m}, \quad m = 2, 3, \ldots$$

For the square array $S_2 = \pi$

(AB = 1)

Eisenstein functions

$$E_1(z) = \sum_{m_1, m_2 \in \mathbb{Z}} \frac{1}{z + m_1 \omega_1 + m_2 \omega_2}$$
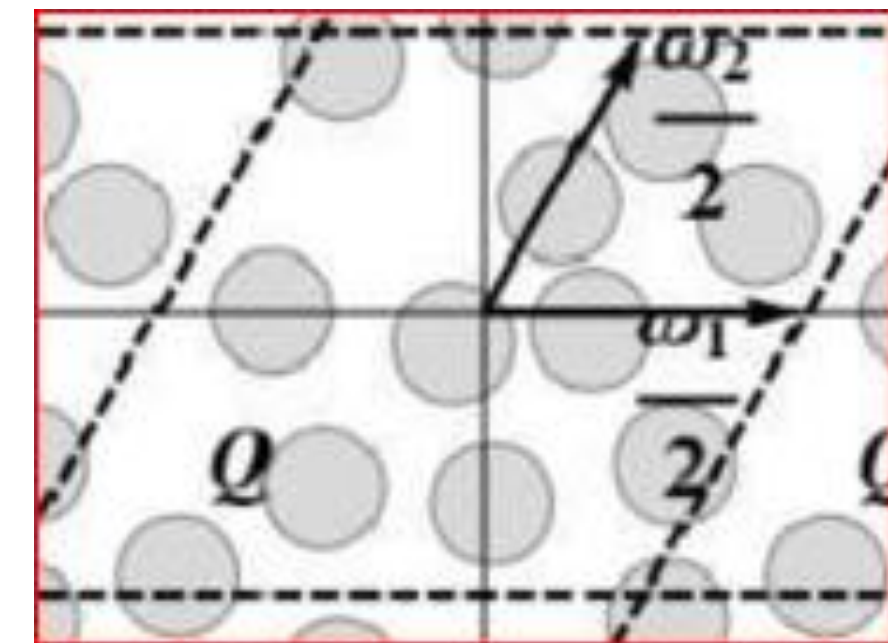
$$E'_m(z) = -m E_{m+1}(z)$$

The high-order Eisenstein functions are expressed in terms of the Weierstrass elliptic function

$$E_2(z) = \wp(z) + S_2, \quad E_m(z) = \frac{(-1)^m}{(m-1)!} \frac{d^{m-2}\wp(z)}{dz^{m-2}}, \quad m = 3, 4, \ldots$$

Structural sums, examples

$$e_2 \;=\; \frac{1}{N^2} \sum_{k_0=1}^{N} \sum_{k_1=1}^{N} E_2(a_{k_0} - a_{k_1}),$$

$$e_{22} \;=\; \frac{1}{N^3} \sum_{k_0=1}^{N} \sum_{k_1=1}^{N} \sum_{k_2=1}^{N} E_2(a_{k_0} - a_{k_1}) \overline{E_2(a_{k_1} - a_{k_2})}$$

Decomposition series for the effective conductivity

(physical constants via contrast parameter, geometry, concentration v):

$$\sigma_{11} - i\sigma_{12} = 1 + 2\rho f(1 + A_1 f + A_2 f^2 + \cdots),$$

where

$$A_1 = \frac{\rho}{\pi} e_2, \quad A_2 = \frac{\rho^2}{\pi^2} e_{22}, \quad A_3 = \frac{1}{\pi^3}\left[-2\rho^2 e_{33} + \rho^3 e_{222}\right],$$

$$A_4 = \frac{1}{\pi^4}\left[3\rho^2 e_{44} - 2\rho^3(e_{332} + e_{233}) + \rho^4 e_{2222}\right],$$

$$A_5 = \frac{1}{\pi^5}\left[-4\rho^2 e_{55} + \rho^3(3e_{442} + 6e_{343} + 3e_{244}) - \right.$$
$$\left. -2\rho^4(e_{3322} + e_{2332} + e_{2233}) + \rho^5 e_{22222}\right],$$

$$A_6 = \frac{1}{\pi^6}\left[5\rho^2 e_{66} - 4\rho^3(e_{255} + 3e_{354} + 3e_{453} + e_{552}) + \right.$$
$$+\rho^4(3e_{2244} + 6e_{2343} + 4e_{3333} + 3e_{2442} + 6e_{3432} + 3e_{4422}) -$$
$$\left. -2\rho^5(e_{22233} + e_{22332} + e_{23322} + e_{33222}) + \rho^6 e_{222222}\right].$$

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Thank you for your
attention

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