Some Mathematical Aspects in the Theory of Inverse Problems.

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This lecture is prepared for the *Mini-Courses in Mathematical Analysis* at the University of Padua, June 2017. As the reference I recommend the monograph *Introduction to the Mathematical Theory of Inverse Problems* by the author of this course (Second Edition, Springer publisher, 2011), where the course material can be found and many more aspects are considered.

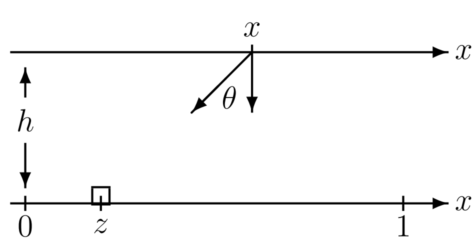
1 Introduction And Examples

We begin with three examples.

Example 1.1 *(Geological prospecting)*
In general, this is the problem of determining the location, shape, and/or some parameters (such as conductivity) of geological anomalies in Earth's interior from measurements at its surface. We consider a simple one-dimensional example and describe the following inverse problem.

Determine changes $\rho = \rho(x)$, $0 \le x \le 1$, of the mass density of an anomalous region at depth h from measurements of the vertical component $f_v(x)$ of the change of force at x. $\rho(z)\Delta z$ is the mass of a "volume element" at z and $\sqrt{(x-z)^2+h^2}$ is its distance from the instrument. The change of gravity is described by Newton's law of gravity $f = \gamma\frac{m}{r^2}$ with gravitational constant γ. For the vertical component, we have

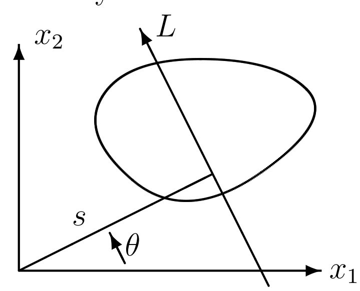
$$\Delta f_v(x) \;=\; \gamma\,\frac{\rho(z)\Delta z}{(x-xz)^2+h^2}\,\cos\theta \;=\; \gamma\,\frac{h\,\rho(z)\Delta z}{\left[(x-xz)^2+h^2\right]^{3/2}}\,.$$



This yields the following integral equation for the determination of ρ:

$$f_v(x) \;=\; \gamma\, h \int_0^1 \frac{\rho(z)}{\big[(x-z)^2 + h^2\big]^{3/2}}\, dz \quad \text{for } 0 \le x \le 1\,. \tag{1.1}$$

Example 1.2 *(Computer tomography)*
The most spectacular application of the Radon transform is in medical imaging. For example, consider a fixed plane through a human body. Let $\rho(x)$ denote the change of density at the point $x \in \mathbb{R}^2$, and let L be any line in the plane. Suppose that we direct a thin beam of X–rays into the body along L and measure how much the intensity is attenuated by going through the body.



Let L be parametrized by (s, δ), where $s \in \mathbb{R}$ and $\delta \in [0, \pi)$. Set $\hat\theta = \binom{\cos\theta}{\sin\theta}$ and $\hat\theta^\perp = \binom{-\sin\theta}{\cos\theta}$. The ray $L_{s,\delta}$ has the coordinates

$$x \;=\; s\,\hat\theta \;+\; t\,\hat\theta^\perp\,, \quad t \in \mathbb{R}\,.$$

The attenuation of the intensity I is approximately described by $dI = -\gamma\rho I\, du$ with some constant γ. Integration along the ray yields

$$\ln I(t) \;=\; -\gamma \int_{t_0}^{t} \rho\left(s\,\hat\theta + \tau\,\hat\theta^\perp\right) d\tau$$

or, assuming that ρ is of compact support, the relative intensity loss is given by

$$\ln I\,(\infty) \;=\; -\gamma \int_{-\infty}^{\infty} \rho\left(s\,\hat\theta + \tau\,\hat\theta^\perp\right) d\tau\,.$$

In principle, from the attenuation factors we can compute all line integrals

$$(R\rho)(s,\delta) \;:=\; \int_{-\infty}^{\infty} \rho\left(s\,\hat\theta + \tau\,\hat\theta^\perp\right) d\tau\,, \quad s \in \mathbb{R},\ \delta \in [0,\pi)\,. \tag{1.2}$$

$R\rho$ is called the **Radon transform** of ρ. The **direct problem** is to compute the Radon transform $R\rho$ when ρ is given. The **inverse problem** is to determine the density ρ for a given Radon transform $R\rho$ (that is, measurements of all line integrals).

Example 1.3 *(Sturm–Liouville eigenvalue problem)*
Let a string of length L and mass density $\rho = \rho(x) > 0$, $0 \le x \le L$, be fixed at the endpoints $x = 0$ and $x = L$. Plucking the string produces tones due to vibrations. Let $v(x,t)$, $0 \le x \le L$, $t > 0$, be the displacement at x and time t. It satisfies the **wave equation**

$$\rho(x)\,\frac{\partial^2 v(x,t)}{\partial t^2} \;=\; \frac{\partial^2 v(x,t)}{\partial x^2}\,, \quad 0 < x < L\,,\; t > 0\,, \tag{1.3}$$

subject to boundary conditions $v(0,t) = v(L,t) = 0$ for $t > 0$.

A periodic displacement of the form

$$v(x,t) \;=\; w(x)\,\big[a\cos\omega t \;+\; b\sin\omega t\big]$$

with frequency $\omega > 0$ is called a **pure tone**. This form of v solves the boundary value problem (1.3) if and only if w and ω satisfy the Sturm–Liouville eigenvalue problem

$$w''(x) \;+\; \omega^2\rho(x)\,w(x) \;=\; 0\,,\; 0 < x < L\,, \quad w(0) = w(L) = 0\,. \tag{1.4}$$

The **direct problem** is to compute the eigenfrequencies ω and the corresponding eigenfunctions for known function ρ. In the **inverse problem**, one tries to determine the mass density ρ from a number of measured frequencies ω.

In all of these examples, we can formulate the direct problem as the evaluation of an operator K acting on a known "model" x in a model space X and the inverse problem as the solution of the equation $K(x) = y$:

Direct problem:	given x (and K), evaluate $K(x)$.
Inverse problem:	given y (and K), solve $K(x) = y$ for x.

In order to formulate an inverse problem, the definition of the operator K, including its domain and range, has to be given. The formulation as an operator equation allows us to distinguish among finite, semifinite, and infinite-dimensional, linear and nonlinear problems.

In general, the evaluation of $K(x)$ means solving a boundary value problem for a differential equation or evaluating an integral.

For all of the pairs of problems presented in the last section, there is a fundamental difference between the direct and the inverse problems. In all cases, the inverse problem is **ill-posed** or **improperly-posed** in the sense of Hadamard, while the direct problem is well-posed. In his lectures in 1923, Hadamard claims that a mathematical model for a physical problem (he was thinking in terms of a boundary valueproblem for a partial differential equation) has to be **properly-posed** or **well-posed** in the sense that it has the following three properties:

1. There exists a solution of the problem (existence).

2. There is at most one solution of the problem (uniqueness).

3. The solution depends continuously on the data (stability).

Mathematically, the existence of a solution can be enforced by enlarging the solution space. The concept of distributional solutions of a differential equation is an example. If a problem has more than one solution, then information about the model is missing. In this case, additional properties, such as sign conditions, can be built into the model. The requirement of stability is the most important one. If a problem lacks the property of stability, then its solution is practically impossible to compute because any measurement or numerical computation is polluted by unavoidable errors: thus the data of a problem are always perturbed by noise! If the solution of a problem does not depend continuously on the data, then in general the computed solution has nothing to do with the true solution.

Definition 1.4 *(well-posedness)*
*Let X and Y be normed spaces, $K : X \to Y$ a (linear or nonlinear) mapping. The equation $Kx = y$ is called **properly-posed** or **well-posed** if the following holds:*

1. ***Existence:*** *For every $y \in Y$ there is (at least one) $x \in X$ such that $Kx = y$.*

2. ***Uniqueness:*** *For every $y \in Y$ there is at most one $x \in X$ with $Kx = y$.*

3. ***Stability:*** *The solution x depends continuously on y; that is, for every sequence $(x_n) \subset X$ with $Kx_n \to Kx$ $(n \to \infty)$, it follows that $x_n \to x$ $(n \to \infty)$.*

*Equations for which (at least) one of these properties does not hold are called **improperly-posed** or **ill-posed**.*

It is important to specify the full triple (X, Y, K) and their norms. Existence and uniqueness depend only on the algebraic nature of the spaces and the operator; that is, whether the operator is onto or one-to-one. Stability, however, depends also on the topologies of the spaces, i.e., whether the inverse operator $K^{-1} : Y \to X$ is continuous.

These requirements are not independent of each other. For example, due to the open mapping theorem, the inverse operator K^{-1} is automatically continuous if K is linear and continuous and X and Y are Banach spaces.

In the numerical treatment of integral equations, a discretization error cannot be avoided. For integral equations of the first kind, a “naive” discretization usually leads to disastrous results as the following simple example shows.

Example 1.5
We look again at the integral equation (1.1) for $\gamma = 1$; that is,

$$h \int_0^1 \frac{\rho(z)}{\big[(x-z)^2 + h^2\big]^{3/2}}\, dz \;=\; f(x) \quad \text{for } 0 \le x \le 1\,, \tag{1.5}$$

where $f(x)$ is given by

$$f(x) \;:=\; h \int_0^1 \frac{1}{\big[(x-z)^2 + h^2\big]^{3/2}}\, dz \;=\; \frac{2}{h}\left[\frac{1}{\exp(2t_0)+1} - \frac{1}{\exp(2t_1)+1}\right]$$

with $t_0 = \ln\big[-x/h + \sqrt{1 + x^2/h^2}\big]$ and $t_1 = \ln\big[(1-x)/h + \sqrt{1+(1-x)^2/h^2}\big]$. (*Hint:* Substitute $z = x + h \sinh t$ in the integral.) Then $\rho(z) = 1$, $z \in (0,1)$, is the (only) solution of the integral equation (1.5).
We approximate the integral by the trapezoidal rule

$$h \int_0^1 \frac{\rho(z)}{\big[(x-z)^2 + h^2\big]^{3/2}}\, dz \;\approx\; \frac{h}{n} \sum_{j=0}^{n}{}' \frac{\rho_j}{\big[(x-j/n)^2 + h^2\big]^{3/2}}$$

where $\sum_{j=0}^{n}{}' a_j = \frac{1}{2} a_0 + \sum_{j=1}^{n-1} a_j + \frac{1}{2} a_n$. For $x = i/n$, we obtain the linear system

$$\frac{h}{n} \sum_{j=0}^{n}{}' \frac{\tilde{\rho}_j}{\big[(i/n - j/n)^2 + h^2\big]^{3/2}} \;=\; f(i/n)\,, \quad i = 0, \dots, n\,. \tag{1.6}$$

which we write as $A\tilde{\rho} = \tilde{f}$ in $\mathbb{R}^{n+1}$ where $\tilde{\rho} = (\tilde{\rho}_j)_{j=0,\dots,n}^\top$ and $\tilde{f} = \big(f(i/n)\big)_{i=0,\dots,n}^\top$.

Then $\tilde{\rho}_j$ should be an approximation to $\rho(j/n)$. The following table (3rd column) lists the error $err2 := \max_{j=0,\dots,n} |\tilde{\rho}_j - 1|$ between the exact solution $\rho_j := \rho(j/n) = 1$ and the approximate solution $\tilde{\rho}_j$ in the maximum norm for $n = 4, \dots, 128$. As comparison, the second column lists the error $err1 := \max_{j=0,\dots,n} |A\rho - \tilde{f}|$ and in the forth column we list the error $err3 := \max_{j=0,\dots,n} |\tilde{\rho}_j - 1|$ of the solution of the integral equation of the *second kind*

$$\rho(x) \;+\; h \int_0^1 \frac{\rho(z)}{\big[(x-z)^2 + h^2\big]^{3/2}}\, dz \;=\; f(x) \;+\; 1 \quad \text{for } 0 \le x \le 1\,,$$

with $\rho = 1$ as solution. We observe that, in contrast to the solution of the first kind integral equation (1.5) the errors decay of order 2 as expected for the trapezoidal rule.

n	$err1$	$err2$	$err3$
4	0.0463232681555	0.1418408834665	0.0183898954803
8	0.0114585130651	0.2530455911927	0.0043111981459
16	0.0028988236549	0.7113811691074	0.0010618436701
32	0.0007244204928	12.7768956652751	0.0002644856746
64	0.0001811609863	489.9135720439897	0.0000660608036
128	0.0000452924859	369.4640129272221	0.0000165218541

We see that the approximations have nothing to do with the true solution and become even worse for finer discretization schemes.

The mathematical reason for an equation $Kx = y$ to be improperly posed is - in many cases - that the operator K is **compact**; that is, it maps bounded sets in relatively compact sets. This is the content of the following theorem.

Theorem 1.6
Let X, Y be normed spaces and $K : X \to Y$ be a linear compact operator with nullspace $\mathcal{N}(K) := \{x \in X : Kx = 0\}$. Let the dimension of the factor space $X/\mathcal{N}(K)$ be infinite. Then there exists a sequence (x_n) in X such that $Kx_n \to 0$ but (x_n) does not converge. We can even choose (x_n) such that $\|x_n\| \to \infty$. In particular, if K is one-to-one, the inverse $K^{-1} : Y \supset \mathcal{R}(K) \to X$ is unbounded. Here, $\mathcal{R}(K) := \{Kx \in Y : x \in X\}$ denotes the range of K.

Proof: We set $\mathcal{N} = \mathcal{N}(K)$ for abbreviation. The factor space $X/\mathcal{N}$ is a normed space with norm $\|[x]\| := \inf\big\{\|x+z\| : z \in \mathcal{N}\big\}$ since the nullspace is closed. The induced operator $\tilde{K} : X/\mathcal{N} \to Y$, defined by $\tilde{K}([x]) := Kx$, $[x] \in X/\mathcal{N}$, is well-defined, compact, and one-to-one. The inverse $\tilde{K}^{-1} : Y \supset \mathcal{R}(K) \to X/\mathcal{N}$ is unbounded since otherwise the identity $I = \tilde{K}^{-1}\tilde{K} : X/\mathcal{N} \to X/\mathcal{N}$ would be compact as a composition of a bounded and a compact operator. This would contradict the assumption that the dimension of $X/\mathcal{N}$ is infinite. Because $\tilde{K}^{-1}$ is unbounded, there exists a sequence $([z_n])$ in $X/\mathcal{N}$ with $Kz_n \to 0$ and $\|[z_n]\| = 1$. We choose $v_n \in \mathcal{N}$ such that $\|z_n + v_n\| \geq \frac{1}{2}$ and set $x_n := (z_n + v_n)/\sqrt{\|Kz_n\|}$. Then $Kx_n \to 0$ and $\|x_n\| \to \infty$. $\square$

2 The Tikhonov Regularization

We saw in the previous section that many inverse problems can be formulated as operator equations of the form

$$Kx \;=\; y\,,$$

where K is a linear compact operator between Hilbert spaces X and Y over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. For simplicity, we **assume throughout this section that the compact operator K is one-to-one**. This is not a serious restriction because we can always replace the domain X by the orthogonal complement of the kernel of K. We make the assumption that there exists a solution $x \in X$ of the unperturbed equation $Kx = y$. In other words, we assume that $y \in \mathcal{R}(K)$. The injectivity of K implies that this solution is unique.

In practice, the exact right-hand side $y^0 \in Y$ is never known exactly but only up to an error of, say, $\delta > 0$. Therefore, we assume that we know $\delta > 0$ and $y^\delta \in Y$ with

$$\big\|y^0 - y^\delta\big\| \leq \delta\,. \tag{2.7}$$

With $x^0 \in X$ we denote the exact solution; that is, $Kx^0 = y^0$. It is our aim to 'solve" the perturbed equation

$$Kx^\delta = y^\delta \,. \tag{2.8}$$

In general, (2.8) is not solvable because we cannot assume that the measured data y^δ are in the range $\mathcal{R}(K)$ of K. Therefore, the best we can hope is to determine an approximation $x^\delta \in X$ to the exact solution x^0.

An additional requirement is that the approximate solution x^δ should depend continuously on the data y^δ. In other words, it is our aim to construct a suitable bounded approximation $R : Y \to X$ of the (unbounded) inverse operator $K^{-1} : \mathcal{R}(K) \to X$.

Definition 2.1
A ***regularization strategy*** *is a family of linear and bounded operators*

$$R_\alpha : Y \longrightarrow X, \quad \alpha > 0\,,$$

such that

$$\lim_{\alpha\to 0} R_\alpha K x \;=\; x \quad \textit{for all } x \in X\,;$$

that is, the operators $R_\alpha K$ converge ***pointwise*** *to the identity.*

From this definition and the compactness of K, we conclude the following.

Theorem 2.2
Let R_α be a regularization strategy for a compact operator $K : X \to Y$ where $\dim X = \infty$. Then we have

(1) The operators R_α are not uniformly bounded; that is, there exists a sequence (α_j) with $\|R_{\alpha_j}\| \to \infty$ for $j \to \infty$.

(2) The sequence $(R_\alpha K x)$ does not converge uniformly on bounded subsets of X; that is, there is no convergence $R_\alpha K$ to the identity I in the operator norm.

Proof: (1) Assume, on the contrary, that there exists $c > 0$ such that $\|R_\alpha\| \le c$ for all $\alpha > 0$. From $R_\alpha y \to K^{-1}y$ $(\alpha \to 0)$ for all $y \in \mathcal{R}(K)$ and $\|R_\alpha y\| \le c\,\|y\|$ for $\alpha > 0$ we conclude that $\|K^{-1}y\| \le c\,\|y\|$ for every $y \in \mathcal{R}(K)$; that is, K^{-1} is bounded. This implies that $I = K^{-1}K : X \to X$ is compact, a contradiction to $\dim X = \infty$.

(2) Assume that $R_\alpha K \to I$ in $\mathcal{L}(X, X)$. From the compactness of $R_\alpha K$ and the closedness of the space of compact operators we conclude that I is also compact, which again would imply that $\dim X < \infty$. $\square$

The notion of a regularization strategy is based on unperturbed data; that is, the regularizer $R_\alpha y$ converges to x for the exact right-hand side $y = Kx$.

Now let $y^0 \in \mathcal{R}(K)$ be the exact right-hand side and $y^\delta \in Y$ be the measured data with $\|y^0 - y^\delta\| \le \delta$. We define

$$x^{\alpha,\delta} \;:=\; R_\alpha y^\delta \tag{2.9}$$

as an approximation of the solution x^0 of $Kx^0 = y^0$. Then the error splits into two parts by the following obvious application of the triangle inequality:

$$\begin{aligned}\left\|x^{\alpha,\delta} - x^0\right\| \;&\le\; \left\|R_\alpha y^\delta - R_\alpha y^0\right\| \;+\; \left\|R_\alpha y^0 - x^0\right\| \\ &\le\; \|R_\alpha\|\,\delta \;+\; \left\|R_\alpha K x^0 - x^0\right\| \,. \end{aligned}\tag{2.10}$$

This is our **fundamental estimate**, which we use often in the following.

We observe that the error between the exact and computed solutions consists of two parts: The first term on the right-hand side describes the error in the data multiplied by the "condition number" $\|R_\alpha\|$ of the regularized problem. By Theorem 2.2, this term tends to infinity as α tends to zero. The second term denotes the approximation error $\left\|\left(R_\alpha - K^{-1}\right)y^0\right\|$ of the exact right-hand side $y^0 = Kx^0$. By the definition of a regularization strategy, this term tends to zero with α. The following figure illustrates the situation.

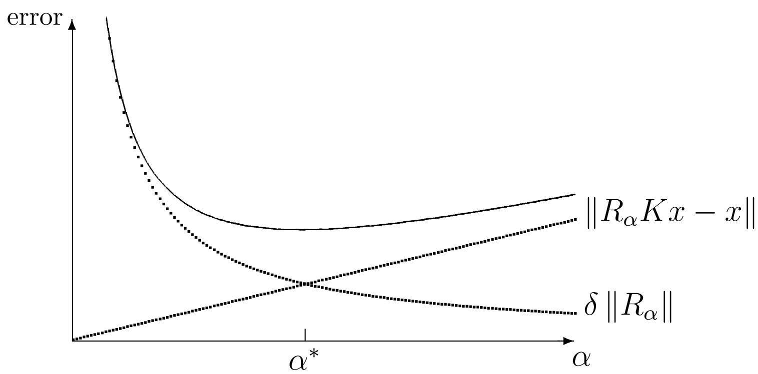


Figure: Behavior of the total error

We need a strategy to choose $\alpha = \alpha(\delta)$ dependent on δ in order to keep the total error as small as possible. This means that we would like to minimize

$$\delta\,\|R_\alpha\| \;+\; \left\|R_\alpha K x^0 - x^0\right\| \,.$$

The procedure is the same in every concrete situation: One has to estimate the quantities $\|R_\alpha\|$ and $\|R_\alpha K x^0 - x^0\|$ in terms of α and then minimize this upper bound with respect to α. Before we carry out these steps for a model example, we introduce the following notation.

Definition 2.3

A regularization strategy $\alpha = \alpha(\delta)$ is called ***admissible*** *if $\alpha(\delta) \to 0$ and*

$$\sup\left\{\left\|R_{\alpha(\delta)} y^\delta - x\right\| : y^\delta \in Y,\ \left\|Kx - y^\delta\right\| \le \delta\right\} \to 0\,, \quad \delta \to 0\,,$$

for every $x \in X$.

Now we introduce the most prominent example of an admissible regularization strategy: Given the linear, bounded operator $K : X \to Y$ and $y \in Y$, determine $x^\alpha \in X$ that minimizes the **Tikhonov functional**

$$J_\alpha(x) \;:=\; \|Kx - y\|^2 \;+\; \alpha\,\|x\|^2 \quad \text{for } x \in X\,. \tag{2.11}$$

We prove the following theorem.

Theorem 2.4
*Let $K : X \to Y$ be a linear and bounded operator between Hilbert spaces and $\alpha > 0$. Then the Tikhonov functional J_α has a unique minimum $x^\alpha \in X$. This minimum x^α is the unique solution of the **normal equation***

$$\alpha\, x^\alpha \;+\; K^*Kx^\alpha \;=\; K^*y\,. \tag{2.12}$$

Proof: Let (x_n) in X be a minimizing sequence; that is, $J_\alpha(x_n) \to I := \inf_{x\in X} J_\alpha(x)$ as n tends to infinity. We show that (x_n) is a Cauchy sequence. Application of the binomial formula yields that

$$\begin{aligned} J_\alpha(x_n) \;+\; J_\alpha(x_m) \;&=\; 2\,J_\alpha\left(\frac{1}{2}(x_n + x_m)\right) \\ &\quad + \frac{1}{2}\,\|K(x_n - x_m)\|^2 \;+\; \frac{\alpha}{2}\,\|x_n - x_m\|^2 \\ &\geq\; 2\,I \;+\; \frac{\alpha}{2}\,\|x_n - x_m\|^2\,. \end{aligned}$$

The left-hand side converges to $2I$ as n, m tend to infinity. This shows that (x_n) is a Cauchy sequence in X and thus convergent. Let $x^\alpha = \lim_{n\to\infty} x_n$, noting that $x^\alpha \in X$. From the continuity of J_α, we conclude that $J_\alpha(x_n) \to J_\alpha(x^\alpha)$; that is, $J_\alpha(x^\alpha) = I$. This proves the existence of a minimum of J_α.

Now we use the following formula:

$$\begin{aligned} J_\alpha(x) \;-\; J_\alpha(x^\alpha) \;&=\; 2\left(Kx^\alpha - y\,,\, K(x - x^\alpha)\right) \;+\; 2\alpha\left(x^\alpha,\, x - x^\alpha\right) \\ &\quad + \;\|K(x - x^\alpha)\|^2 \;+\; \alpha\,\|x - x^\alpha\|^2 \\ &=\; 2\left(K^*(Kx^\alpha - y) + \alpha x^\alpha\,,\, x - x^\alpha\right) \\ &\quad + \;\|K(x - x^\alpha)\|^2 \;+\; \alpha\,\|x - x^\alpha\|^2 \end{aligned} \tag{2.13}$$

for all $x \in X$. From this, we see the equivalence of the normal equation with the minimization problem for J_α. Indeed, if x^α satisfies (2.12), then $J_\alpha(x) - J_\alpha(x^\alpha) = \|K(x - x^\alpha)\|^2 + \alpha\,\|x - x^\alpha\|^2 \geq 0$; that is, x^α minimizes J_α. If, on the other hand, x^α minimizes J_α, then we substitute $x = x^\alpha + tz$ for any $t > 0$ and $z \in X$ and arrive at

$$0 \;\leq\; 2t\left(K^*(Kx^\alpha - y) + \alpha x^\alpha,\, z\right) \;+\; t^2\,\|Kz\|^2 \;+\; t^2\alpha\,\|z\|^2\,.$$

Division by $t > 0$ and $t \to 0$ yields $\big(K^*(Kx^\alpha - y) + \alpha x^\alpha,\, z\big) \ge 0$ for all $z \in X$; that is, $K^*(Kx^\alpha - y) + \alpha x^\alpha = 0$, and x^α solves (2.12).

Finally, we show that $\alpha\, I + K^*K$ is one-to-one for every $\alpha > 0$. Let $\alpha x + K^*Kx = 0$. Multiplication by x yields $\alpha(x,x) + (Kx, Kx) = 0$; that is, $x = 0$. $\square$

The solution x^α of equation (2.12) can be written in the form $x^\alpha = R_\alpha y$ with

$$R_\alpha \;:=\; (\alpha I + K^*K)^{-1} K^* \;:\; Y \longrightarrow X\,. \tag{2.14}$$

We show that this family of operators R_α is an admissible regularization strategy.

Theorem 2.5

Let $K : X \to Y$ be a linear, compact, and injective operator and $\alpha > 0$. The operator $\alpha I + K^*K$ is boundedly invertible. The operators $R_\alpha : Y \to X$ from (2.14) form a regularization strategy with $\|R_\alpha\| \le 1/\big(2\sqrt{\alpha}\big)$. It is called the ***Tikhonov regularization method****. $R_\alpha y^\delta$ is determined as the unique solution $x^{\alpha,\delta} \in X$ of the equation of the second kind*

$$\alpha\, x^{\alpha,\delta} \;+\; K^*Kx^{\alpha,\delta} \;=\; K^*y^\delta\,. \tag{2.15}$$

Every choice $\alpha(\delta) \to 0$ ($\delta \to 0$) with $\delta^2/\alpha(\delta) \to 0$ ($\delta \to 0$) is admissible.

Proof: First we show $\|R_\alpha\| \le 1/\big(2\sqrt{\alpha}\big)$. Let $y \in Y$ and $x = R_\alpha y$. Then $(\alpha I + K^*K)x = K^*y$. We multiply this equation by x and use the property of the adjoint of K. This yields

$$\alpha\|x\|^2 \;+\; \|Kx\|^2 \;=\; (K^*y, x) \;=\; (y, Kx) \;\le\; \|y\|\|Kx\|\,.$$

Completing the squares yields

$$\alpha\|x\|^2 \;\le\; \alpha\|x\|^2 \;+\; \left[\|Kx\| - \frac{1}{2}\,\|y\|\right]^2 \;\le\; \frac{1}{4}\,\|y\|^2$$

which implies $\sqrt{\alpha}\,\|x\| \le \|y\|/2$; that is, $\|R_\alpha y\| \le \|y\|/\big(2\sqrt{\alpha}\big)$.

Next we show that $R_\alpha Kx \to x$ for every $x \in X$ as α tends to zero. Let again $x^\alpha := R_\alpha Kx$. Then $(\alpha I + K^*K)x^\alpha = K^*Kx$; that is,

$$(\alpha I + K^*K)(x^\alpha - x) \;=\; -\alpha x\,. \tag{2.16}$$

Multiplication with $x^\alpha - x$ yields as before

$$\alpha\|x^\alpha - x\|^2 \;+\; \|K(x^\alpha - x)\|^2 \;=\; -\alpha\,(x, x^\alpha - x)\,.$$

Now we use the fact that the range $\mathcal{R}(K^*)$ of K^* is dense in X. This follows from the injectivity of K. Therefore, for given $\varepsilon > 0$ there exists $z \in Y$ with $\|x - K^*z\| \le \varepsilon$. Now we continue with (2.17) in the form

$$\begin{aligned}\alpha\|x^\alpha - x\|^2 \;+\; \|K(x^\alpha - x)\|^2 \;&=\; -\alpha\,(x - K^*z, x^\alpha - x) \;-\; \alpha\,\big(z, K(x^\alpha - x)\big)\\ &\le\; \alpha\,\varepsilon\,\|x^\alpha - x\| \;+\; \alpha\|z\|\|K(x^\alpha - x)\|\,.\end{aligned}$$

Completing the squares yields

$$\alpha \left[\|x^\alpha - x\| - \frac{\varepsilon}{2}\right]^2 \;\leq\; \alpha \left[\|x^\alpha - x\| - \frac{\varepsilon}{2}\right]^2 + \left[\|K(x^\alpha - x)\| - \frac{\alpha\|z\|}{2}\right]^2 \;\leq\; \alpha\,\frac{\varepsilon^2}{4} + \frac{\alpha^2\|z\|^2}{4}\,,$$

and thus

$$\left|\|x^\alpha - x\| - \frac{\varepsilon}{2}\right| \;\leq\; \frac{1}{2}\sqrt{\varepsilon^2 + \alpha\|z\|^2}\,;$$

that is,

$$\|x^\alpha - x\| \;\leq\; \frac{\varepsilon}{2} \;+\; \frac{1}{2}\sqrt{\varepsilon^2 + \alpha\|z\|^2}\,. \tag{2.17}$$

Now we choose $\alpha = \alpha(\varepsilon)$ so small such that $\alpha\|z\|^2 \leq 3\varepsilon^2$. For this α we have $\|x^\alpha - x\| \leq \varepsilon/2 + \varepsilon = 3\varepsilon/2$. Since $\varepsilon > 0$ was arbitrary the convergence $x^\alpha \to x$ follows. Now we use the fundamental estimate (2.10) to obtain

$$\left\|x^{\alpha(\delta),\delta} - x\right\| \;\leq\; \frac{\delta}{2\sqrt{\alpha(\delta)}} \;+\; \left\|R_{\alpha(\delta)}Kx - x\right\| \tag{2.18}$$

which yields the assertion. □

Estimate (2.18) yields only convergence of $x^{\alpha(\delta),\delta}$ to x as δ tends to zero but no order of convergence. If we make stronger assumptions on the true solution on x^0 we obtain the orders $\sqrt{\delta}$ and $\delta^{2/3}$:

Theorem 2.6
(a) Let in addition to the assumptions of the previous theorem $x^0 = K^*z \in \mathcal{R}(K^*)$ for some $z \in Y$. We choose $\alpha(\delta) = c\,\delta$ for some $c > 0$. Then the following estimate holds:

$$\left\|x^{\alpha(\delta),\delta} - x^0\right\| \;\leq\; \frac{1}{2}\sqrt{\delta}\left[1/\sqrt{c} + \sqrt{c}\,\|z\|\right]. \tag{2.19a}$$

(b) Let $x^0 = K^*Kz \in \mathcal{R}(K^*K)$ for some $z \in X$. The choice $\alpha(\delta) = c\,\delta^{2/3}$ for some $c > 0$ leads to the error estimate

$$\left\|x^{\alpha(\delta),\delta} - x^0\right\| \;\leq\; \delta^{2/3}\left[1/(2\sqrt{c}) + 2c\,\|z\|\right]. \tag{2.19b}$$

Proof: (a) In the second part of the previous proof (for $x = x^0$) we used the fact that x^0 can be approximated arbitrarily well by elements in $\mathcal{R}(K^*)$. By our assumption that $x^0 = K^*z$ for some $z \in Y$ we can now copy the proof with $\varepsilon = 0$ and have by (2.17) that $\|R_\alpha Kx^0 - x^0\| \leq \frac{1}{2}\sqrt{\alpha}\|z\|$. The fundamental estimate (2.10) yields

$$\left\|x^{\alpha,\delta} - x^0\right\| \;\leq\; \frac{\delta}{2\sqrt{\alpha}} \;+\; \frac{1}{2}\sqrt{\alpha}\,\|z\|\,.$$

The choice $\alpha(\delta) = c\,\delta$ yields (2.19a) and ends the proof of part (a).

(b) We write (2.16) for $x = x^0$ in the form $(\alpha I + K^*K)(x^\alpha - x^0) = -\alpha x^0 = -\alpha K^*Kz$; that is, $(\alpha I + K^*K)(x^\alpha - x^0 + \alpha z) = \alpha^2 z$. Multiplication with $x^\alpha - x^0 + \alpha z$ yields

$$\alpha\|x^\alpha - x^0 + \alpha z\|^2 \;+\; \|K(x^\alpha - x^0 + \alpha z)\|^2 \;=\; \alpha^2\,(z, x^\alpha - x + \alpha z) \;\leq\; \alpha^2\|z\|\|x^\alpha - x^0 + \alpha z\|\,,$$

and thus $\|x^\alpha - x^0 + \alpha z\| \leq \alpha\|z\|$. From this $\|x^\alpha - x^0\| \leq 2\alpha\|z\|$ follows. Again, the fundamental estimate (2.10) yields

$$\left\|x^{\alpha,\delta} - x^0\right\| \;\leq\; \frac{\delta}{2\sqrt{\alpha}} \;+\; 2\alpha\|z\|\,.$$

The choice $\alpha(\delta) = c\,\delta^{2/3}$ lead to the estimate (2.19b). □

From Theorem 2.5, we observe that α has to be chosen to depend on δ in such a way that it converges to zero as δ tends to zero but not as fast as δ^2. The additional assumptions of Theorem 2.6 that x^0 belongs to the range of K^* or K^*K, respectively, turns into a smoothness assumption when applying this theorem to examples where K - and thus K^* - is an integral operator. We observe that the smoother the solution x is the slower α has to tend to zero. On the other hand, the convergence can be arbitrarily slow if no a priori assumption about the solution x (such as (a) or (b)) is available.

It is surprising to note that the order of convergence of Tikhonov's regularization method cannot be improved. Indeed, the following result can be shown.

Theorem 2.7
Let $K : X \to Y$ be linear, compact, and one-to-one such that the range $\mathcal{R}(K)$ is infinite-dimensional. Furthermore, let $x^0 \in X$, and assume that there exists a continuous function $\alpha : [0,\infty) \to [0,\infty)$ with $\alpha(0) = 0$ such that

$$\lim_{\delta\to 0}\left\|x^{\alpha(\delta),\delta} - x^0\right\|\delta^{-2/3} \;=\; 0$$

for every $y^\delta \in Y$ with $\left\|y^\delta - Kx^0\right\| \leq \delta$, where $x^{\alpha(\delta),\delta} \in X$ solves (2.15). Then $x^0 = 0$.

3 The Landweber Iteration

Also for this section we make the assumptions that K is one-to-one with dense range. Then also K^* is one-to-one. We can rewrite the equation $Kx = y$ in the form $x = (I - a\,K^*K)\,x \,+\, a\,K^*y$ for some $a > 0$ and iterating this equation; that is, computing

$$x_0 \;:=\; 0 \quad \text{and} \quad x_m \;=\; (I - a\,K^*K)\,x_{m-1} \;+\; a\,K^*y \tag{3.20}$$

for $m = 1, 2, \ldots$. This iteration scheme can be interpreted as the steepest descent algorithm applied to the quadratic functional $x \mapsto \|Kx - y\|^2$ as the following lemma shows.

Lemma 3.1
Let the sequence (x_m) *be defined by (3.20) and define the functional* $\psi : X \to \mathbb{R}$ *by* $\psi(x) = \frac{1}{2}\|Kx - y\|^2$, $x \in X$. *Then* ψ *is Fréchet differentiable in every* $z \in X$ *and*

$$\psi'(z)x \;=\; (Kz - y, Kx) \;=\; \mathrm{Re}\big(K^*(Kz-y), x\big)\,, \quad x \in X\,. \tag{3.21}$$

The linear functional $\psi'(z)$ *can be identified with* $K^*(Kz - y) \in X$ *in the Hilbert space* X *over the field* $\mathbb{R}$. *Therefore,* $x^m = x^{m-1} - a\,K^*(Kx^{m-1} - y)$ *is the steepest descent step with stepsize* a.

Proof: The binomial formula yields

$$\psi(z + x) \;-\; \psi(z) \;-\; (Kz - y, Kx) \;=\; \frac{1}{2}\,\|Kx\|^2$$

and thus

$$\big|\psi(z + x) \;-\; \psi(z) \;-\; (Kz - y, Kx)\big| \;\le\; \frac{1}{2}\,\|K\|^2\,\|x\|^2\,,$$

which proves that the mapping $x \mapsto (Kz - y, Kx)$ is the Fréchet derivative of ψ at z. □

Equation (3.20) is a linear recursion formula for x_m. By induction with respect to m, it is easily seen that x^m has the explicit form $x_m = R_m y$, where the operator $R_m : Y \to X$ is defined by

$$R_m \;:=\; a\sum_{k=0}^{m-1}(I - aK^*K)^k K^* \quad \text{for } m = 1, 2, \dots\,. \tag{3.22}$$

Therefore, we note that m plays the role of the regularization parameter α (or, better, $1/m$ because the regularization parameter should tend to zero).

Theorem 3.2
Again let $K : X \to Y$ *be a compact operator and let* $0 < a < 1/\|K\|^2$. *Define the linear and bounded operators* $R_m : Y \to X$ *by (3.22). These operators* R_m *define a regularization strategy with discrete regularization parameter* $\alpha = 1/m$, $m \in \mathbb{N}$, *and* $\|R_m\| \le \sqrt{a\,m}$. *The sequence* $x_m^\delta = R_m y^\delta$ *is computed by the iteration (3.20); that is,*

$$x_0^\delta \;=\; 0 \quad \textit{and} \quad x_m^\delta \;=\; (I - a\,K^*K)x_{m-1}^\delta \;+\; a\,K^*y^\delta \tag{3.23}$$

for $m = 1, 2, \dots$. *Every strategy* $m(\delta) \to \infty$ $(\delta \to 0)$ *with* $\delta^2\,m(\delta) \to 0$ $(\delta \to 0)$ *is admissible.*
Furthermore, $\|KR_m - I\| \le 1$ *for all* m *and* $Kx_m^\delta \to y^\delta$ *as* $m \to \infty$.

Before we prove this theorem we recall the notion of a **singular system** $\{\sigma_j, \psi_j, \phi_j : j \in \mathbb{N}\}$ for the operator K: It is $\sigma_j > 0$ for all j and $\{\psi_j : l \in \mathbb{N}\}$ and $\{\phi_j : l \in \mathbb{N}\}$ are orthonormal systems in X and Y, respectively, and $K\psi_j = \sigma_j\phi_j$ and $K^*\phi_j = \sigma_j\psi_j$ for all $j \in \mathbb{N}$. The injectivity of K and K^* imply that both systems are also complete. In particular, it holds that $x = \sum_j (x, \psi_j)\psi_j$ and $y = \sum_j (y, \phi_j)\phi_j$ for all $x \in X$ and $y \in Y$ and $\|x\|^2 = \sum_j (x, \psi_j)^2$ and $\|y\|^2 = \sum_j (y, \phi_j)^2$.

Proof of Theorem 3.2: By the properties of the singular system we have for $m = 1, 2, \ldots$:

$$R_m \phi_j \;=\; a \sum_{k=0}^{m-1} (1 - a\sigma_j^2)^k \sigma_j \, \psi_j \;=\; a\, \sigma_j \frac{1 - (1 - a\sigma_j^2)^m}{a\sigma_j^2}\, \psi_j \;=\; \frac{1 - (1 - a\sigma_j^2)^m}{\sigma_j}\, \psi_j$$

and thus for $y = \sum_j \beta_j \phi_j$:

$$R_m y \;=\; \sum_j \beta_j \frac{1 - (1 - a\sigma_j^2)^m}{\sigma_j}\, \psi_j \, .$$

Therefore, with $(1 - a\sigma_j^2)^m \ge 1 - am\, \sigma_j^2$ (Bernoulli's inequality),

$$\|R_m y\|^2 \;=\; \sum_j \beta_j^2 \frac{\left[1 - (1 - a\sigma_j^2)^m\right]^2}{\sigma_j^2} \;\le\; \sum_j \beta_j^2 \frac{1 - (1 - a\sigma_j^2)^m}{\sigma_j^2} \;\le\; am\, \|y\|^2$$

which proves that $\|R_m\| \le \sqrt{am}$. By the fundamental estimate it remains to show that $R_m K x \to x$ as m tends to infinity. With $x = \sum_j \alpha_j \psi_j$ we compute $Kx = \sum_j \alpha_j \sigma_j \phi_j$ and thus

$$R_m K x \;=\; \sum_j \alpha_j \left[1 - (1 - a\sigma_j^2)^m\right] \psi_j \quad \text{and thus} \quad \|R_m K x - x\|^2 \;=\; \sum_j \alpha_j^2 (1 - a\sigma_j^2)^{2m} \, .$$

From $\sum_j \alpha_j^2 = \|x\|^2$ and $(1 - a\sigma_j^2)^{2m} \to 0$, $m \to \infty$, for every j we conclude by a simple argument (exercise) that $\|R_m K x - x\| \to 0$ as $m \to \infty$. Analogously, for $y^\delta = \sum_j \beta_j \phi_j$ we compute

$$K R_m y^\delta \;=\; \sum_j \beta_j \left[1 - (1 - a\sigma_j^2)^m\right] \phi_j \quad \text{and thus} \quad \|K R_m y^\delta - y^\delta\|^2 \;=\; \sum_j \beta_j^2 (1 - a\sigma_j^2)^{2m} \, ,$$

which shows that $\|K R_m - I\| \le 1$ for all m and that $\|K R_m y^\delta - y^\delta\|$ converges to zero as $m \to \infty$. □

The *a-priori* choice $\delta^2 \cdot m(\delta) \to 0$ $(\delta \to 0)$ is not useful in practise. For iterative methods it is very natural to use a stopping rule. We suggest the following

Stopping rule: Let $r > 1$ be a fixed number. Stop the algorithm at the first occurrence of $m \in \mathbb{N}_0$ with $\left\|K x_m^\delta - y^\delta\right\| \le r\, \delta$. If $\left\|y^\delta\right\| > r\, \delta$ then $m = m(\delta) \in \mathbb{N}$ is well defined since $\left\|K x_m^\delta - y^\delta\right\| \to 0$ as $m \to \infty$ by the previous theorem. $m(\delta)$ is characterized by the inequalities

$$\left\|K x_{m(\delta),}^\delta - y^\delta\right\| \;\le\; r\delta \;<\; \left\|K x_m^\delta - y^\delta\right\| \quad \text{for all } m = 0, \ldots, m(\delta) - 1 \, .$$

The following theorem shows that this choice of $m(\delta)$ is possible for Landweber's method and leads to an admissible regularization strategy.

Theorem 3.3
Let, in addition to the assumptions of the previous theorem, $x^0 = K^* z \in \mathcal{R}(K^*)$ for some $z \in Y$. Then we have the following order of convergence:

$$\left\|x^\delta_{m(\delta)} - x^0\right\| \;\le\; c\,\sqrt{\|z\|}\,\sqrt{\delta} \tag{3.24}$$

for some $c > 0$.

Proof: We remind the reader of the fundamental estimate (2.10), which we need in the following form (see Theorem 3.2):

$$\left\|x^\delta_m - x^0\right\| \;\le\; \delta\,\sqrt{a\,m} \;+\; \left\|R_m K x^0 - x^0\right\|\,. \tag{3.25}$$

Writing $m = m(\delta)$ for abbreviation we have

$$\begin{aligned}\left\|KR_{m-1}y^0 - y^0\right\| \;&\ge\; \left\|KR_{m-1}y^\delta - y^\delta\right\| \;-\; \left\|(KR_{m-1} - I)(y^0 - y^\delta)\right\| \\ &\ge\; r\,\delta - \|KR_{m-1} - I\|\;\delta \;\ge\; (r-1)\,\delta,\end{aligned}$$

and hence for $y^0 = Kx^0 = KK^*z = \sum_j (z,\phi_j)\,\sigma_j^2\,\phi_j$:

$$m^2\,(r-1)^2\,\delta^2 \;\le\; m^2\,\left\|KR_{m-1}y^0 - y^0\right\|^2 \;=\; \sum_{j=1}^\infty m^2\big(1 - a\sigma_j^2\big)^{2m-2}\,\sigma_j^4\,(z,\phi_j)^2\,.$$

Now we use the elementary estimate

$$m^2\sigma^4\big(1 - a\sigma^2\big)^{2m-2} \le 1/a^2 \quad \text{for all } m \ge 2 \text{ and } 0 \le \sigma \le 1/\sqrt{a}$$

which yields

$$(r-1)^2\,\delta^2\,m^2 \;\le\; \frac{1}{a^2}\;\|z\|^2\;;$$

that is, we have shown the upper bound

$$m(\delta) \;\le\; \frac{1}{a(r-1)}\,\frac{\|z\|}{\delta}\,.$$

Now we estimate the second term on the right-hand side of (3.25). From the Cauchy–Schwarz inequality, we conclude that

$$\begin{aligned}\left\|(I - R_m K)x^0\right\|^2 \;&=\; \sum_{j=1}^\infty \sigma_j^2\,\big(1 - a\sigma_j^2\big)^{2m}|(z,\phi_j)|^2 \\ =\;& \sum_{j=1}^\infty \left[\sigma_j^2\big(1 - a\sigma_j^2\big)^m|(z,\phi_j)|\right]\left[\big(1 - a\sigma_j^2\big)^m|(z,\phi_j)|\right] \\ \le\;& \sqrt{\sum_{j=1}^\infty \sigma_j^4\big(1 - a\sigma_j^2\big)^{2m}|(z,\phi_j)|^2}\;\sqrt{\sum_{j=1}^\infty \underbrace{\big(1 - a\sigma_j^2\big)^{2m}}_{\le 1}|(z,\phi_j)|^2} \\ \le\;& \left\|KR_m y^0 - y^0\right\|\,\|z\| \;\le\; \|z\|\;\left[\left\|(I - KR_m)(y^0 - y^\delta)\right\| + \left\|(I - KR_m)y^\delta\right\|\right] \\ \le\;& \|z\|\;(1 + r)\,\delta.\end{aligned}$$

Therefore, we conclude from (3.25) that

$$\left\|x^\delta_{m(\delta)} - x^0\right\| \;\leq\; \delta\,\sqrt{a\,m(\delta)} \;+\; \left\|R_{m(\delta)}Kx^0 - x^0\right\| \;\leq\; c\,\sqrt{\|z\|\,\delta}\,,$$

which ends the proof. $\square$

We note that the stopping rule is independent of the smoothness of the solution x^0. One can show by very similar arguments that, with the same stopping rule, one has the error estimate (for any $\ell \in \mathbb{N}$)

$$\left\|x^\delta_{m(\delta)} - x^0\right\| \;=\; \mathcal{O}\left(\delta^{2\ell/(2\ell+1)}\right)$$

provided $x^0 \in \mathcal{R}\big((A^*A)^\ell\big)$. Therefore, in contrast to Tikhonov's method the Landweber iteration provides an order of convergence which is arbitrarily close to δ provided the solution is smooth enough.

4 The Factorization Method for a Problem of Impedance Tomography

Electrical impedance tomography (EIT) is a medical imaging technique in which an image of the conductivity (or permittivity) of part of the body is determined from electrical surface measurements. Typically, conducting electrodes are attached to the skin of the subject and small alternating currents are applied to some or all of the electrodes. The resulting electrical potentials are measured, and the process may be repeated for numerous different configurations of applied currents.

Applications of EIT as an imaging tool can be found in fields such as medicine (monitoring of the lung function or the detection of skin cancer or breast cancer), geophysics (locating of underground deposits, detection of leaks in underground storage tanks), or nondestructive testing (determination of cracks in materials).

To derive the EIT model we start from the time-harmonic Maxwell system in the form

$$\operatorname{curl} H \;+\; (i\omega\varepsilon - \sigma)\,E \;=\; 0\,, \quad \operatorname{curl} E \;-\; i\omega\mu\,H \;=\; 0$$

in some domain which we take as a cylinder of the form $B \times \mathbb{R} \subset \mathbb{R}^3$ with bounded cross-section $B \subset \mathbb{R}^2$. Here, ω, ε, σ, and μ denote the frequency, electric permittivity, conductivity, and magnetic permeability, respectively, which are all assumed to be constant along the axis of the cylinder; that is, depend on x_1 and x_2 only. We note that the real parts $\operatorname{Re}\big[\exp(-i\omega t)\,E(x)\big]$ and $\operatorname{Re}\big[\exp(-i\omega t)\,H(x)\big]$ are the physically meaningful electric and magnetic field, respectively. For low frequencies ω (i.e., for small $(\omega\mu\sigma)\cdot L^2$ where L is a typical length scale of B), one can show that the Maxwell system is approximated by

$$\operatorname{curl} H \;-\; \sigma\,E \;=\; 0\,, \quad \operatorname{curl} E \;=\; 0\,.$$

The second equation yields the existence[1] of a scalar potential u such that $E = -\nabla u$. Substituting this into the first equation and taking the divergence yields

$$\operatorname{div}\big(\sigma\nabla u\big) \;=\; 0 \quad \text{in } \Omega\,, \tag{4.26}$$

where $\Omega \in \mathbb{R}^n$ is some connected and bounded domain with Lipschitz boundary. There are several possibilities for modeling the attachment of the electrodes on the boundary $\partial\Omega$ of Ω. The simplest of these is the *continuum model* in which the potential $u|_{\partial\Omega}$ and the boundary current distribution $f = \sigma\,\nabla u \cdot \nu = \sigma\,\partial u/\partial\nu$ are both given on the boundary $\partial\Omega$. Here, $\nu = \nu(x)$ is the unit normal vector at $x \in \partial\Omega$ directed into the exterior of Ω. First, we observe that,[2] by the divergence theorem,

$$0 \;=\; \int_\Omega \operatorname{div}\big(\sigma\nabla u\big)\,dx \;=\; \int_{\partial\Omega} \sigma\,\frac{\partial u}{\partial\nu}\,ds \;=\; \int_{\partial\Omega} f\,ds\,;$$

that is, the boundary current distribution f has zero mean; that is, we assume that $f \in L^2_*(\partial\Omega) := \big\{f \in L^2(\partial\Omega) : \int_{\partial\Omega} f\,ds = 0\big\}$.

In the *inverse problem of EIT* the conductivity function σ is unknown and has to be determined from simultaneous measurements of the boundary voltages $u|_{\partial\Omega}$ and current densities f, respectively.

Let $\sigma \in L^\infty(\Omega)$ and $f \in L^2_*(\partial\Omega)$ be given real-valued functions such that $\sigma(x) \geq \sigma_0$ on Ω for some $\sigma_0 > 0$. The direct problem is to determine $u \in H^1(\Omega)$ such that

$$\operatorname{div}\big(\sigma\nabla u\big) \;=\; 0 \quad \text{in } \Omega\,, \qquad \sigma\,\frac{\partial u}{\partial\nu} \;=\; f \quad \text{on } \partial\Omega\,. \tag{4.27}$$

We note that the solution is only unique up to an additive constant. Therefore, we normalize the solution u such that it has vanishing mean on the boundary; that is, $u \in H^1_*(\Omega)$ where

$$H^1_*(\Omega) \;=\; \left\{u \in H^1(\Omega) : \int_{\partial\Omega} u\,ds = 0\right\}. \tag{4.28}$$

The formulation (4.27) of the boundary value problem includes that the normal derivative $\sigma\partial u/\partial\nu$ is continuous on every interface where σ jumps. The solution u has to be understood in the variational (or weak) sense; that is, $u \in H^1_*(\Omega)$ is a solution of (4.27) provided that

$$\int\limits_\Omega \sigma\,\nabla\psi\cdot\nabla u\,dx \;=\; \int\limits_{\partial\Omega} \psi\,f\,ds \quad \text{for all } \psi \in H^1_*(\Omega)\,. \tag{4.29}$$

Existence and uniqueness follows from the representation theorem due to Riesz–Fischer because the left hand side is an inner product of $H^1_*(\Omega)$ which is equivalent to the ordinary

[1] If the domain is simply connected.

[2] Provided σ, f, and u are smooth enough.

one. Therefore, the **Neumann-Dirichlet operator** Λ_σ from $L^2_*(\partial\Omega)$ into itself is well defined. It maps[3] $f \in L^2_*(\partial\Omega)$ into the trace $u|_{\partial\Omega} \in L^2_*(\partial\Omega)$.

In this lecture we restrict ourselves to the more modest problem to determine only the shape of the region D where σ differs from the known background medium which we assume to be homogeneous with conductivity 1.

We sharpen the assumption on σ.

Assumption 4.1 *In addition to the above assumtion on σ let there exist finitely many domains D_j, $j = 1, \ldots, m$, such that $\overline{D_j} \subset \Omega$ and $\overline{D_j} \cap \overline{D_k} = \emptyset$ for $j \neq k$ and such that the complement $\Omega \setminus \overline{D}$ of the closure of the union $D = \bigcup_{j=1}^m D_j$ is connected. Furthermore, there exists $q_0 > 0$ such that $\sigma = 1$ on $\Omega \setminus D$ and $\sigma \geq 1 + q_0$ on D. We define the contrast q by $q = \sigma - 1$.*

The **inverse problem** of this section is to determine the shape of D from the Neumann–Dirichlet operator Λ_σ.

In the following we use the *relative* data $\Lambda_\sigma - \Lambda_1$ where $\Lambda_1 : L^2_*(\partial\Omega) \to L^2_*(\partial\Omega)$ corresponds to the known background medium; that is, to $\sigma = 1$. The information that $\Lambda_\sigma - \Lambda_1$ does not vanish simply means that the background is perturbed by some contrast $q = \sigma - 1$. In the Factorization Method we develop a criterion to decide whether or not a given point $z \in \Omega$ belongs to D. The idea is then to take a fine grid in Ω and to check this criterion for every grid point z. This provides a pixel-based picture of D.

We recall that $\Lambda_\sigma f = u|_{\partial\Omega}$ and $\Lambda_1 f = u_1|_{\partial\Omega}$, where $u, u_1 \in H^1_*(\Omega)$ solve

$$\int_\Omega (1+q)\,\nabla u \cdot \nabla\psi\,dx \;=\; (f,\psi)_{L^2(\partial\Omega)} \text{ for all } \psi \in H^1_*(\Omega)\,, \tag{4.30}$$

$$\int_\Omega \nabla u_1 \cdot \nabla\psi\,dx \;=\; (f,\psi)_{L^2(\partial\Omega)} \quad \text{for all } \psi \in H^1_*(\Omega)\,. \tag{4.31}$$

For the difference we have $(\Lambda_1 - \Lambda_\sigma)f = (u_1 - u)|_{\partial\Omega}$, and $u_1 - u \in H^1_*(\Omega)$ satisfies the variational equation

$$\int_\Omega (1+q)\,\nabla(u_1-u)\cdot\nabla\psi\,dx \;=\; \int_D q\,\nabla u_1 \cdot \nabla\psi\,dx \quad \text{for all } \psi \in H^1_*(\Omega)\,, \tag{4.32}$$

which is the variational form of

$$\operatorname{div}\big(\sigma\nabla(u_1-u)\big) = \operatorname{div}(q\nabla u_1) \text{ in } \Omega\,, \quad \frac{\partial(u_1-u)}{\partial\nu} = 0 \text{ on } \partial\Omega\,.$$

[3] Actually, it maps $H^{-1/2}_*(\partial\Omega)$ into $H^{1/2}_*(\partial\Omega)$ but we don't need these Sobolev spaces of fractional oders here.

It is the aim to *factorize* the operator $\Lambda_1 - \Lambda$ in the form

$$\Lambda_1 - \Lambda \;=\; H^* T\, H\,,$$

where the operators $H : L^2_*(\partial\Omega) \to L^2(D)^n$ and $T : L^2(D)^n \to L^2(D)^n$ are defined as follows:[4]

- $Hf = \nabla u_1|_D$, where $u_1 \in H^1_*(\Omega)$ solves the variational equation (4.31), and
- $Th = q(h - \nabla w|_D)$ where $w \in H^1_*(\Omega)$ solves

  $$\operatorname{div}(\sigma \nabla w) = \operatorname{div}(qh) \text{ in } \Omega\,, \quad \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial\Omega\,;$$

  that is,

  $$\int_\Omega (1+q)\, \nabla w \cdot \nabla \psi \, dx \;=\; \int_D q\, h \cdot \nabla \psi \, dx \quad \text{for all } \psi \in H^1_*(\Omega)\,, \tag{4.33}$$

  in variational form.

We note that the solution w of (4.33) exists and is unique. This is seen again by the representation theorem of Riesz-Fischer because the right-hand side again defines a linear and bounded functional $F(\psi) = \int_D q\, h \cdot \nabla \psi \, dx$ on $H^1_*(\Omega)$. The left-hand side of (4.33) is again the inner product in $H^1_*(\Omega)$.

Theorem 4.2

Let the operators $H : L^2_*(\partial\Omega) \to L^2(D)^n$ and $T : L^2(D)^n \to L^2(D)^n$ be defined as above by (4.31) and (4.33), respectively. Then

$$\Lambda_1 - \Lambda_\sigma \;=\; H^*\, T\, H\,. \tag{4.34}$$

Proof: We define the auxiliary operator $G : L^2(D)^n \to L^2_*(\partial\Omega)$ by $Gh = w|_{\partial\Omega}$ where $w \in H^1_*(\Omega)$ solves (4.33). Obviously, we conclude from (4.32) that $\Lambda_1 - \Lambda_\sigma = GH$.
We determine the adjoint $H^* : L^2(D)^n \to L^2_*(\partial\Omega)$ of H and prove that $H^* h = v|_{\partial\Omega}$ where $v \in H^1_*(\Omega)$ solves

$$\Delta v = \operatorname{div} h \text{ in } \Omega\,, \quad \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega\,;$$

that is,

$$\int_\Omega \nabla v \cdot \nabla \psi \, dx \;=\; \int_D h \cdot \nabla \psi \, dx \quad \text{for all } \psi \in H^1_*(\Omega) \tag{4.35}$$

[4] Here, $L^2(D)^n$ denotes the space of vector-valued functions $D \to \mathbb{R}^n$ such that all components are in $L^2(D)$.

in variational form. The solution v exists and is unique by the same arguments as above. To prove the representation of H^*h we conclude from (4.35) for $\psi = u_1$ and (4.31) that

$$(Hf,h)_{L^2(D)} \;=\; \int\limits_D \nabla u_1 \cdot h\,dx \;=\; \int\limits_\Omega \nabla u_1 \cdot \nabla v\,dx \;=\; (f,v)_{L^2(\partial\Omega)}\,,$$

and thus $v|_{\partial\Omega}$ is indeed the value H^*h of the adjoint.

Now it remains to show that $G = H^*T$. Let $h \in L^2(D)^n$ and $w \in H^1_*(\Omega)$ solve (4.33). Then $Gh = w|_{\partial\Omega}$. We rewrite (4.33) as

$$\int\limits_\Omega \nabla w \cdot \nabla\psi\,dx \;=\; \int\limits_D q\,(h - \nabla w)\cdot\nabla\psi\,dx \quad \text{for all } \psi \in H^1_*(\Omega)\,. \tag{4.36}$$

The comparison with (4.35) yields $H^*\big(q(h-\nabla w)\big) \;=\; w|_{\partial\Omega} \;=\; Gh$; that is, $H^*T = G$. Substituting this into $\Lambda_1 - \Lambda_\sigma = GA$ yields the assertion. □

Properties of the operators H and T are listed in the following theorem.

Theorem 4.3 *(a) The operator $H : L^2_*(\partial\Omega) \to L^2(D)^n$ is compact and one-to-one.*

(b) The operator $T : L^2(D)^n \to L^2(D)^n$ is self-adjoint and coercive; that is,

$$(Th,h)_{L^2(D)} \;\geq\; c\,\|h\|^2_{L^2(D)} \quad \textit{for all } h \in L^2(D)^n\,, \tag{4.37}$$

where $c = q_0\big(1 - q_0/(1+q_0)\big) > 0$.

Proof: (a) Choose an open set U such that $\overline{D} \subset U$ and $\overline{U} \subset \Omega$ and choose a non-negative function $\varphi \in C^\infty(\Omega)$ with $\varphi = 1$ on D and $\varphi = 0$ on $\Omega\backslash U$. Let $f \in L^2_*(\partial\Omega)$ and $u_1 \in H^1_*(\Omega)$ the corresponding solution of (4.31). Taking $\psi = \varphi u_1$ yields $\int_U \nabla u_1 \cdot \nabla(\varphi u_1)\,dx = 0$; that is,

$$\int_U \varphi\,|\nabla u_1|^2 dx \;=\; -\int_U u_1\,\nabla\varphi\cdot\nabla u_1\,dx \;\leq\; \|\nabla\varphi\|_\infty\,\|u_1\|_{L^2(\Omega)}\|\nabla u_1\|_{L^2(\Omega)}\,. \tag{4.38}$$

Now we begin with the proof of compactness. Let (f_j) be bounded in $L^2_*(\partial\Omega)$ and $u_j \in H^1_*(\Omega)$ the corresponding solutions of (4.31). The boundedness of the solution operator $f \mapsto u_1$ yields boundedness of (u_j) in $H^1_*(\Omega)$. Since this space is compactly imbedded in $L^2(\Omega)$ we can extract a subsequence, also denoted by (u_j), which converges in $L^2(\Omega)$. By (4.38) for $u_j - u_m$ instead of u_1 we have

$$\int_D |\nabla(u_j - u_m)|^2 dx \;\leq\; \int_U \varphi\,|\nabla(u_j - u_m)|^2 dx \;\leq\; c\,\|u_j - u_m\|_{L^2(\Omega)}$$

where c is a bound on $\|\nabla\varphi\|_\infty\,\|\nabla(u_j - u_m)\|_{L^2(\Omega)}$. This shows that $(\nabla u_j|_D)$ is a Cauchy sequence in $L^2(D)^n$ and proves compactness of H.

To show injectivity, let $0 = Hf = \nabla u_1|_D$ where $u_1 \in H^1_*(\Omega)$ denotes the weak solution of $\Delta u_1 = 0$ in Ω and $\partial u_1/\partial\nu = f$ on $\partial\Omega$. Without proof we use the regularity result that u_1 is analytic in Ω and therefore also its derivatives. The unique continuation property yields $\nabla u_1 = 0$ in all of Ω. Therefore, u_1 is constant in Ω and thus $f = \partial u_1/\partial\nu = 0$.
(b) Let $h_1, h_2 \in L^2(D)^2$ with corresponding solutions $w_1, w_2 \in H^1_*(\Omega)$ of (4.33). Then, with (4.33) for h_2, w_2 and $\psi = w_1$:

$$
\begin{aligned}
(Th_1, h_2)_{L^2(D)} &= \int_D q\,(h_1 - \nabla w_1)\cdot h_2\,dx \\
&= \int_D q\,h_1\cdot h_2\,dx \;-\; \int_D q\,\nabla w_1\cdot h_2\,dx \\
&= \int_D q\,h_1\cdot h_2\,dx \;-\; \int_\Omega (1+q)\,\nabla w_1\cdot\nabla w_2\,dx\,.
\end{aligned}
$$

This expression is symmetric with respect to h_1 and h_2. Therefore, T is self-adjoint.
For $h \in L^2(D)^n$ and corresponding solution $w \in H^1_*(\Omega)$ of (4.33) we conclude that

$$
\begin{aligned}
(Th, h)_{L^2(D)} &= \int_D q\,|h - \nabla w|^2\,dx \;+\; \int_D q\,(h - \nabla w)\cdot\nabla w\,dx \\
&= \int_D q\,|h - \nabla w|^2\,dx \;+\; \int_\Omega |\nabla w|^2\,dx \quad \text{(with the help of (4.36))} \\
&\geq \int_D \left[q_0\,|h|^2 - 2\,q_0\,h\cdot\nabla w + (1+q_0)\,|\nabla w|^2\right] dx \\
&= \int_D \left[\left|\sqrt{1+q_0}\,\nabla w - \frac{q_0}{\sqrt{1+q_0}}\,h\right|^2 + q_0\left(1 - \frac{q_0}{1+q_0}\right)|h|^2\right] dx \\
&\geq q_0\left(1 - \frac{q_0}{1+q_0}\right)\|h\|^2_{L^2(D)}\,.
\end{aligned}
$$

□

From this result and the factorization (4.34) we note that $\Lambda_1 - \Lambda_\sigma$ is compact, self-adjoint, and nonnegative.

Now we derive the binary criterion on a point $z \in \Omega$ to decide whether or not this point belongs to D. First, for every point $z \in \Omega$ we define a particular function $G(\cdot, z)$ such that $\Delta G(\cdot, z) = 0$ in $\Omega \setminus \{z\}$ and $\partial G(\cdot, z)/\partial\nu = 0$ on $\partial\Omega$ such that $G(x, z)$ becomes singular as x tends to z. We construct G from the Green's function N for Δ in Ω with respect to the Neumann boundary conditions.

We make an ansatz for N in the form $N(x, z) = \Phi(x, z) - \tilde{N}(x, z)$ where $\Phi(x, z)$ is the fundamental solution of the Laplace equation in $\mathbb{R}^n$ and determine $\tilde{N}(\cdot, z) \in H^1_*(\Omega)$ as the unique solution of the Neumann problem

$$\Delta\tilde{N}(\cdot,z)=0 \text{ in } \Omega \quad \text{and} \quad \frac{\partial\tilde{N}}{\partial\nu}(\cdot,z)=\frac{\partial\Phi}{\partial\nu}(\cdot,z)+\frac{1}{|\partial\Omega|} \text{ on } \partial\Omega\,.$$

We note that the solution exists because $\int_{\partial\Omega}\big[\partial\Phi/\partial\nu(\cdot,z)+1/|\partial\Omega|\big]\,ds=0$. This is seen by Green's first theorem in the region $\Omega\setminus K(z,\varepsilon)$:

$$\begin{aligned}\int\limits_{\partial\Omega}\frac{\partial\Phi}{\partial\nu}(\cdot,z)\,ds &= \int\limits_{|x-z|=\varepsilon}\frac{\partial\Phi}{\partial\nu}(x,z)\,ds(x)\\ &= -\frac{1}{2\pi}\int\limits_{|x-z|=\varepsilon}\frac{x-z}{|x-z|^2}\cdot\frac{x-z}{|x-z|}\,ds(x) \;=\; -1\,.\end{aligned}$$

Then $N=\Phi-\tilde{N}$ is the Green's function in Ω with respect to the Neumann boundary conditions; that is, N satisfies

$$\Delta N(\cdot,z)=0 \text{ in } \Omega\setminus\{z\} \quad \text{and} \quad \frac{\partial N}{\partial\nu}(\cdot,z)=-\frac{1}{|\partial\Omega|} \text{ on } \partial\Omega\,.$$

From the differentiable dependence of the solution $\tilde{N}(\cdot,z)$ on the parameter $z\in\Omega$ we conclude that, for any fixed $a\in\mathbb{R}^n$ with $|a|=1$, the function $G(\cdot,z)=a\cdot\nabla_z N(\cdot,z)$ satisfies

$$\Delta G(\cdot,z)=0 \text{ in } \Omega\setminus\{z\} \quad \text{and} \quad \frac{\partial G}{\partial\nu}(\cdot,z)=0 \text{ on } \partial\Omega\,. \tag{4.39}$$

The function $G(\cdot,z)$ has the following desired properties.

Lemma 4.4 *Let $z\in\Omega$, $R>0$, $\hat{\theta}_0\in\mathbb{R}^n$ with $|\hat{\theta}_0|=1$, and $\delta>0$ be kept fixed. For $\varepsilon\in[0,R)$ define the set (part of a cone)*

$$C_\varepsilon \;=\; \Big\{z+r\hat{\theta}:\varepsilon<r<R,\ \hat{\theta}\cdot\hat{\theta}_0>\delta\Big\}$$

with vertex in z. Let R be so small such that $\overline{C_\varepsilon}\subset\Omega$. Then

$$\lim_{\varepsilon\to 0}\|G(\cdot,z)\|_{L^2(C_\varepsilon)} \;=\; \infty\,.$$

We observe that the functions $\phi_z(x)=G(\cdot,z)|_{\partial\Omega}$ are traces of harmonic functions in $\Omega\setminus\{z\}$ with vanishing normal derivatives on $\partial\Omega$. Comparing this with the classical formulation of the adjoint H^* of H it seems to be plausible that the "source region" D can be determined by moving the source point z in ϕ_z. This is confirmed in the following theorem.

Theorem 4.5 *Let Assumptions 4.1 hold and let $a \in \mathbb{R}^n$ with $|a| = 1$ be fixed. For every $z \in \Omega$ define $\phi_z \in L^2_*(\partial\Omega)$ by*

$$\phi_z(x) \;=\; G(x,z) \;=\; a \cdot \nabla_z N(x,z)\,, \quad x \in \partial\Omega\,, \tag{4.40}$$

where N denotes the Green's function with respect to the Neumann boundary condition. Then

$$z \in D \quad \Longleftrightarrow \quad \phi_z \in \mathcal{R}(H^*)\,, \tag{4.41}$$

where $H^* : L^2(D)^n \to L^2_*(\partial\Omega)$ is the adjoint of H, given by (4.35), and $\mathcal{R}(H^*)$ its range.

Proof: First let $z \in D$. Choose a disc $K[z,\varepsilon] = \{x \in \mathbb{R}^n : |x - z| \le \varepsilon\}$ with center z and radius $\varepsilon > 0$ such that $K[z,\varepsilon] \subset D$. Furthermore, choose a function $\varphi \in C^\infty(\mathbb{R}^n)$ such that $\varphi(x) = 0$ for $|x - z| \le \varepsilon/2$ and $\varphi(x) = 1$ for $|x - z| \ge \varepsilon$ and set $w(x) = \varphi(x)G(x,z)$ for $x \in \Omega$. Then $w \in H^1_*(\Omega)$ and $w = G(\cdot,z)$ in $\Omega \setminus D$, thus $w|_{\partial\Omega} = \phi_z$.

Next we determine $u \in H^1_*(D)$ as a solution of $\Delta u = \Delta w$ in D, $\partial u/\partial\nu = 0$ on ∂D; that is, in weak form

$$\int\limits_D \nabla u \cdot \nabla\psi\, dx \;=\; \int\limits_D \nabla w \cdot \nabla\psi\, dx \;-\; \int\limits_{\partial D} \psi\, \frac{\partial}{\partial\nu} G(\cdot,z)\, ds\,, \quad \psi \in H^1_*(D)\,,$$

because $\partial w/\partial\nu = \partial G(\cdot,z)/\partial\nu$ on ∂D. Again, the solution exists and is unique. Application of Green's first theorem in $\Omega \setminus D$ yields

$$\int\limits_{\partial D} \frac{\partial}{\partial\nu} G(\cdot,z)\, ds \;=\; \int\limits_{\partial\Omega} \frac{\partial}{\partial\nu} G(\cdot,z)\, ds \;=\; 0\,.$$

Therefore, the previous variational equation holds for all $\psi \in H^1(D)$. Now let $\psi \in H^1_*(\Omega)$ be a test function on Ω. Then

$$\begin{aligned} \int\limits_D \nabla u \cdot \nabla\psi\, dx \;&=\; \int\limits_D \nabla w \cdot \nabla\psi\, dx \;-\; \int\limits_{\partial D} \psi\, \frac{\partial}{\partial\nu} G(\cdot,z)\, ds \\ &=\; \int\limits_D \nabla w \cdot \nabla\psi\, dx \;+\; \int\limits_{\Omega\setminus D} \nabla G(\cdot,z) \cdot \nabla\psi\, dx \;=\; \int\limits_\Omega \nabla w \cdot \nabla\psi\, dx\,. \end{aligned}$$

Therefore, the definition $h = \nabla u$ in D yields $H^*h = w|_{\partial\Omega} = \phi_z$ and thus $\phi_z \in \mathcal{R}(H^*)$.

Now we prove the opposite direction. Let $z \notin D$. We have to show that ϕ_z is not contained in the range of H^* and assume, on the contrary, that $\phi_z = H^*h$ for some $h \in L^2(D)^n$. Let $v \in H^1_*(\Omega)$ be the corresponding solution of (4.35). Therefore, the function $w = v - G(\cdot,z)$ vanishes on $\partial\Omega$ and solves the following equations in the weak form

$$\Delta w = 0 \text{ in } \Omega \setminus D_\varepsilon(z)\,, \quad \frac{\partial w}{\partial\nu} = 0 \text{ on } \partial\Omega\,,$$

for every $\varepsilon > 0$ such that $D_\varepsilon(z) = D \cup K(z,\varepsilon) \subset \Omega$, i.e

$$\int\limits_{\Omega\setminus D_\varepsilon(z)} \nabla w \cdot \nabla\psi \, dx \;=\; 0$$

for all $\psi \in H^1\big(\Omega \setminus D_\varepsilon(z)\big)$ with $\psi = 0$ on $\partial D_\varepsilon(z)$. We extend w by zero into the outside of Ω. Then $w \in H^1\big(\mathbb{R}^n \setminus D_\varepsilon(z)\big)$ because $w = 0$ on $\partial\Omega$ and

$$\int\limits_{\mathbb{R}^n\setminus D_\varepsilon(z)} \nabla w \cdot \nabla\psi \, dx \;=\; 0$$

for all $\psi \in H^1\big(\mathbb{R}^n \setminus D_\varepsilon(z)\big)$ which vanish on $\partial D_\varepsilon(z)$. Therefore, $\Delta w = 0$ in the exterior $\Omega = \mathbb{R}^n \setminus \big(\overline{D} \cup \{z\}\big)$ of $\overline{D} \cup \{z\}$. Now we use without proof that w is analytic in Ω and thus satisfies the unique continuation principle. Therefore, because it vanishes in the exterior of Ω it has to vanish in all of the connected[5] set Ω; that is, $v = G(\cdot, z)$ in $\Omega \setminus \big(\overline{D} \cup \{z\}\big)$. The point z can either be on the boundary ∂D or in the exterior of $\overline{D}$. In either case there is a cone C_0 of the form $C_0 = \big\{z + r\hat\theta : 0 < r < R,\ \hat\theta \cdot \hat\theta_0 > \delta\big\}$ with $C_0 \subset \Omega \setminus \overline{D}$. It is $v|_{C_0} \in L^2(C_0)$ inasmuch as $v \in H^1_*(\Omega)$. However, Lemma 4.4 yields that $\|G(\cdot,z)\|_{L^2(C_\varepsilon)} \to \infty$ for $\varepsilon \to 0$ where $C_\varepsilon = \big\{x \in C_0 : |x - z| > \varepsilon\big\}$. This is a contradiction because $v = G(\cdot, z)$ in C_0 and ends the proof. □

Therefore, we have shown an explicit characterization of the unknown domain D by the range of the operator H^*. This operator, however, is also unknown: only $\Lambda_1 - \Lambda_\sigma$ is known! The operators H^* and $\Lambda_1 - \Lambda_\sigma$ are connected by the factorization $\Lambda_1 - \Lambda_\sigma = H^* T H$. We can easily derive a second factorization of $\Lambda_1 - \Lambda_\sigma$. The operator $\Lambda_1 - \Lambda_\sigma$ is self-adjoint and compact as an operator from $L^2_*(\partial\Omega)$ into itself. Therefore, there exists a spectral decomposition of the form

$$(\Lambda_1 - \Lambda_\sigma) f \;=\; \sum_{j=1}^{\infty} \lambda_j \, (f, \psi_j)_{L^2(\partial\Omega)} \, \psi_j \,,$$

where $\lambda_j \in \mathbb{R}$ denote the eigenvalues and $\psi_j \in L^2_*(\partial\Omega)$ the corresponding orthonormal eigenfunctions of $\Lambda_1 - \Lambda_\sigma$. Furthermore, from the factorization and the coercivity of T it follows that $\big((\Lambda_1 - \Lambda_\sigma) f, f\big)_{L^2(\partial\Omega)} \geq 0$ for all $f \in L^2_*(\partial\Omega)$. This implies $\lambda_j \geq 0$ for all j. Therefore, we can define the square root

$$(\Lambda_1 - \Lambda_\sigma)^{1/2} f \;=\; \sum_{n=1}^{\infty} \sqrt{\lambda_j} \, (f, \psi_j)_{L^2(\partial\Omega)} \, \psi_j \,,$$

and have

$$(\Lambda_1 - \Lambda_\sigma)^{1/2} (\Lambda_1 - \Lambda_\sigma)^{1/2} \;=\; \Lambda_1 - \Lambda_\sigma \;=\; H^* T\, H \,. \qquad (4.42)$$

We show that the ranges of $(\Lambda_1 - \Lambda_\sigma)^{1/2}$ and H^* coincide. This follows directly from the following functional analytic result.

[5] Here we make use of the assumption that $\Omega \setminus \overline{D}$ is connected.

Lemma 4.6 *Let X and Y be Hilbert spaces, $B : X \to X$, $A : X \to Y$, and $T : Y \to Y$ linear and bounded such that $B = A^*TA$. Furthermore, let T be self-adjoint and coercive; that is, there exists $c > 0$ such that $(Ty, y)_Y \geq c\|y\|_Y^2$ for all $y \in Y$. Then, for any $\phi \in X$, $\phi \neq 0$,*

$$\phi \in \mathcal{R}(A^*) \iff \inf\big\{(Bx, x)_X : x \in X\,,\ (x, \phi)_X = 1\big\} \ > \ 0\,.$$

Proof: (i) First, let $\phi = A^*y \in \mathcal{R}(A^*)$ for some $y \in Y$. Then $y \neq 0$, and we estimate for arbitrary $x \in X$ with $(x, \phi)_X = 1$:

$$\begin{aligned} (Bx, x)_X &= (A^*TAx, x)_X \ = \ (TAx, Ax)_Y \ \geq \ c\|Ax\|_Y^2 \\ &= \frac{c}{\|y\|_Y^2}\,\|Ax\|_Y^2\|y\|_Y^2 \ \geq \ \frac{c}{\|y\|_Y^2}\,\big|(Ax, y)_Y\big|^2 \\ &= \frac{c}{\|y\|_Y^2}\,\big|(x, A^*y)_X\big|^2 \ = \ \frac{c}{\|y\|_Y^2}\,\big|(x, \phi)_X\big|^2 \ = \ \frac{c}{\|y\|_Y^2}\,. \end{aligned}$$

Therefore, we have found a positive lower bound for the infimum.

(ii) Second, let $\phi \notin \mathcal{R}(A^*)$. Define the closed subspace

$$V \ = \ \big\{x \in X : (\phi, x)_X = 0\big\} \ = \ \{\phi\}^\perp\,.$$

We show that the image $A(V)$ is dense in the closure $\mathrm{cl}(\mathcal{R}(A))$ of the range of A. Indeed, let $y \in \mathrm{cl}(\mathcal{R}(A))$ such that $y \perp Ax$ for all $x \in V$; that is, $0 = (Ax, y)_Y = (x, A^*y)$ for all $x \in V$; that is, $A^*y \in V^\perp = \mathrm{span}\{\phi\}$. Because $\phi \notin \mathcal{R}(A^*)$ we conclude that $A^*y = 0$. Therefore, $y \in \mathrm{cl}(\mathcal{R}(A)) \cap \mathcal{N}(A^*)$. This yields $y = 0$.[6] Therefore, $A(V)$ is dense in $\mathrm{cl}(\mathcal{R}(A))$. Because $A\phi/\|\phi\|_X^2$ is in the range of A there exists a sequence $\tilde{x}_n \in V$ such that $A\tilde{x}_n \to -A\phi/\|\phi\|_X^2$. We define $x_n := \tilde{x}_n + \phi/\|\phi\|_X^2$. Then $(x_n, \phi)_X = 1$ and $Ax_n \to 0$ for $n \to \infty$, and we estimate

$$(Bx_n, x_n)_X \ = \ (TAx_n, Ax_n)_Y \ \leq \ \|T\|\|Ax_n\|_Y^2 \ \longrightarrow \ 0\,, \quad n \to \infty\,,$$

and thus $\inf\big\{(Bx, x)_X : x \in X\,,\ (x, \phi)_X = 1\big\} = 0$. $\square$

We apply this result to both of the factorizations of (4.42). In both cases, $B = \Lambda_1 - \Lambda_\sigma$ and $X = L^2_*(\partial\Omega)$. First we set $Y = L^2(D)^n$ and $A = H : L^2_*(\partial\Omega) \to L^2(D)^n$ and $T : L^2(D)^n \to L^2(D)^n$ as in the second factorization of (4.42). Because T is self-adjoint and coercive we conclude for any $\phi \in L^2_*(\partial\Omega)$, $\phi \neq 0$, that

$$\phi \in \mathcal{R}(H^*) \ \Leftrightarrow \ \inf\Big\{\big((\Lambda_1 - \Lambda)f, f\big)_{L^2(\partial\Omega)} : f \in L^2_*(\partial\Omega)\,,\ (f, \phi)_{L^2(\partial\Omega)} = 1\Big\} \ > \ 0\,.$$

Second, we consider the first factorization of (4.42) with T being the identity. For $\phi \in L^2_*(\partial\Omega)$, $\phi \neq 0$, we conclude that

$$\phi \in \mathcal{R}\big((\Lambda_1 - \Lambda_\sigma)^{1/2}\big) \ \Leftrightarrow \ \inf\Big\{\big((\Lambda_1 - \Lambda_\sigma)f, f\big)_{L^2(\partial\Omega)} : (f, \phi)_{L^2(\partial\Omega)} = 1\Big\} \ > \ 0\,.$$

[6]Take a sequence (x_j) in X such that $Ax_j \to y$. Then $0 = (A^*y, x_j)_X = (y, Ax_j)_Y \to (y, y)_Y$; that is, $y = 0$.

The right-hand sides of the characterizations only depend on $\Lambda_1 - \Lambda_\sigma$, therefore we conclude that

$$\mathcal{R}\big((\Lambda_1 - \Lambda_\sigma)^{1/2}\big) \;=\; \mathcal{R}(H^*)\,. \tag{4.43}$$

Application of Theorem 4.5 yields the main result of the Factorization Method:

Theorem 4.7 *Let Assumptions 4.1 be satisfied. For fixed $a \in \mathbb{R}^n$ with $a \neq 0$ and every $z \in \Omega$ let $\phi_z \in L^2_*(\partial\Omega)$ be defined by (4.40); that is, $\phi_z(x) = a \cdot \nabla_z N(x,z)$, $x \in \partial\Omega$, where N denotes the Green's function for Δ with respect to the Neumann boundary conditions. Then*

$$z \in D \quad\Longleftrightarrow\quad \phi_z \in \mathcal{R}\big((\Lambda_1 - \Lambda_\sigma)^{1/2}\big)\,. \tag{4.44}$$

We now rewrite the right-hand side with Picard's theorem. First we show injectivity of the operator $\Lambda_1 - \Lambda_\sigma$. Indeed, let $f \in L^2_*(\partial\Omega)$ with $(\Lambda_1 - \Lambda_\sigma)f = 0$. Then

$$\begin{aligned} 0 &= \big((\Lambda_1 - \Lambda_\sigma)f, f\big)_{L^2(\partial\Omega)} \;=\; (H^*THf, f)_{L^2(\partial\Omega)} \\ &= (THf, Hf)_{L^2(D)^n} \;\geq\; c\|Hf\|^2_{L^2(D)^n} \end{aligned}$$

and thus $Hf = 0$. Injectivity of H yields $f = 0$. $\square$

Therefore, the operator $\Lambda_1 - \Lambda_\sigma$ is self-adjoint, compact, one-to-one, and all eigenvalues are positive. Let again $\{\lambda_j, \psi_j\}$ be an eigensystem of $\Lambda_1 - \Lambda_\sigma$; that is, $\lambda_j > 0$ are the eigenvalues of $\Lambda_1 - \Lambda_\sigma$ and $\psi_j \in L^2_*(\partial\Omega)$ are the corresponding orthonormal eigenfunctions. The set $\{\psi_j : j = 1, 2, \ldots\}$ is complete by the spectral theorem. Then we have:

Theorem 4.8 *Let Assumptions 4.1 be satisfied. For fixed $a \in \mathbb{R}^n$ with $a \neq 0$ and for every $z \in \Omega$ let $\phi_z \in L^2_*(\partial\Omega)$ be defined by (4.40); that is, $\phi_z(x) = a \cdot \nabla_z N(x,z)$, $x \in \partial\Omega$. Then*

$$z \in D \quad\Longleftrightarrow\quad \sum_{j=1}^{\infty} \frac{(\phi_z, \psi_j)^2_{L^2(\partial\Omega)}}{\lambda_j} \;<\; \infty \tag{4.45}$$

or, equivalently,

$$z \in D \quad\Longleftrightarrow\quad W(z) \;:=\; \left[\sum_{j=1}^{\infty} \frac{(\phi_z, \psi_j)^2_{L^2(\partial\Omega)}}{\lambda_j}\right]^{-1} \;>\; 0\,. \tag{4.46}$$

Here we agreed on the setting that the inverse of the series is zero in the case of divergence. Therefore, W vanishes outside of D and is positive in the interior of D. The function

$$\text{sign}\; W(z) \;=\; \begin{cases} 1, & W(z) > 0, \\ 0, & W(z) = 0, \end{cases}$$

is thus the characteristic function of D.

We finish this section with some further remarks.

- We leave it to the reader to show that in the case of $B = K(0,1)$ being the unit disk in $\mathbb{R}^2$ and $D = K(0,R)$ the disk of radius $R < 1$ the ratios $(\phi_z, \psi_j)^2_{L^2(\partial\Omega)}/\lambda_j$ behave as $(|z|/R)^{2j}$. Therefore, convergence holds if and only if $|z| < R$ which confirms the assertion of the last theorem.

- The same characterization as in Theorem 4.8 has been justified for different types of inpenetrable objects D; that is, where u lives only in $\Omega \setminus D$ and satisfies some hhomogeneous boundary condition on ∂D. In particular, the Factorization Method provides a proof of uniqueness of D independent of the nature of D; that is, whether it is finitely conducting, a perfect conductor (Dirichlet boundary conditions on ∂D), a perfect insulator (Neumann boundary conditions on ∂D), or a boundary condition of Robin-type.