Mathematical Analysis of Problems in Complex Media Electromagnetics

Ioannis G. Stratis*

Mini-courses in Mathematical Analysis
Universitá degli Studi di Padova – July 2018

Abstract

At the macroscopic level, Maxwell's equations read

$$\mathrm{curl} H = \partial_t D + J\,, \tag{1}$$

$$\mathrm{curl} E = -\,\partial_t B\,, \tag{2}$$

$$\mathrm{div} E = \varrho\,, \tag{3}$$

$$\mathrm{div} B = 0\,, \tag{4}$$

where E and H are the electric and the magnetic field, D, B are the electric and magnetic flux densities, respectively, J is the electric current density, and ϱ is the density of the (externally impressed) electric charge.

In this mini-course we will only consider harmonic time dependence ($\exp(-\,\mathrm{i}\,\omega\,\mathrm{t})$, with angular frecuency $\omega > 0$) of all the involved fields.

Constitutive relations (that in general have the form $D = D(E, H), B = B(E, H)$) must be introduced into Maxwell's equations.
We will focus on a very interesting linear case, describing the so-called "reciprocal chiral (or Pasteur) media", and use the "Drude-Born-Fedorov" (DBF) constitutive relations

$$D = \varepsilon(E + \beta\,\mathrm{curl}E)\,, \;\; B = \mu(H + \beta\,\mathrm{curl}H)\,, \tag{5}$$

where ε, μ, β are appropriately regular functions – the electric permittivity, the magnetic permeability and the chirality measure of the complex (chiral) material filling $\mathcal{O}$, which is an open, connected and bounded subset of $\mathbb{R}^3$, with sufficiently smooth boundary $\partial\mathcal{O}$.
By eliminating H we obtain the equation

$$\mathrm{curl}\,(\alpha\,\mathrm{curl}E) = \omega^2\,\left(\mathrm{curl}(\beta\,\varepsilon\,E)) + \beta\,\varepsilon\,\mathrm{curl}E + \varepsilon\,E\right)\,. \tag{6}$$

*Department of Mathematics, National and Kapodistrian University of Athens, Panepistimiopolis, GR 15784 Zographou, Athens, Greece. E-mail: istratis@math.uoa.gr

We consider the (non-homogeneous version of the) perfect conductor boundary condition

$$\widehat{\nu} \times E = f\,, \text{ on } \partial\mathcal{O}\,, \tag{7}$$

where $\widehat{\nu}$ is the outward normal on $\partial\mathcal{O}$, and f is a prescribed electric field on $\partial\mathcal{O}$.

Similar problems arise in many important applications, e.g., diamagnetic structures, Hall-effect devices, magnetohydrodynamics (MHD), toroidal coils, plasma physics, hydrodynamic fluctuations in fluids, magnetic resonance imaging (MRI), etc.

We will first obtain an appropriate weak formulation of the problem ((6),(7)). By introducing a suitable bilinear form we will express this interior problem as a variational problem, and then use standard arguments to establish its solvability.
In order to properly define the functional framework, we will introduce and briefly refer to the required properties of appropriate spaces, e.g., $H(\mathrm{curl}, \mathcal{O})$, $H_0(\mathrm{curl}, \mathcal{O})$, $H(\mathrm{div}, \mathcal{O})$, $H^{-1/2}(\mathrm{curl}, \mathcal{O})$, $H^{-1/2}(\mathrm{curl}, \mathcal{O})$, etc.

Additionally, we intend to discuss briefly the corresponding exterior problem.

If time allows, some remarks on (i) the discretised version, and (ii) the eigenvalue problem (in the case where β is a small non-negative constant) will also be made.

References

[1] Assous, F., Ciarlet, P., Labrunie, S., *Mathematical Foundations of Computational Electromagnetism*, Springer, 2018.

[2] Bossavit, A., *Computational Electromagnetism*, Academic Press, 1998.

[3] Brezis, H., *Sobolev Spaces and Partial Differential Equations*, Springer, 2010.

[4] Cessenat, M., *Mathematical Methods in Electromagnetism*, World Scientific, 1996.

[5] Dautray, R., Lions J.-L., *Mathematical Analysis and Numerical Methods for Science and Technology*, 6 volumes, Springer, 1988 - 1992.

[6] Evans, L. C., *Partial Differential Equations*, 2nd ed., American Mathematical Society, 2010.

[7] Kirsch, A., Hettlich, F., *The Mathematical Theory of Time-Harmonic Maxwell's Equations*, Springer, 2015.

[8] Kristensson, G., *Scattering of Electromagnetic Waves by Obstacles*, SciTech Publishing, an imprint of the IET, 2016.

[9] Nédélec, J.-C., *Acoustic and Electromagnetic Equations*, Springer, 2001.

[10] Roach, G. F., Stratis, I. G., Yannacopoulos, A. N., *Mathematical Analysis of Deterministic and Stochastic Problems in Complex Media Electromagnetics*, Princeton University Press, 2012.