ISTITUTO DI MATEMATICA «R. CACCIOPPOLI»
UNIVERSITÀ DI NAPOLI

METHODS OF FUNCTIONAL ANALYSIS AND THEORY OF ELLIPTIC EQUATIONS

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1. Agmon-Allegretto-Piepenbrink positivity principle

$-\Delta u + V(x)u = f(x)$ in Ω — a domain in $\mathbb{R}^N$, $N \geq 2$

$V(x)$ — a potential, $V = V^+ - V^-$, $V^+ \in L^\infty_{loc}(\Omega)$

$f(x) \in L^1_{loc}(\Omega)$ — R.H.S. $\quad V^- \in L^1_{loc}(\Omega)$

$u \in H^1_{loc}(\Omega) \cap L^1_{loc}(\Omega, V dx)$ — weak supersolution:

$$\int_\Omega \nabla u \nabla \varphi + \int_\Omega V(x) u \varphi \geq \int_\Omega f \varphi \quad \forall \varphi \in H^1_c(\Omega) \cap L^\infty(\Omega),\ \varphi \geq 0$$

"$\leq$" — subsolution

"$=$" — solution

$u \geq 0$ — supersolution in Ω $\Rightarrow$ $\operatorname{ess\,inf}_K u > 0$, $K \subset \Omega$.

$$E_V(\varphi) = \int_\Omega |\nabla\varphi|^2 + \int_\Omega V(x)\varphi^2$$ — energy

We wanto understand when $E_V(\varphi) \geq 0$

$\forall \varphi \in C_c^\infty(\Omega)$

Note that $V = V^+ - V^-$!

Theorem (AAP positivity principle ~ 74')

Assume $g(x) \geq 0$. (*) has a positive supersol. $\boxed{u_*}$

$$\Rightarrow E_V(\varphi) = \underbrace{\int_\Omega \left|\nabla \frac{\varphi}{u_*}\right|^2 u_*^2 \, dx}_{\geq 0} + \underbrace{\int_\Omega \frac{g}{u_*} \varphi^2 \, dx}_{\geq 0} \quad \forall \varphi \in C_c^\infty(\Omega)$$

Corollary: Assume $\exists \varphi_0 \in C_0^\infty$:

$E_V(\varphi_0) < 0 \Rightarrow$ (*) has no positive supersol.

▲ Let $\varphi \in C_c^\infty(\Omega)$. Take $\psi = \underbrace{\frac{\varphi^2}{u_*}}_{\text{ground state transform}} \in H^1_c(\Omega) \cap L^\infty_c(\Omega)$

$u_*^{-1} \in L^\infty_{loc}(\Omega)$

$$\int_\Omega \nabla u_* \nabla \psi + \int V u_* \psi =$$

$$= 2\int_\Omega \nabla u_* \nabla \varphi \frac{\varphi}{u_*} - \int_\Omega |\nabla u_*|^2 \frac{\varphi^2}{u_*^2} + \int_\Omega V \varphi^2 \geq \int_\Omega f \cancel{u_*} \frac{\varphi^2}{\cancel{u_*}}$$

$$\int |\nabla\varphi|^2 + V\varphi^2 - \int \left|\nabla \frac{\varphi}{u_*}\right|^2 u_*^2 =$$

$$= \int |\nabla\varphi|^2 + V\varphi^2 - \int \left(\frac{|\nabla\varphi|^2}{u_*^2} - 2\varphi\nabla\varphi \frac{\nabla u_*}{u_*^3} + \varphi^2 \frac{|\nabla u_*|^2}{u_*^4} \right) u_*^2 =$$

$$= \underbrace{\int |\nabla\varphi|^2 + V\varphi^2}_{E(v)} - \int |\nabla\varphi|^2 + 2\varphi\nabla\varphi \frac{\nabla u_*}{u_*} - \varphi^2 \frac{|\nabla u_*|^2}{u_*^2} \geq \int \lambda_1 \varphi^2$$

Remark: If $V^- \in L^\infty_{loc}(\Omega) \Longrightarrow$
In AAP-principle we have $\Longleftrightarrow$.
[Agmon, Thm. 3.1].

Example (Hardy inequality in $\mathbb{R}^N$, $N \geq 3$)

$$\int_{\mathbb{R}^N} |\nabla \varphi|^2 \geq \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{\varphi^2}{|x|^2} \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$$

◀ Consider $-\Delta u - \underbrace{\left(\frac{N-2}{2}\right)^2}_{c_H} \frac{u}{|x|^2} = 0$ in $\underbrace{\mathbb{R}^N \setminus \{0\}}_{\Omega}$

$$V(x) = \frac{c_H}{|x|^2}$$

Take $u_* = |x|^{-\frac{N-2}{2}}$ — solution! ▶

Example ("Hardy ineq." in $\mathbb{R}^2$)

$$\int_\Omega |\nabla\varphi|^2 \geq \frac{1}{4}\int_\Omega \frac{\varphi^2}{|x|^2(\log|x|)^2} \quad \forall \varphi\in C_c^\infty(\Omega)$$

$\Omega = \mathbb{R}^2 \setminus B_1$.

◀ $u_* = (\log|x|)^{\frac{1}{2}} > 0$ in $|x|>1$

$$-\Delta u_* = \frac{1}{4}\frac{1}{|x|^2(\log|x|)^{3/2}} =$$

$$= \frac{1}{4}\frac{u_*}{|x|^2(\log|x|)^2}$$

▷

Exercise (Improved Hardy ineq. in $\mathbb{R}^N$)

$$\int_\Omega |\nabla\varphi|^2 \geq \left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{\varphi^2}{|x|^2} + \frac{1}{4}\int_\Omega \frac{\varphi^2}{|x|^2(\log|x|)^2}$$

$\forall \varphi \in C^\infty(\Omega)$,

$\Omega = \mathbb{R}^N \setminus B_1$, $N \geq 2$.

$$u_* = |x|^{-\frac{N-2}{2}} (\log|x|)^{\frac{1}{2}}.$$