Mathematical Analysis of problems in Complex Media Electromagnetics

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Mini-course (handwritten) Notes
Lectures 2 – 5

Ω : open, bounded, connected and simply connected in $\mathbb{R}^3$

$\partial\Omega$: bounded, connected, $C^{1,1}$ (graph of an everywhere differentiable function with locally Lipschitz continuous gradient)

$n(x)$: the unit outward normal at $x \in \partial\Omega$

THE INTERIOR PROBLEM

$$\begin{cases} \operatorname{curl} E(x) = \beta(x)\gamma^2(x)\, E(x) + i\omega\mu(x)\left(\dfrac{\gamma(x)}{k(x)}\right)^2 H(x) \\ \operatorname{curl} H(x) = \beta(x)\gamma^2(x)\, H(x) - i\omega\varepsilon(x)\left(\dfrac{\gamma(x)}{k(x)}\right)^2 E(x) \end{cases}, \quad x \in \Omega$$

$$n(x) \times E(x) = f(x), \quad x \in \partial\Omega$$

- $\omega > 0$: angular frequency
- $k^2(x) := \omega^2 \varepsilon(x)\mu(x)$, $\gamma^2(x) := \dfrac{k^2(x)}{1 - \beta^2(x)k^2(x)}$
- $\varepsilon, \mu, \beta : \overline{\Omega} \to \mathbb{R}$ positive, C^2-functions
- $\dfrac{1 - \omega^2 \varepsilon(x)\mu(x)\beta^2(x)}{\mu(x)} > 0$

If $f = 0$ the problem is homogeneous; as long as the other parameters are fixed, the homogeneous problem will have nontrivial solutions for specific values of ω (eigenvalues/ resonant frequencies of Ω / "cavity problem").

Assume that we are away from such an eigenvalue.

Then, we will prove that the interior problem is well-posed.

Before we state the corresponding theorem, we will discuss FUNCTION SPACES

$$H^m(\Omega) = W^{m,2}(\Omega) \qquad m \in \mathbb{N}_0 \quad (W^{0,2} = L^2)$$

$H^m_0(\Omega)$ = closure of $C^\infty_0(\Omega)$ in the $H^m(\Omega)$ norm

$\left.\begin{matrix} s \in \mathbb{R} \\ 0 < s < 1 \end{matrix}\right\}$ $H^s(\Omega) = \left\{ u \in L^2(\Omega) : \frac{|u(x) - u(y)|}{|x-y|^{s+\frac{3}{2}}} \in L^2(\Omega \times \Omega) \right\}$

$\left.\begin{matrix} s \in \mathbb{R} \\ s > 1 \end{matrix}\right\}$ $H^s(\Omega) = \left\{ u \in H^{[s]}(\Omega) : D^\alpha u \in H^{s-[s]}(\Omega), \ \forall \alpha \in \mathbb{N}_0^3 : |\alpha| = [s] \right\}$

($[\sigma]$: the biggest integer that is less or equal than σ)

$$\|u\|_{H^s(\Omega)} = \left(\|u\|^2_{H^{[s]}(\Omega)} + \sum_{|\alpha| = [s]} \int_{\Omega \times \Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{3+2(s-[s])}} \, dx dy \right)^{1/2}$$

$H^s_0(\Omega)$ = closure of C^∞_0 in the $H^s(\Omega)$ norm

$H^{-s}(\Omega)$ = the dual space of $H^s_0(\Omega)$

With our assumptions on Ω and $\partial\Omega$, there exists a unique linear continuous map (the "trace" of u on $\partial\Omega$) $\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega)$: for any $u \in H^1(\Omega) \cap C(\overline{\Omega})$

$$\gamma_0(u) = u\big|_{\partial\Omega} .$$

$$H^{1/2}(\partial\Omega) := \gamma_0(H^1(\Omega))$$

$H^{-1/2}(\partial\Omega)$:= the dual space of $H^{1/2}(\partial\Omega)$

To define H^s on Γ, assume first that Γ is of the special form

$$\Gamma_0 = \{ (y, \psi(y)) : y \in \mathbb{R}^2 \}$$

$$\psi : \mathbb{R}^2 \to \mathbb{R} \text{ a } C^{k-1,1} \text{ function } (k \in \mathbb{N})$$

Then set

$$u_\psi(y) = u(y, \psi(y)), \quad y \in \mathbb{R}^2$$

and then

$$H^s(\Gamma_0) := \{ u \in L^2(\Gamma_0) : u_\psi \in H^s(\mathbb{R}^2) \}, \quad 0 \le |s| \le k.$$

This is a Hilbert space, equipped with the inner product

$$(u, v)_{H^s(\Gamma_0)} := (u_\psi, v_\psi)_{H^s(\mathbb{R}^2)}.$$

Further, $H^{-s}(\Gamma_0)$ = the dual space of $H^s(\Gamma_0)$.

When Γ is a general $C^{k-1,1}$ surface, we consider a finite cover with each Γ_j transformed by rotation and translation to the form of Γ_0, and consider a partition of unity associated to this cover. Then

$$u \in H^s(\Gamma) \iff \theta_j u \in H^s(\Gamma_j)$$

and

$$\|u\|_{H^s(\Gamma)} = \sum \|\theta_j u\|_{H^s(\Gamma_j)}$$

which is proved to be independent of the choice of the partition of unity

$$H^{-s}(\Gamma) = \text{the dual space of } H^s(\Gamma)$$

Note that

$H^s(\partial\Omega)$ can be seen as the space of traces of $H^{s+\frac{1}{2}}(\Omega)$.

Fractional index Sobolev spaces also arise as interpolation spaces.

Spaces of tangential fields

$$v : \overline{\Omega} \to \mathbb{R}^3 \quad \rightsquigarrow \quad v\big|_{\partial\Omega} = \left(n \cdot v\big|_{\partial\Omega}\right) n + \left(n \times v\big|_{\partial\Omega}\right) \times n$$

NORMAL TRACE OPERATOR $\quad \gamma_n : v \longmapsto n \cdot v\big|_{\partial\Omega}$

TANGENTIAL TRACE OPERATOR $\quad \gamma_\tau : v \longmapsto n \times v\big|_{\partial\Omega}$

TANGENTIAL COMPONENTS TRACE OPERATOR $\quad \pi_{\partial\Omega} : v \mapsto \left(n \times v\big|_{\partial\Omega}\right) \times n$

- $TL^2(\partial\Omega) = \left\{ u \in \left(L^2(\partial\Omega)\right)^3 : \ \gamma_n(u) = 0 \right\}$
- $TH^s(\partial\Omega) = \left\{ u \in \left(H^s(\partial\Omega)\right)^3 : \ \gamma_n(u) = 0 \right\}$

Tangential Differential Operators

- Vector Valued acting on Scalar Fields
  - $\mathrm{Grad}\, \varphi := \pi_{\partial\Omega}\left(\mathrm{grad}\, \Phi\right)$ $\qquad \varphi : \partial\Omega \to \mathbb{R}$, $\quad \underset{\text{(extension)}}{\Phi : \overline{\Omega} \to \mathbb{R}} : \Phi\big|_{\partial\Omega} = \varphi$
  - $\overrightarrow{\mathrm{Curl}}\, \varphi := -\gamma_\tau\left(\mathrm{Grad}\, \varphi\right)$
- Scalar acting on (tangential) Vector Fields
  - $\mathrm{Curl}\, v := \gamma_n\left(\mathrm{curl}\, V\right)$ $\qquad v : \partial\Omega \to \mathbb{R}^3$, $\quad \underset{\text{(extension)}}{V : \overline{\Omega} \to \mathbb{R}^3} : V\big|_{\partial\Omega} = v$
  - $\left\langle \mathrm{Div}\, v, \tilde{\varphi} \right\rangle_{L^2(\partial\Omega)} := -\left\langle v, \mathrm{Grad}\, \tilde{\varphi} \right\rangle_{\left(L^2(\partial\Omega)\right)^3}, \quad \forall\, \tilde{\varphi} \in C_0^\infty(\partial\Omega)$

Laplace-Beltrami Operator

$$\Delta_{\partial\Omega} \varphi := \mathrm{Div}\, \mathrm{Grad}\, \varphi = -\mathrm{Curl}\, \overrightarrow{\mathrm{Curl}}\, \varphi$$

We also need the following Hilbert space

$$H^s(\mathrm{div},\partial\Omega)=\{u\in TH^s(\partial\Omega):\ \mathrm{Div}\,u\in H^s(\partial\Omega)\}$$

endowed with the norm

$$\|u\|^2_{H^s(\mathrm{div},\partial\Omega)}=\|u\|^2_{TH^s(\partial\Omega)}+\|\mathrm{Div}\,u\|_{H^s(\partial\Omega)}.$$

For $s=-\frac{1}{2}$ the space $H^{-1/2}(\mathrm{div},\partial\Omega)$ is the natural energy space which occurs in E/M potential theory.
Similarly: $H^{-1/2}(\mathrm{curl},\partial\Omega)$.

Spaces for Electromagnetics

- $H(\mathrm{curl},\Omega)=\{u\in(L^2(\Omega))^3:\ \mathrm{curl}\,u\in(L^2(\Omega))^3\}$ Hilbert sp

$$\|u\|^2_{H(\mathrm{curl},\Omega)}=\|u\|^2_{(L^2(\Omega))^3}+\|\mathrm{curl}\,u\|_{(L^2(\Omega))^3}$$

$\left((C_0^{0,1}(\bar\Omega))^3 \text{ and } (C_0^\infty(\bar\Omega))^3 \text{ are dense in } H(\mathrm{curl},\Omega)\right)$

This is the energy space in E/M. Further,

- $H_0(\mathrm{curl},\Omega)=\{u\in H(\mathrm{curl},\Omega):\ n\times u|_{\partial\Omega}=0\}$: for Ω:bdd
- $H(\mathrm{div},\Omega)=\{u\in(L^2(\Omega))^3:\ \mathrm{div}\,u\in L^2(\Omega)\}$

$$\|u\|^2_{H(\mathrm{div},\Omega)}=\|u\|^2_{(L^2(\Omega))^3}+\|\mathrm{div}\,u\|_{L^2(\Omega)}$$

- $H_0(\mathrm{div},\Omega)=\{u\in H(\mathrm{div},\Omega):\ n\cdot u|_{\partial\Omega}=0\}$ (for Ω: bounded)
- $H(\mathrm{div}0,\Omega)=\{u\in(L^2(\Omega))^3:\ \mathrm{div}\,u=0\}$

(and $H(\mathrm{curl}0,\Omega)$, $H_0(\mathrm{div}0,\Omega)$, $H_0(\mathrm{curl}0,\Omega)$, $H(\mathrm{curl},\mathrm{div},\Omega)$, ...)

The corresponding spaces for exterior problems, in $\Omega_{ext}=\mathbb{R}^3\setminus\Omega$, are indicated by X_{loc}, but we won't deal with them.

THEOREM

Assume that ω is not an eigenvalue of

$$(1) \quad \operatorname{curl} E = \beta\gamma^2 E + i\omega\mu\left(\frac{\gamma}{k}\right)^2 H$$
$$(2) \quad \operatorname{curl} H = \beta\gamma^2 H - i\omega\varepsilon\left(\frac{\gamma}{k}\right)^2 E \quad , \text{ in } \Omega,$$
$$(3) \quad n \times E = 0 \quad , \text{ on } \partial\Omega.$$

Then the interior problem (1), (2)

$$(4) \quad n \times E = f \quad , \text{ on } \partial\Omega,$$

has a unique solution in $H(\operatorname{curl}, \Omega) \times H(\operatorname{curl}, \Omega)$ for every $f \in H^{-1/2}(\operatorname{div}, \partial\Omega)$, under the assumptions

(5) $\varepsilon, \mu, \beta : \overline{\Omega} \to \mathbb{R}$ are positive $C^1(\overline{\Omega})$ functions

(6) $\dfrac{1-\omega^2\varepsilon\mu\beta}{\mu} > 0$, in $\overline{\Omega}$.

and the considered assumptions on Ω and $\partial\Omega$.

PROOF

$\boxed{1}$ Rewrite (1), (2) in terms of E only:

$$\operatorname{curl}\left(\frac{1-\omega^2\varepsilon\mu\beta}{\mu}\operatorname{curl} E\right) = \omega^2\left[\operatorname{curl}(\beta\varepsilon E) + \beta\varepsilon \operatorname{curl} E + \varepsilon E\right]$$

Denote $\quad \theta := \dfrac{1-\omega^2\varepsilon\mu\beta}{\mu}$.

Weak formulation : take dot product of above eqn with a vector function φ, integrate over Ω and use the Green's 2^{nd} formula

$$\int_\Omega \operatorname{curl} v \cdot w \, dx - \int_\Omega v \cdot \operatorname{curl} w \, dx = \left\langle \gamma_\tau(v), \pi_{\partial\Omega}(w) \right\rangle \quad , \forall v, w \in H(\operatorname{curl}, \Omega)$$

The first term becomes

$$\int_\Omega \operatorname{curl}(\theta \operatorname{curl} E) \cdot \overline{\varphi} \, dx = \int_\Omega (\theta \operatorname{curl} E) \cdot \operatorname{curl} \overline{\varphi} \, dx +$$
$$+ \int_{\partial\Omega} (n \times \theta \operatorname{curl} E) \cdot \Big(\underbrace{(n \times \overline{\varphi})}_{=0} \times n\Big) ds(x)$$

If $\varphi \in H_0(\operatorname{curl}, \Omega)$, then $\int_{\partial\Omega} = 0$

so we get

$$\int_{\Omega} (\theta \operatorname{curl} E) \cdot \operatorname{curl} \bar{\psi} = \omega^2 \int_{\Omega} \left(\varepsilon E + \operatorname{curl}(\beta \varepsilon E) + \beta \varepsilon \operatorname{curl} E \right) \cdot \bar{\psi} \, dx$$

and hence we introduce the bilinear form

$$a^{(\beta)}(\cdot,\cdot) : H(\operatorname{curl},\Omega) \times H_0(\operatorname{curl},\Omega) \to \mathbb{C}$$

$$a^{(\beta)}(E,\psi) := \left(\theta \operatorname{curl} E, \operatorname{curl} \psi\right) - \omega^2(\varepsilon E, \psi) - \omega^2\left(\operatorname{curl}(\beta \varepsilon E), \psi\right) - \omega^2\left(\beta \varepsilon \operatorname{curl} E, \psi\right),$$

where $(\cdot,\cdot)$ is the $(L^2(\Omega))^3$ inner product.

$\boxed{2}$ We make homogeneous the boundary condition $n \times E = f$.

Seek $E = U + E_0$, $U \in (H^1(\Omega))^3$: $\gamma_t(U) = f$.

E_0 satisfies

- $\operatorname{curl}(\theta \operatorname{curl} E_0) - \omega^2\left[\operatorname{curl}(\beta \varepsilon E_0) + \beta \varepsilon \operatorname{curl} E_0 + \varepsilon E_0\right] = F$, in Ω,

where $F := -\operatorname{curl}(\theta \operatorname{curl} U) + \omega^2\left[\operatorname{curl}(\beta \varepsilon U) + \beta \varepsilon \operatorname{curl} U + \varepsilon U\right]$

- $n \times E_0 = 0$, on $\partial\Omega$.

Since $f \in H^{-1/2}(\operatorname{div},\partial\Omega)$, U must be in $H(\operatorname{curl},\Omega)$ so that $F \in (H^{-1}(\Omega))^3$.

Considering the duality pairing between $H(\operatorname{curl},\Omega)$ and $(H^{-1}(\Omega))^3$, the equation for E_0 can be written as

$$a^{(\beta)}(E_0,\psi) = \langle F, \psi \rangle.$$

3 So far we have

$$E = U + E_0 \quad , \quad U \in (H^1(\Omega))^3 ; \; n \times U = f \quad , \; f \in H^{-1/2}(\mathrm{div}, \partial\Omega)$$

$$a^{(\beta)}(E_0, \psi) = \langle F, \psi \rangle \qquad (\#)$$

Now, seek E_0 in the form (Helmholtz decomposition)

$$E_0 = e + \operatorname{grad}\phi \, , \quad \phi \in H_0^1(\Omega), \; e \in H_0(\mathrm{curl}, \Omega) \cap H(\mathrm{div}, \Omega)\}$$

So $(\#)$ becomes

$$a^{(\beta)}(e, \psi) + a^{(\beta)}(\operatorname{grad}\phi, \psi) = \langle F, \psi \rangle ,$$

whereby

$$a^{(\beta)}(e, \psi) + b^{(\beta)}(\phi, \psi) = \langle F, \psi \rangle , \quad \forall \psi \in H_0(\mathrm{curl}, \Omega)$$

where

$b^{(\beta)} : H_0^1(\Omega) \times H_0(\mathrm{curl}, \Omega)$ is the bilinear form defined as

$$b^{(\beta)}(\phi, \psi) := -\omega^2 \int_\Omega \left(\operatorname{curl}(\beta\varepsilon \operatorname{grad}\phi) + \varepsilon \operatorname{grad}\phi \right) \cdot \bar{\psi} \, dx$$

4 Eliminate the electrostatic potential ϕ.

Select the test function $\psi \in H_0(\mathrm{curl}, \Omega)$, so that

$$\psi = \operatorname{grad}\xi \, , \quad \xi \in H_0^1(\Omega)$$

(hence $\operatorname{curl}\psi = 0$), integrate by parts using the divergence theorem $\int_\Omega \operatorname{div} v \, dx = \int_{\partial\Omega} \gamma_n(v) \, ds$, use also that $\operatorname{div}\operatorname{curl} = 0$, and after some algebra, get

$$\omega^2 \left[\int_\Omega \operatorname{div}(\beta\varepsilon \operatorname{curl} e + \varepsilon e) \cdot \xi \, dx + \int_\Omega \varepsilon \operatorname{grad}\phi \cdot \operatorname{grad}\xi \, dx \right) =$$

$$= \int_\Omega F \cdot \operatorname{grad}\xi \, dx$$

if we choose e so that

$$\operatorname{div}(\beta\varepsilon \operatorname{curl} e + \varepsilon e) = 0 \quad \left(\Leftrightarrow \operatorname{div}(\varepsilon e) = = \operatorname{grad}(\beta\varepsilon)\cdot \operatorname{curl} e\right)$$

or, its equivalent weak form

$$\int_\Omega (\beta\varepsilon \operatorname{curl} e + \varepsilon e)\cdot \operatorname{grad}\xi \, dx = 0 \ , \quad \forall \xi \in H_0^1(\Omega)$$

the previous relation becomes

$$\omega^2 \int_\Omega \varepsilon \operatorname{grad}\phi \cdot \operatorname{grad}\xi \, dx = \int_\Omega F\cdot \operatorname{grad}\xi \, dx, \qquad (\ddagger)$$

which is the weak form of an elliptic equation for ϕ.

- Let us make a subtle remark here: although the embeddings of $H(\operatorname{curl},\Omega)$ (as well as of $H(\operatorname{div},\Omega)$, and of their intersections with ker curl or ker div) are NOT compact for "regular" bounded Ω, it is known that the space

$$N^{(\beta)} = \{u \in H_0(\operatorname{curl},\Omega) : \operatorname{div}(\varepsilon u) = \operatorname{grad}(\beta\varepsilon)\cdot \operatorname{curl} u \in L^2(\Omega)\}$$

is compact in $(L^2(\Omega))^3$ (Leis for smooth domains, Weck / Weber / R. Picard for more general)

under our assumptions on ε, β.

Return to $(\ddagger)$ and write it as

$$\tilde{a}(\phi,\xi) := \omega^2 \int_\Omega \varepsilon \operatorname{grad}\phi \cdot \operatorname{grad}\xi \, dx = \int_\Omega F\cdot \operatorname{grad}\xi \, dx$$

where the form

$$\tilde{a} : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{C}$$ is (clearly) bounded & bilinear.

Then use Lax-Milgram to obtain that there exists a unique $\phi \in H_0^1(\Omega)$ solving $\tilde{a}(\phi,\xi) = \int_\Omega F\cdot \operatorname{grad}\xi \, dx$, and that it satisfies

$$\|\operatorname{grad}\phi\|_{(L^2(\Omega))^3} \leq \overbrace{c\,\|F\|_{(H^{-1}(\Omega))^3} \leq \tilde{C}\,\|f\|_{H^{-1/2}(\operatorname{div},\partial\Omega)}}^{\text{follows from the defn of } F \text{ and the properties of } R}$$

Classical Lax-Milgram Lemma

H : Hilbert space

$a : H \times H \to \mathbb{C}$ bounded (continuous) $|a(x,y)| \leq \gamma |x||y|$

coercive $a(x,x) \geq \delta \|x\|^2$

sesquilinear form • $a(x+y, z+w) = a(x,z) + a(x,w) + a(y,z) + a(y,w)$

• $a(\rho x, \sigma y) = \bar{\rho}\sigma\, a(x,y)$

▶ Then, there exists $A : H \to H$, $A \in \mathcal{L}(H,H)$ defined by

$$a(x,y) = \langle Ax, y\rangle$$

with $|A| = \|a\| := \sup\limits_{x,y\neq 0} \frac{|a(x,y)|}{|x||y|} \leq \gamma$, $\forall x,y \in H$.

Moreover,

$$a(A^{-1}x, y) = \langle x, y\rangle,$$

with $|A^{-1}| \leq \frac{1}{\delta}$, $\forall x,y \in H$.

▶ Then, the variational problem $a(u,v) = \langle f, v\rangle$ has a solution in H, for all $f \in H'$.

$\begin{cases} -\Delta u = f, & \Omega \\ u = 0, & \partial\Omega \end{cases}$ WEAK FORM : $\hat{a}(u,v) = \langle f, v\rangle$, $\forall v \in H_0^1(\Omega)$

$$\hat{a}(u,v) = \int_\Omega \operatorname{grad} u(x) \cdot \operatorname{grad} v(x)\, dx \;\Big\}\; \langle f, v\rangle = \int_\Omega f(x) v(x)\, dx$$

Gårding's Inequality : $\exists\, G > 0$: $\hat{a}(u,v) \geq c\|u\|^2_{H^1(\Omega)} - G\|u\|^2_{L^2(\Omega)}$,

for all $u \in H_0^1(\Omega)$

Poincaré's Inequality : $\|u\|_{L^2(\Omega)} \leq \tilde{c}\, \|\operatorname{grad} u\|_{L^2(\Omega)}$

$\tilde{c} = \tilde{c}(\Omega)$

$\boxed{5}$ Determination of e.

Substitute ϕ in

$$\mathrm{curl}(\theta\, \mathrm{curl}(\overbrace{e + \mathrm{grad}\phi}^{E_\#})) - \omega^2 \Big[\mathrm{curl}(\beta\varepsilon(e + \mathrm{grad}\phi)) + \beta\varepsilon\, \mathrm{curl}(e + \mathrm{grad}\phi) + \varepsilon(e + \mathrm{grad}\phi)\Big] =$$

$$= \underbrace{-\mathrm{curl}(\theta\, \mathrm{curl}\, U) - \omega^2\Big[\mathrm{curl}(\beta\varepsilon U) + \beta\varepsilon\, \mathrm{curl}\, U + \varepsilon U\Big]}_{F}$$

so, get (curl grad = 0)

$$\mathrm{curl}(\theta\, \mathrm{curl}\, e) - \omega^2\Big[\mathrm{curl}(\beta\varepsilon e) + \beta\varepsilon\, \mathrm{curl}\, e + \varepsilon e\Big] =$$

$$= F + \omega^2\Big[\mathrm{curl}(\beta\varepsilon\, \mathrm{grad}\phi) + \varepsilon\, \mathrm{grad}\phi\Big]$$

or, in variational notation

$$a^{(\beta)}(e,\psi) = (F,\psi) + \omega^2\big(\mathrm{curl}(\beta\varepsilon\, \mathrm{grad}\phi), \psi\big) + \omega^2\big(\varepsilon\, \mathrm{grad}\phi, \psi\big), \quad \forall\, \psi \in H_0(\mathrm{curl}, \Omega).$$

$\boxed{6}$ Rewrite the above as

$$\overbrace{(\theta\, \mathrm{curl}\, e, \mathrm{curl}\,\psi) + \omega^2(\beta\varepsilon\, \mathrm{curl}\, e, \psi) + \omega^2(\varepsilon e, \psi)}^{a_+^{(\beta)}(e,\psi)}$$

$$-\omega^2\big(\mathrm{curl}(\beta\varepsilon e), \psi\big) - 2\omega^2(\beta\varepsilon\, \mathrm{curl}\, e, \psi) - 2\omega^2(\varepsilon e, \psi) =$$

$$= (F,\psi) + \omega^2\big(\mathrm{curl}(\beta\varepsilon\, \mathrm{grad}\phi), \psi\big) + \omega^2\big(\varepsilon\, \mathrm{grad}\phi, \psi\big),$$

and note that, by our assumptions on $\theta, \varepsilon, \beta, \mu$, $a_+^{(\beta)}(e,\psi)$ is a bounded, coercive, bilinear form on $H(\mathrm{curl},\Omega) \times H_0(\mathrm{curl},\Omega)$.

Hence, by Lax-Milgram, for any $\mathcal{F} \in (H_0^{-1}(\Omega))^3$, the variational problem

$$a_+^{(\beta)}(e,\psi) = (\mathcal{F},\psi), \quad \forall\, \psi \in H_0(\mathrm{curl},\Omega)$$

has a unique solution.

$\boxed{7}$ Perturbation of coercive form – Fredholm Theory

Define the space $\mathcal{N}^{(\beta)}$ as in page 9, and recall that it is it is compactly embedded in $(L^2(\Omega))^3$.

Let ϕ be the solution of the variational problem

$$a_+^{(\beta)}(\phi,\psi) = -\omega^2\Big[(\mathrm{curl}(\beta\varepsilon e),\psi) + 2(\beta\varepsilon\,\mathrm{curl}\,e,\psi) + 2(\varepsilon e,\psi)\Big],$$

for all $\psi \in \mathcal{N}^{(\beta)}$.

Define the operator $\mathbb{K} : (L^2(\Omega))^3 \to (L^2(\Omega))^3 : \mathbb{K}e = \phi$.

i) $\mathbb{K}$ is well-defined (by Lax-Milgram)

ii) $e \in \mathcal{N}^{(\beta)} \Rightarrow \mathbb{K}e \in \mathcal{N}^{(\beta)}$

- $\mathrm{div}\,RHS = 0$, since $e \in \mathcal{N}^{(\beta)} \underset{\text{defn}}{\Rightarrow} \mathrm{div}(\beta\varepsilon\,\mathrm{curl}\,e + \varepsilon e) = 0$

  and $\mathrm{div}\,\mathrm{curl} = 0$

- $\mathrm{div}\,LHS = \mathrm{div}(\varepsilon\mathbb{K}e + \beta\varepsilon\,\mathrm{curl}\,\mathbb{K}e)$, since $\mathrm{div}\,\mathrm{curl} = 0$

Hence $\mathrm{div}(\varepsilon\mathbb{K}e + \beta\varepsilon\,\mathrm{curl}\,\mathbb{K}e) = 0 \underset{\text{defn}}{\Leftrightarrow} \mathbb{K}e \in \mathcal{N}^{(\beta)}$

(This formal argument becomes rigorous, since – by the regularity of the solution of the above variational problem – $e \in H_0(\mathrm{curl},\Omega)$.)

iii) $\mathbb{K}$ maps bounded subsets of $(L^2(\Omega))^3$ into bounded sets of $\mathcal{N}^{(\beta)}$.

iv) $\mathcal{N}^{(\beta)} \underset{c}{\hookrightarrow} (L^2(\Omega))^3$

So, eventually, $\mathbb{K}$ is compact.

Now define the map $\mathbb{G}$ as: given ϕ, let

$\mathbb{G}\phi$ be the solution of the variational problem

$$a_+^{(\beta)}(\mathbb{G}\phi, \psi) = (F,\psi) + \omega^2\Big[\big(\mathrm{curl}(\beta\varepsilon\,\mathrm{grad}\phi),\psi\big) + (\varepsilon\,\mathrm{grad}\phi,\psi)\Big],$$

for all $\psi \in N^{(\beta)}$.

is well-defined (by Lax-Milgram).

In view of the relations defining $\mathbb{K}$ and $\mathbb{G}$, the relation

$$a_+^{(\beta)}(e,\psi) - \omega^2\Big[\big(\mathrm{curl}(\beta\varepsilon e),\psi\big) - 2\big(\beta\varepsilon\,\mathrm{curl}\,e,\psi\big) - 2(\varepsilon e,\psi)\Big] =$$

$$= (F,\psi) + \omega^2\Big[\big(\mathrm{curl}(\beta\varepsilon\,\mathrm{grad}\phi),\psi\big) + (\varepsilon\,\mathrm{grad}\phi,\psi)\Big]$$

of page 11, can be written as

$$a_+^{(\beta)}(e,\psi) + a_+^{(\beta)}(\mathbb{K}e,\psi) = a_+^{(\beta)}(\mathbb{G}\phi,\psi)$$

for all $\psi \in H_0(\mathrm{curl},\Omega)$.

Since $a_+^{(\beta)}$ is bilinear, the above relation can be written as

$$(\mathbb{I} + \mathbb{K})e = \mathbb{G}\phi.$$

Since $\mathbb{K}$ is a compact operator, the Fredholm Alternative ascertains that for every $\mathbb{G}\phi$, the inhomogeneous equation $(\mathbb{I}+\mathbb{K})e = \mathbb{G}\phi$ has a unique solution depending continuously on $\mathbb{G}\phi$, if the homogeneous equation

$$(\mathbb{I}+\mathbb{K})e = 0$$

has only the trivial solution, which is the same as

$$a_+^{(\beta)}(e + \mathbb{K}e,\psi) = (0,\psi).$$

QED (cf. page 11)

Remark

A – not very different – approach for the proof of the solvability result uses a mixed variational formulation and the Babuška-Brezzi generalisation of the Lax-Milgram Lemma (next page).

A suitable variational formulation is needed:

E is decomposed as

$$E = \mathcal{V} + e + \operatorname{grad}\phi$$

$\mathcal{V} \in (H_0^1(\Omega))^3 \; : \; n \times \mathcal{V} = f$

$e \in H_0(\operatorname{curl}, \Omega) \cap H(\operatorname{div}, \Omega)$

$\phi \in H_0^1(\Omega)$

$$\left| \begin{array}{l} a^{(P)}(e, w) + b^{(P)}(\phi, w) = a^{(P)}(\mathcal{V}, w) \quad , \; \forall w \in H_0(\operatorname{curl}, \Omega) \\ b^{(P)}(\psi, e) = 0 \quad , \; \forall \psi \in H_0^1(\Omega) \end{array} \right.$$

$$|a^{(P)}(e,e)| \geq c_1 \|\operatorname{curl} e\|^2_{(L^2(\Omega))^3} - c_2 \|e\|^2_{(L^2(\Omega))^3} \; , \; \begin{array}{l} c_1 > 0 \\ c_2 > 0 \end{array}$$

$$|b^{(P)}(\psi, \operatorname{grad}\psi)| \geq c_3 \|\psi\|^2_{H_0^1(\Omega)} \quad , \; c_3 > 0$$

The Babuška-Brezzi inf-sup condition is satisfied for $b^{(P)}$. Further,

$\ker\{b^{(P)}\} = \mathcal{N}^{(P)} \underset{c}{\hookrightarrow} (L^2(\Omega))^3$, so $a^{(P)}$ is a compact perturbation of a coercive form on $\ker\{b^{(P)}\}$.

The Fredholm Alternative completes the proof.

Generalization of Lax-Milgram Lemma

H_1, H_2 : Hilbert spaces

$a : H_1 \times H_2 \to \mathbb{C}$: bounded, sesquilinear form such that

(i) $\exists\, c > 0$:

$$\inf_{u \in A_1} \sup_{v \in A_2} |a(u,v)| \geq \alpha > 0$$

where $A_1 := \{ u \in H_1 : \|u\|_{H_1} = 1 \}$,

$A_2 := \{ v \in H_2 : \|v\|_{H_2} \leq 1 \}$.

INF-SUP or BABUŠKA-BREZZI CONDITION

(ii) $\forall\, 0 \neq v \in H_2$: $\sup_{u \in H_1} |a(u,v)| > 0$.

Then for every $f \in H_2'$, there exists a unique $u \in H_1$:

$$a(u,v) = (f,v)_{H_2}, \ \forall v \in H_2.$$

The solution to this problem satisfies the bound

$$\|u\|_{H_1} \leq \frac{c}{\alpha} \|f\|_{H_2'}$$

We also have the following result (mixed variational problem)

Let H and S be Hilbert spaces, and let $a : H \times H \to \mathbb{C}$ and $b : H \times S \to \mathbb{C}$ be bounded, sesquilinear forms such that $\exists\, \alpha > 0$: (T-coercivity of $a(\cdot,\cdot)$)

$$|a(u,u)| \geq \alpha \|u\|_H^2 \quad \text{for all } u \in T := \{ u \in H : b(u,\xi) = 0 \ \forall \xi \in S \}, \quad (*)$$

where α is independent of u, and $\exists\, \beta > 0$:

$$\sup_{w \in H} \frac{b(w,p)}{\|w\|_H} \geq \beta \|p\|_S, \qquad \text{(Babuška-Brezzi condition)} \qquad (**)$$

where β is independent of p.

Let $f \in H'$ and $g \in S'$. Then, the problem of finding $u \in H$ and $p \in S$ such that

$$a(u,\phi) + b(\phi,p) = f(\phi), \ \forall \phi \in H$$
$$b(u,\xi) = g(\xi), \ \forall \xi \in S$$

has a unique solution, and $\|u\|_H + \|p\|_S \leq C \left[\|f\|_{H'} + \|g\|_{S'} \right]$.

The Eigenvalue Problem

Ω : simply connected bounded, open, connected in $\mathbb{R}^3$
$\partial\Omega$: bdd, connected, $C^{1,1}$

$$L^{(\beta)} : H_0(\mathrm{curl},\Omega) \to (H^{-1}(\Omega))^3$$

$$v \mapsto L^{(\beta)} v := \mathrm{curl}(\theta\, \mathrm{curl}\, v) - \omega^2\left[\mathrm{curl}(\beta\varepsilon v) + \beta\varepsilon\, \mathrm{curl}\, v + \varepsilon v\right]$$

$$\left(a^{(\beta)}(E,\psi) = \langle L^{(\beta)}E, \psi\rangle \quad , \forall \psi \in H_0(\mathrm{curl},\Omega)\right)$$

In terms of the above, the eigenvalue problem is to find ω such that the problem

$$L^{(\beta)}E = 0,$$

or,

$$a^{(\beta)}(E,\psi;\omega^2) = 0 \quad , \forall \psi \in H_0(\mathrm{curl},\Omega),$$

admits nontrivial solutions.

Assume that $\beta \geq 0$ is a constant and, further, that it is small (physically relevant assumption).

- First consider the case $\beta = 0$.

$$L^{(0)}v = \mathrm{curl}(\mu^{-1}\mathrm{curl}\, v) - \omega^2 \varepsilon v$$

$$E = e + \mathrm{grad}\,\phi \qquad \left((L^2(\Omega))^3 = \mathrm{grad}\, H_0^1(\Omega) + H(\mathrm{div}\,0,\Omega)\right)$$ Helmholtz decomposition

$$e \in N^{(0)} = \{u \in H_0(\mathrm{curl},\Omega) : \mathrm{div}(\varepsilon u) \in L^2(\Omega)\}$$

$$\phi \in H_0^1(\Omega)$$

$$a^{(0)}(e,\psi;\omega^2) = 0 \quad , \forall \psi \in N^{(0)}$$

Friedrichs's Inequality : Ω : bounded, simply connected, Lipschitz with $\partial\Omega = \Sigma \cup \Gamma$, $\Sigma \cap \Gamma = \emptyset$, Σ,Γ : connected.
Then $\exists c>0 : \forall w \in N^{(0)}$ we have

$$\|w\|_{(L^2(\Omega))^3} \leq c\left[\|\mathrm{curl}\, w\|_{(L^2(\Omega))^3} + \|n \times w\|_{(L^2(\Sigma))^3}\right]$$

Note that in our case $\Sigma = \emptyset$.

If $\omega^2 = 0$ we have that

$$\left(\mathrm{curl}(\mu^{-1}\mathrm{curl}E), \psi\right) = 0$$

$$\Rightarrow \left(\mu^{-1}\mathrm{curl}E, \mathrm{curl}\psi\right) = 0 \quad \forall \psi \in N^{(0)}$$

$$\Rightarrow \left(\mu^{-1}\mathrm{curl}e, \mathrm{curl}\psi\right) = 0 \quad \left.\begin{array}{l}\forall \psi \in N^{(0)} \\ e \in N^{(0)}\end{array}\right\} \xRightarrow{\text{Friedrich}} e = 0.$$

So $\omega^2 = 0$ is an eigenvalue of infinite multiplicity, with corresponding eigenfunctions $E = \mathrm{grad}\,\phi$, $\phi \in H_0^1(\Omega)$.
These eigenvalues are not considered to be physically relevant: since no sources are present $\mathrm{div}(\varepsilon E)$ must be 0. Then $\mathrm{div}(\varepsilon\, \mathrm{grad}\phi) = 0$, and since $\phi \in H_0^1(\Omega) \Rightarrow \phi = 0$.

- Rewrite the eigenvalue problem ($\omega^2 > 0$ now) as

$$a_+^{(0)}(e, \psi; 1) = -\, b^{(0)}(e, \psi; \omega^2 + 1), \quad \forall \psi \in N^{(0)}$$

and define the operator $\mathcal{K} : (L^2(\Omega))^3 \to (L^2(\Omega))^3$ which acts on a function w, giving the solution of

$$a_+^{(0)}(\mathcal{K}w, \psi; 1) = -\, b^{(0)}(w, \psi; 1), \quad \forall \psi \in N^{(0)}.$$

Similarly as for K, it follows that $\mathcal{K}$ is compact. Further, using the equivalent, weighted by ε, norm of $(L^2(\Omega))^3$, $\mathcal{K}$ becomes self-adjoint. $\boxed{(\varepsilon u, v)_{(L^2(\Omega))^3}}$

The eigenvalue problem can be rewritten as

$$\mathcal{K}e = \frac{1}{1+\omega^2} e,$$

and then, by the Hilbert-Schmidt Theory, we get that

- There is an infinite, discrete set of eigenvalues ω_j^2, $j \in \mathbb{N}$, such that $\lim_{j\to\infty} \omega_j^2 = \infty$, $0 < \omega_1^2 \leq \omega_2^2 \leq \cdots$

Consider now the case $\beta \neq 0$.

LEMMA : $\exists\ \omega > 0$ and $\beta_0 > 0$: $\forall \beta \in [0, \beta_0]$,

0 is not an eigenvalue of $L^{(\beta)}$ in Ω
(with the PEC boundary condition on $\partial\Omega$)

Indeed, since $\theta - \mu^{-1} = -\omega^2 \varepsilon \beta^2$, we have

$$\left(\left(L^{(\beta)} - L^{(0)} \right) v, v \right) = -\beta^2 \omega^2 \int_\Omega \varepsilon \, |\mathrm{curl}\, v|^2 dx \, +$$

$$- \beta \omega^2 \int_\Omega \varepsilon \, \mathrm{curl}\, v \cdot \bar{v} \, dx - \beta \omega^2 \int_\Omega \varepsilon v \cdot \mathrm{curl}\, \bar{v} \, dx$$

$$\left(\int_\Omega \mathrm{curl}\, V \cdot W \, dx - \int_\Omega V \cdot \mathrm{curl}\, W \, dx = \int_{\partial\Omega} (n \times V) \cdot ((n \times W) \times n) \, ds \right)$$

whereby,

$$\left| \left(\left(L^{(\beta)} - L^{(0)} \right) v, v \right) \right| \leq C \beta \, \| v \|_{H_0(\mathrm{curl}, \Omega)}$$

for an appropriate constant C.

Hence $$\lim_{\beta \to 0} \| L^{(\beta)} - L^{(0)} \|_{H_0(\mathrm{curl}, \Omega)} = 0 .$$

The result follows, based on the properties of the problem for $\beta = 0$.

Main References

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