References – nonlinear Liouville's theorems

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4. Nonlinear Liouville type theorems.

Consider semilinear equation

$$(*) \quad -\Delta u = u^p \quad \text{in } \Omega = \mathbb{R}^N \setminus \overline{B}_1 \qquad N \geq 3, \; p \in \mathbb{R}$$

$0 \leq u \in H^1_{loc}(\Omega)$ is a nonnegative weak supersolution if

$$\int_\Omega \nabla u \nabla \varphi \geq \int_\Omega u^p \varphi, \qquad \forall \varphi \in C_c^\infty(\Omega), \; \varphi \geq 0.$$

If $u \geq 0$ is a supersolution $\Rightarrow$ $-\Delta u \geq 0$ in Ω

$\Rightarrow$ $u > 0$ a.e. in Ω.

Serrin's exponent

Theorem (Serrin $\simeq$ 76')

(*) has a positive supersolution $\Leftrightarrow$ $p > \frac{N}{N-2}$

Difficult part is nonexistence. We consider separately $p>1$ and $p<1$.

If $p=1$ (*) is linear: $-\Delta u = u$ in Ω

Then $E(\varphi) = \int |\nabla\varphi|^2 - \int |\varphi|^2 < 0$ for some $\varphi \in C_0^\infty$

$\Rightarrow$ (*) has no positive supersol by AAP

Case $p>1$. Assume $\exists\, u>0$, $-\Delta u \geq u^p$ in Ω

Then $-\Delta u \geq 0$ in Ω $\Longrightarrow$ $u \geq c|x|^{-(N-2)}$ in B_2^c (3

($|x|^{-(N-2)}$ is a small solution to $-\Delta$!)

$\Longrightarrow$ $-\Delta u - \underbrace{V_1(x)}_{u^{p-1}}\, u \geq 0$ in Ω — linearisation of (*)

By (1), $V_1(x) = u^{p-1} \geq c_1 |x|^{-(N-2)(p-1)}$

$\Longrightarrow$ $-\Delta u - c_1 |x|^{-(N-2)(p-1)} u \geq 0$ in Ω

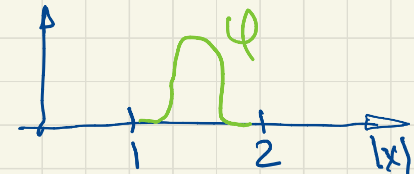
If $1<p<\frac{N}{N-2}$ $\Longrightarrow$ $\underbrace{-(N-2)(p-1)}_{-2+\varepsilon} > -2$

$$\Rightarrow \quad -\Delta u - c_1 |x|^{-2+\varepsilon} u \geq 0 \text{ in } B_1^c, \quad \varepsilon > 0 \qquad (**)$$

Lemma 1. (**) has no positive supersolutions

◁ Consider $E(\varphi) = \int_\Omega |\nabla \varphi|^2 - c_1 \int_\Omega \frac{\varphi^2}{|x|^{2-\varepsilon}}$

Take $\varphi \geq 0$, $\varphi \in C_0^\infty(A_{1,2})$



Set $\varphi_R(x) = \varphi\left(\frac{x}{R}\right)$. Then

$R \gg 1$

$$E(\varphi_R) = R^{N-2} \int |\nabla \varphi|^2 - c_1 R^{N-2+\varepsilon} \int \frac{\varphi^2}{|x|^{2-\varepsilon}}$$

$\to -\infty$ as $R \to \infty$

$\Rightarrow$ (**) has no positive supersol. by AAP ▷

$1 < p < \frac{N}{N-2} \Longrightarrow -\Delta u \geq u^p$ has no pos. supersol in Ω

Consider critical case $p = \frac{N}{N-2}$ $\left(-(N-2)(p-1) = -2 \right)$

Then $-\Delta u - \frac{c_1}{|x|^2} u \geq 0$ in Ω $(**)$

for some $c_1 > 0$.

If $c_1 > c_H = \left(\frac{N-2}{2}\right)^2 \Longrightarrow (**)$ has no pos. supers.

by AAP.

Assume c_1 is small!

Then $-\Delta u - \frac{c_1}{|x|^2} u \geq 0$ in B_1^c, $0 < c_1 < C_H$.

$\Rightarrow$ $u \geq c_2 |x|^{\alpha_-}$ — small solution of $-\Delta - \frac{c_1}{|x|^2}$

α_- is the smallest root of $-\alpha(\alpha+N-2) = c_1$

Note that $-(N-2) < \alpha_1 < -\frac{N-2}{2}$! Then

$$-\Delta u - \underbrace{V_2(x)}_{u^{p-1}} u \geq 0 \text{ in } \Omega,$$

$$V_2(x) = u^{p-1} \geq c_3 |x|^{\underbrace{\alpha_-(p-1)}_{<-2}} = c_3 |x|^{-2+\varepsilon}$$

By Lemma 1 — no pos supersol!

Case $p<1$. $\quad -\Delta u - \underbrace{u^{p-1}} u \geq 0$

$p-1<0 \Longrightarrow$ upper bound on u is needed!

But upper bound on superharmonics $-\Delta u \geq 0$ from large solution is not pointwise!

$$-\Delta u \geq 0 \text{ in } B_1^c \Longrightarrow \liminf_{|x|\to\infty} u \leq +\infty \,!$$

Lemma 2. $-\Delta u \geq u^p$ in B_1^c, $p<1$

$$\Rightarrow \quad u \geq c|x|^{\frac{2}{1-p}}$$

See [3, Lemma 6.1] — uses AAP + weak Harnack

But by Phragmen-Lindelöf,

$$-\Delta u \geq 0 \text{ in } B_1^c \quad \Rightarrow \quad \liminf_{|x|\to\infty} u < +\infty,$$

— incompatible with $u \geq c|x|^{\underbrace{\frac{2}{p-1}}_{>0}}$, $p<1$!

We proved nonexistence $\forall\ p \le \frac{N}{N-2}$:

$p=1$, $1<p<\frac{N}{N-2}$, $p=\frac{N}{N-2}$, $p<1$.

Lemma. Assume $p > \frac{N}{N-2}$. Then

$c|x|^{-\frac{2}{p-1}}$ is a solution of $-\Delta u = u^p$ in $\mathbb{R}^N \setminus \{0\}$,

for some explicit $c = c(N,p)$.

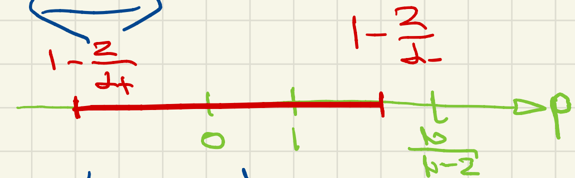
Existence proved. $-(N-2) < -\frac{2}{p-1}$ ◮

Remark: $\exists$ sol. $u>0$, $u \simeq |x|^{-(N-2)}$ as $|x| \to \infty$

Exercise: $-\Delta u + \frac{c}{|x|^2} u \geq u^p$ in B_1^c, $c>0$

has a positive supersolution $\iff$

$$p \in \Big[\underbrace{1-\frac{2}{\alpha_+}}_{<0}, \underbrace{1-\frac{2}{\alpha_-}}_{>1}\Big],$$



where $\alpha_- < \alpha_+$ are roots of $-\alpha(\alpha+N-2)+c=0$

$c>0$

Hint: $p \geq 1$ — the same

$p<1$ — the same + Lemma 2 remains valid.

$$-\Delta V + \frac{c}{|x|^2} V = 0 \quad \text{in } B_1^c \,, \quad c > 0$$

$V_0 = |x|^{\alpha_+} - 1$ $\quad$ ($V_0(x) = 0$ on $|x| = 1$)

– large solution

$V_1 = |x|^{\alpha_-}$

$$-\Delta u + u^p = 0$$

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$$-\Delta u - \frac{c}{x_1} u + \frac{B}{|x|^2} u \quad \text{in } \mathbb{R}^N_+$$