Positivity principles for p-Laplacian

$$(*) \quad -\Delta_p u + V u^{p-1} = f \quad \text{in } \Omega \subseteq \mathbb{R}^N \text{ - domain}$$

$$\mathcal{E}_V(u) = \int_\Omega |\nabla u|^p + \int_\Omega V(x)|u|^p, \quad p>1,$$

$$V \in L^\infty_{loc}(\Omega), \quad V = V^+ - V^-, \quad f \in L^\infty_c(\Omega)$$

$$\partial \mathcal{E}_V(u,\varphi) = \frac{d}{d\tau}\mathcal{E}_V(u+\tau\varphi)\Big|_{\tau=0} =$$

$$= \lim_{\tau\to 0} \frac{\mathcal{E}_V(u+\tau\varphi) - \mathcal{E}_V(u)}{\tau}, \quad \tau\in\mathbb{R}, \quad \varphi\in C_c^\infty(\Omega)$$

$$\partial \mathcal{E}_V(u,\varphi) = p\int_\Omega \underbrace{|\nabla u|^{p-2}\nabla u\cdot\nabla\varphi}_{\underbrace{-\nabla\cdot\left(|\nabla u|^{p-2}\nabla u\right)}_{-\Delta_p u}\varphi} + p\int V|u|^{p-2}u\varphi$$

$$\int_\Omega |\nabla u|^p + \int |u|^p = \|u\|^p_{W^{1,p}} \text{ — norm on } W^{1,p}(\Omega)$$

$$p = 2 \implies W^{1,2}(\Omega) = H^1(\Omega)$$

$$-\Delta_p u = -\nabla \cdot |\nabla u|^{p-2} \nabla u = -\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)$$

Def. (weak sol. to (*)) $u \in W^{1,p}_{loc}(\Omega)$ is a weak solution of (*) if

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi + \int_\Omega V |u|^{p-2} u \varphi = \int_\Omega f \varphi \quad \forall \varphi \in C_c^\infty(\Omega)$$

Remark: We always assume $u \in C_{loc}(\Omega)$

$\implies$ "solution" = weak sol. in $W^{1,p}_{loc}(\Omega) \cap C_{loc}(\Omega)$

For sub / supersolutions replace $=$ by $\leq$ / $\geq$

Radial ansatz: take $u = r^\gamma$, $\Omega = \mathbb{R}^N \setminus \{0\}$

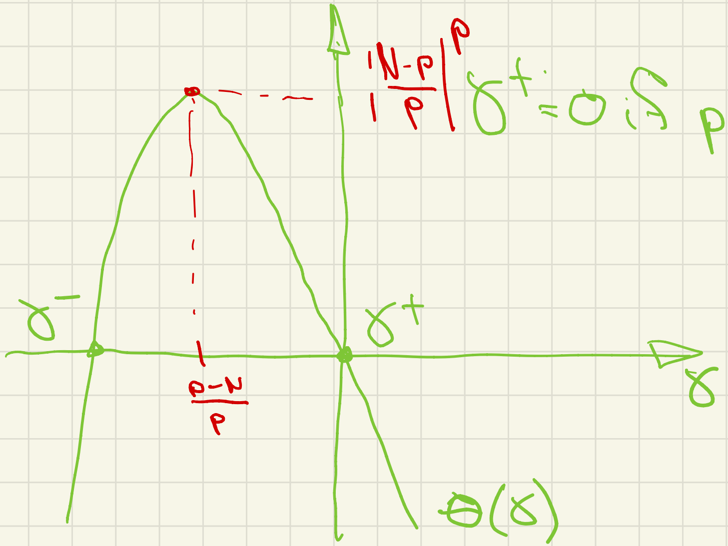
$$-\Delta_p u(|x|) = -r^{1-N}\left(r^{N-1}|u_r|^{p-2}u_r\right)_r =$$

$$= -r^{1-N}\left(r^{N-1}\left|\gamma r^{\gamma-1}\right|^{p-2}\gamma r^{\gamma-1}\right)_r =$$

$$= -r^{1-N}\left(r^{N-1}|\gamma|^{p-2}\gamma\, r^{(\gamma-1)(p-2)+\gamma-1}\right)_r =$$

$$= \underbrace{-\gamma|\gamma|^{p-2}\left(\gamma(p-1)+N-p\right)}\, r^{(\gamma-1)(p-1)-1} = 0\,?$$

$$-\Delta_p |x|^{\gamma} = 0 \quad \text{if} \quad \underbrace{-\gamma|\gamma|^{p-2}\left(\gamma(p-1)+N-p\right)}_{\theta(\gamma)} = 0$$



$\gamma^+ = 0$ if $p<N$

γ^- – smallest root

γ^+ – largest root

$$\gamma^- = -\frac{N-p}{p-1}, \quad \text{if } p \leq N$$

$p=1 \Rightarrow \gamma^- \simeq -\infty$

$p=2 \Rightarrow \gamma^- = -(N-2)$

$p=N \Rightarrow \gamma^- = 0$

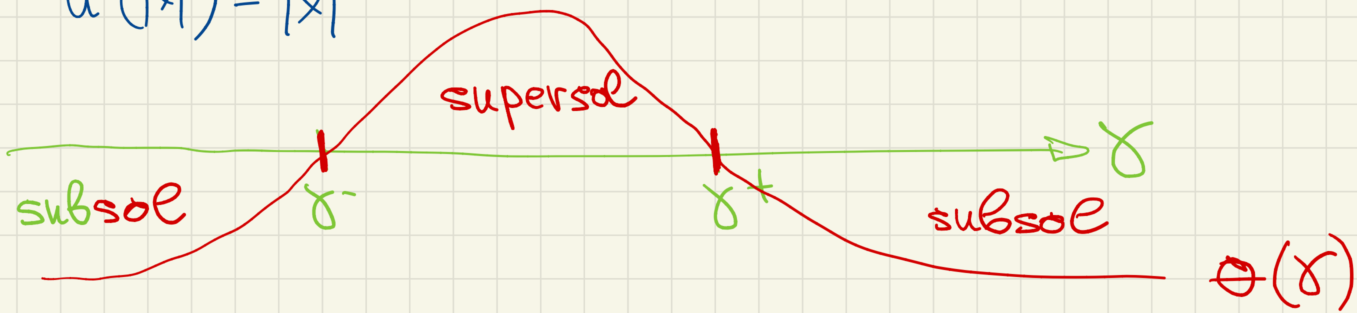
$p>N \Rightarrow \gamma^- = 0,$

$$\gamma^+ = \frac{p-N}{p-1} > 0$$

$$U(x) = |x|^{-\frac{N-p}{p-1}}$$

– "fundamental sol."

$$u(|x|) = |x|^\gamma$$



Picone identity and AAP-principle

Lemma (AAP-principle) $f \geq 0$,

Let $\phi > 0$ be a solution of $-\Delta_p \phi + V\phi^{p-1} \underset{=}{\geq} f$ in Ω

Then $\mathcal{E}_V(u) \underset{=}{\geq} \int_\Omega \underbrace{R(u,\phi)}_{\geq 0} + \int \frac{f}{\phi^{p-1}} |u|^p \quad \forall u \in C_c^\infty(\Omega)$

$$R(u,\phi) = |\nabla u|^p - \nabla\left(\frac{u^p}{\phi^{p-1}}\right) |\nabla \phi|^{p-2} \nabla \phi .$$

Cor. $\mathcal{E}_V(u) \geq \int \frac{f}{\phi^{p-1}} |u|^p$

Take $\psi = \frac{u^p}{\phi^{p-1}}$ as a test function, $\underset{\geq 0}{u} \in C_c^\infty(\Omega)$

$$\int |\nabla\phi|^{p-2} \nabla\phi \, \nabla \frac{u^p}{\phi^{p-1}} + \int V \cancel{\phi^{p-1}} \frac{u^p}{\cancel{\phi^{p-1}}} \overset{\geq}{=} \int f \frac{u^p}{\phi^{p-1}}$$

$$\int V u^p \overset{\geq}{=} (p-1)\int |\nabla\phi|^p \frac{u^p}{\phi^p} + p\int \frac{u}{\phi} \nabla\phi \left|\frac{u\nabla\phi}{\phi}\right|^{p-2} + \int f \frac{u^p}{\phi^{p-1}}$$

$$E_V(u) \overset{\geq}{=} \int |\nabla u|^p + \int V|u|^p \overset{\geq}{=}$$

$$= \int_\Omega \underbrace{\cancel{R(u,\phi)}}_{\geq 0} + \int f \frac{u^p}{\phi^{p-1}}.$$

Corollary (nonexistence principle)

Assume $\mathcal{E}_V(\varphi) < 0$ for a $\varphi \in C_c^\infty(\Omega)$

$\Longrightarrow$ $-\Delta_p u + Vu \geq 0$ has no positive supersolutions!

Exercise (Hardy ineq. for p-Lapl)

$$\int_{\mathbb{R}^N} |\nabla u|^p \geq C_H \int_{\mathbb{R}^N} \frac{|u|^p}{|x|^p} \quad \forall u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$$

$\underbrace{C_H = \left|\frac{N-p}{p}\right|^p}_{\text{sharp}}$ Take $\phi = |x|^{-\frac{N-p}{p}}$ and use AAP

Then $-\Delta_p \underbrace{|x|^{-\frac{N-p}{p}}}_{\phi} - C_H \frac{\overbrace{|x|^{-\frac{N-p}{p}(p-1)}}^{\phi^{p-1}}}{|x|^p} = 0$ in $\mathbb{R}^N \setminus \{0\}$

$$\Rightarrow \int |\nabla u|^p - C_H \int \frac{|u|^p}{|x|^p} \geq 0$$

$$\phi = |x|^{-\frac{N-p}{p}} > 0$$

$$-\Delta_p \phi - C_H \frac{\phi^{p-1}}{|x|^p} = 0$$

$$-\Delta_p u - c \frac{u^{p-1}}{|x|^p} u \geq 0 \quad , \quad c > C_H$$

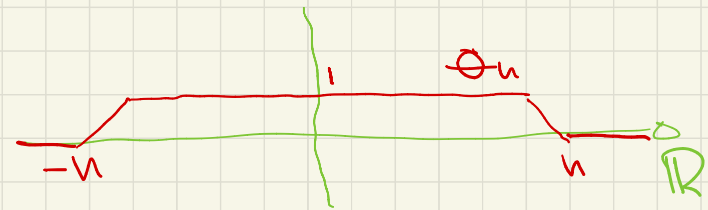
$\Longrightarrow$ no positive sol.!

Comparison principle:

Assumption (S) $\exists \ (\theta_n)_{n\in\mathbb{N}} \subset \overbrace{W^{1,\infty}_c}^{\text{Lipschitz}}(\mathbb{R}^N)$ such that

$$0 \le \theta_n \to 1 \quad \text{a.e. in } \mathbb{R}^N \text{ as } n \to \infty,$$

$$\int_\Omega R(\theta_n u, u) \to 0$$



Then $u \in W^{1,p}_{loc}(\Omega)$ satisfies (S).

Ω — exterior domain in $\mathbb{R}^N$ or $\mathbb{R}^N$.

Comparison principle. Let $f \geq 0$, $f \in L^1_{loc}(\Omega)$

$$u > 0, \quad -\Delta_p u + V u^{p-1} \geq f$$

$$-\Delta_p v + V|v|^{p-2}v \leq f \quad \oplus \quad v \text{ satisfies } (S).$$

Then $u \geq v$ on $\partial\Omega \Rightarrow u \geq v$ in Ω.

Rem. If $p = 2$ and $(u-v)^- \in \mathcal{D}^1_V(\Omega)$
$\Rightarrow$ v satisfies (S).