Some references:

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3. λ-property and nonconvexity for $p \neq 2$

$p=2$ $\quad -\Delta\phi + V\phi = f > 0$ in Ω, $\phi > 0$ in Ω

$\Rightarrow$ AAP-principle: $\mathcal{E}_V(u) = \int \left|\nabla \frac{u}{\phi}\right|^2 \phi^2 + \int \underbrace{\frac{f}{\phi}}_{\lambda(x)>0} u^2 \quad \forall u \in C_c^\infty(\Omega)$

$\mathcal{D}_V^1(\Omega) \subset L^2(\Omega, \lambda(x)dx)$

$p \neq 2$. We say λ-property holds for $-\Delta_p + V$ if $\exists\, \phi > 0 : -\Delta_p\phi + V\phi^{p-1} \overset{(\geq)}{=} f > 0$ in Ω

$\Rightarrow \mathcal{E}_V(u) = \int |\nabla u|^p + \int V|u|^p \overset{(\geq)}{=} \int \underbrace{R(u,\phi)}_{\geq 0} + \int \underbrace{\frac{f}{\phi^{p-1}}}_{\lambda(x)>0} |u|^p$

$\mathcal{E}_V(u) \geq \int \lambda(x)|u|^p$

Lemma. Assume $V^- \neq 0$. Then $\mathcal{E}_V$ is nonconvex!

$p \neq 2$

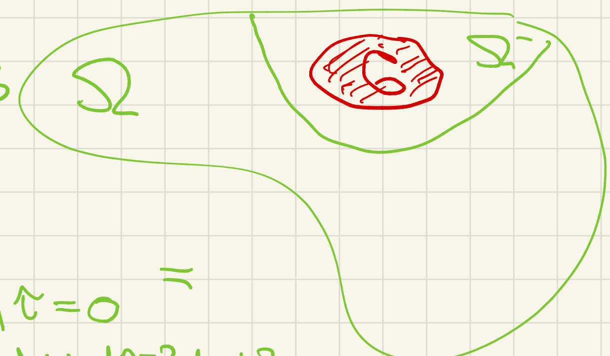
Remark: in fact $\int \lambda(x)|u|^p \le \mathcal{E}_V(u) \le c \int \lambda(x)|u|^p$

◀ del Pino, Elgueta, Manasevich, 1989, $p>2$

Set $\Omega^- = \{x \in \Omega,\ V(x) < 0\}$ – open (assume $V \in C(\Omega)$)

Take $G \Subset \Omega^-$.

Take $u \in C_c^\infty(\Omega)$, $u \equiv 1$ on G



Let $\varphi \in C_c^\infty(G)$.

$$\delta^2 \mathcal{E}_V(u,\varphi) = \frac{d^2}{d\tau^2} \mathcal{E}_V(u+\tau\varphi)\Big|_{\tau=0} =$$

$$= p(p-1)\int_G \underbrace{|\nabla u|^{p-2}}_{=0} |\nabla \varphi|^2 + p(p-1)\int_G V \underbrace{|u|^{p-2}}_{=1} |\varphi|^2 =$$

$$= p(p-1)\int_G \underbrace{V}_{<0} |\varphi|^2 < 0$$ ▶

Remark: 1) $p<2$ Fleckinger, Takac... 1995 (see [3])

2) As a consequence, comparison principle fails "sometimes" and

$$-\Delta_p u + V u^{p-1} = f$$ has more than one solution in general! (only if f changes sign)

if $p \neq 2$, $V^- \neq 0$.

3) no way to define $\cancel{\mathcal{D}^{1,p}_V(\Omega)}$!

4. Small and large solutions

Assume $-\Delta_p + V$ satisfies λ-property, $\Omega = B_1^c$.

Def. We say u is a small (sub) solution if

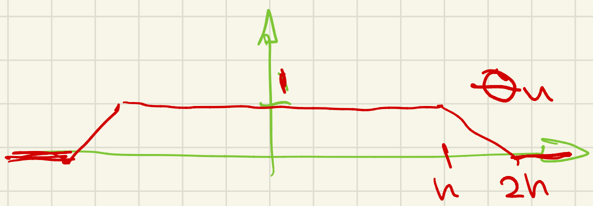
$-\Delta_p u + V u \leq 0$ in Ω, $u > 0$ in Ω and

u satisfies assumption (S):

$\exists\ \theta_n$ - Lipschitz, comp. supp., $0 \leq \theta_n \leq 1$,

$\theta_n \to 1$ as $n \to \infty$ a.e. on Ω,

$$\int_\Omega R(\theta_n u, u) \to 0$$



Lemma. $-\Delta_p w + V w^{p-1} \geq 0$ in B_1^c, $w > 0$
and V is a small sub. sol. $\Longrightarrow$
$\exists\, c > 0$: $w \geq cV$ in $|x| > 2$

Def. We say $U > 0$ is a large (sub)sol. for $-\Delta_p + V f$

$$\begin{cases} -\Delta_p U + V U^{p-1} = 0 & \text{in } \Omega \\ U = 0 & \text{on } \partial\Omega \end{cases}$$

Remark: if U has finite energy: test $\varphi = U$

$$\mathcal{E}_V(U) = \int |\nabla U|^{p-2} \nabla U \cdot \underbrace{\nabla U}_{\nabla \varphi} + \int V |U|^{p-2} U \underbrace{U}_{\varphi} = 0$$

$\Longrightarrow U$ must have ∞ energy!

Lemma (Phragmen-Lindelöf property)

Let $U > 0$ is a large (sub) solution to $-\Delta_p + V$

Let $-\Delta_p w + V w^{p-1} \geq 0$ in Ω, $w > 0 \Longrightarrow$

$$\liminf_{|x| \to \infty} \frac{w}{U} < +\infty .$$

Examples for $V=0$

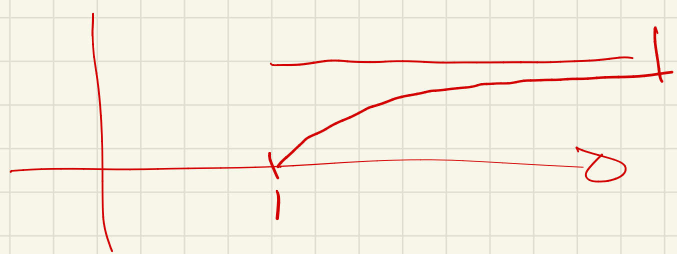
1) $p<N$. Then $|x|^{\gamma}$, $\gamma \le -\frac{N-p}{p-1}$ is small sub-sol.

$1-|x|^{\frac{p-N}{p-1}}$ — a large solution

For small solution — check (S)

For large solution:

$$-\Delta_p\left(1-|x|^{\frac{p-N}{p-1}}\right) = -\nabla\cdot\left|\nabla\left(1-|x|^{\frac{p-N}{p-1}}\right)\right|^{p-2}\nabla\left(1-|x|^{\frac{p-N}{p-1}}\right)=0$$



$p > N$: $|x|^{\gamma}$ with $\gamma \le 0$ — small (sub) sol.

1 — small solution

$|x|^{\frac{p-N}{p-1}} - 1$ — large solution in B_1^c

$p = N$: 1 — small solution

$\log|x|$ — large solution in B_1^c

5. Nonlinear Liouville's theorem

(*) Consider $-\Delta_p u = u^q$ in B_1^c, $1<p<+\infty$, $q \in \mathbb{R}$

- quasilinear

Thm. (*) has no positive supersol. $\Longleftrightarrow$

1) $p<N$: $q \leq q_S := \dfrac{N(p-1)}{N-p}$ $> p-1$

2) $p=N$: $\forall q \in \mathbb{R}$

3) $p>N$: $q \geq q_S$ <0 (Exercise)

$\triangle$ $p<N$, i) $q>p-1$; assume $\exists u>0$: $-\Delta_p u \geq \underbrace{u^q}_{\geq 0}$ in B_1^c

Then $-\Delta_p u \geq 0$ in B_1^c $\Longrightarrow$ $\underline{u \geq c|x|^{-\frac{N-p}{p-1}}}$

$\Longrightarrow$ $-\Delta_p u - \underbrace{V(x)}_{u^{q-(p-1)}} u^{p-1} \geq 0$

$$V(x) = u^{q-(p-1)} \geq c_1 |x|^{(q-(p-1))\frac{p-N}{p-1}} = c_1 |x|^{-p+\varepsilon}$$

$$q-(p-1)\frac{p-N}{p-1} > -p \iff q < q_s = \frac{N(p-1)}{N-p}$$

$\Longrightarrow \varepsilon > 0$

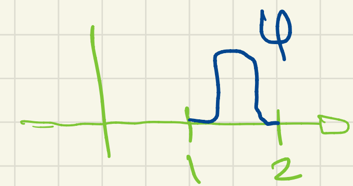
$$\Rightarrow \quad -\Delta_p u - \frac{c_1}{|x|^{p-\varepsilon}} u^{p-1} \geq 0 \quad \text{in } B_1^c, \ \varepsilon > 0.$$

$\Rightarrow$ $u \equiv 0$ by Nonexist. principle,

Because $\exists\, \varphi \in C_c^\infty(\Omega)$:

$$E_V(\varphi) = \int |\nabla \varphi|^p - \int \frac{|\varphi|^p}{|x|^{p-\varepsilon}} < 0.$$

Hint: take $\varphi_R(x) = \varphi\left(\frac{x}{R}\right)$



Critical case: $q = q_s$ – difficult!

$q < p-1$ ($q = p-1$ – Exercise!)

Lemma: $u \geq c_* |x|^{\frac{p}{(p-1)-q}}$ if $q < p-1$

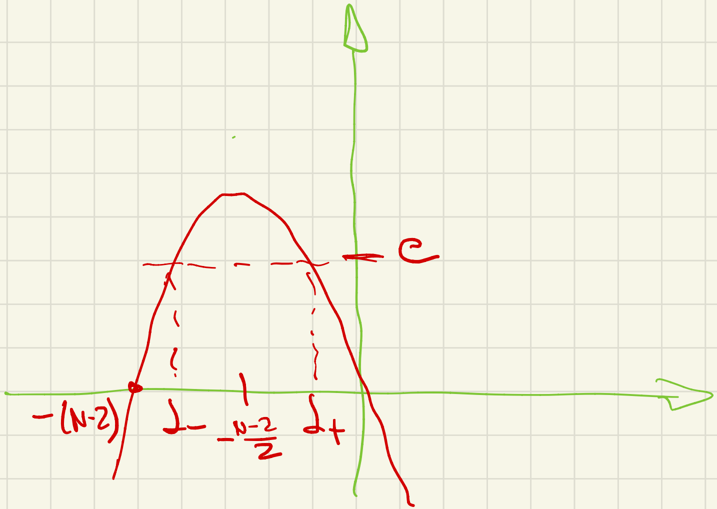
and $-\Delta_p u \geq u^q$ in B_1^c

$\triangle$ $q < p-1$, Assume $-\Delta_p u \geq u^q$ in B_1^c.

$\Rightarrow -\Delta_p u \geq 0$ in B_1^c $\Rightarrow$ (Phragm.-Lind.)

$\limsup\limits_{|x| \to \infty} u < \infty$ $\quad \left(U = 1 - |x|^{-\frac{N-p}{p-1}} \right)$

$\frac{p}{(p-1)-q} > 0$ – contradiction $\Rightarrow u \equiv 0$ $\blacktriangleright$



$-\Delta u - \frac{c}{|x|^2} u$ in $|x|>1$

$c < C_H = \left(\frac{N-2}{2}\right)^2$

r^{α_-} – small sol.

r^{α_+} – large

r^{α} – supersol. if $\alpha_- < \alpha < \alpha_+$