HARNACK AND POINTWISE ESTIMATES FOR NONNEGATIVE SOLUTIONS TO A CLASS OF DEGENERATE/SINGULAR PARABOLIC EQUATIONS

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Abstract.

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2. Elliptic Harnack inequality

2.1. **Original Harnack.** In 1887, The German mathematician C.G. Axel von Harnack proved the following result in [39].

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Theorem 2.1. *Let u be a nonnegative harmonic function in $B_R(x_0) \subseteq \mathbb{R}^2$. Then for all $x \in B_r(x_0) \subset B_R(x_0)$ it holds*

$$\frac{R-r}{R+r}u(x_0) \le u(x) \le \frac{R+r}{R-r}u(x_0).$$

The estimate can be generalized to any dimension $N \ge 1$, resulting in

$$\left(\frac{R}{R+r}\right)^{N-2}\frac{R-r}{R+r}u(x_0) \le u(x) \le \left(\frac{R}{R-r}\right)^{N-2}\frac{R+r}{R-r}u(x_0), \tag{2.1}$$

however, the modern version of the Harnack inequality for harmonic functions is the following special case of the previous one.

Theorem 2.2. *Let $N \ge 1$. Then there exists a constant $C = C(N) > 1$, such that if u is a nonnegative, harmonic function in $B_{2r}(x_0)$, then*

$$\sup_{B_r(x_0)} u \le C \inf_{B_r(x_0)} u. \tag{2.2}$$

The proof of this latter form of the Harnack inequality is an easy consequence of the mean value theorem, while the more precise form (2.1) can be derived through Poisson representation formula. The Harnack inequality has several deep and powerful consequences. On the local side, Harnack himself in [39] derived from it a precisely quantified oscillation estimate. Due to the ubiquity of this argument we recall its elementary proof. Let $x_0 = 0$ and

$$M_r(u) = \sup_{B_r} u, \qquad m_r(u) = \inf_{B_r} u, \quad \mathrm{osc}_r(u) = M_r(u) - m_r(u).$$

Both $M(2r) - u$ and $u - m(2r)$ are nonnegative and harmonic in B_{2r}, so (2.2) holds for them, thus

$$M(2r) - m_r(u) \le C(M(2r) - M_r(u)), \quad M_r(u) - m(2r) \le C(m_r(u) - m(2r)),$$

which added together give

$$(M(2r) - m(2r)) + (M_r(u) - m_r(u)) \le C\left((M(2r) - m(2r)) - (M_r(u) - m_r(u))\right).$$

Rearranging, we obtain

$$\mathrm{osc}_r(u) \le \frac{C-1}{C+1}\mathrm{osc}_{2r}(u),$$

which is the claimed quantitive estimate of decrease in oscillation.

Removable singularity results can also be obtained through the Harnack inequality, as well as two classical convergence criterions for sequences of harmonic functions. At the global level, it implies Liouville and Picard type theorems. In particular, Liouville's theorem asserts that any globally defined harmonic function bounded from below must be constant, as can be clearly seen by applying (2.2) to $u - \inf_{\mathbb{R}^N} u$ and letting $r \to +\infty$.

2.2. **Modern developements.** In his celebrated paper [49], Moser extended the Harnack inequality to weak solutions of linear elliptic operators

$$L(u) := \sum_{i,j=1}^{N} D_i(a_{ij}(x)D_j u) = 0 \tag{2.3}$$

with measurable coefficients.

Theorem 2.3. *Suppose $u \ge 0$ solves (2.3) in a ball $B_{2r}(x_0)$ where a_{ij} are measurable functions satisfying the ellipticity condition*

$$\lambda|\xi|^2 \le \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2, \qquad 0 \le \lambda \le \Lambda < +\infty. \tag{2.4}$$

Then there exists a constant $C > 1$ depending only on N and the ellipticity ratio Λ/λ *such that*

$$\sup_{B_r(x_0)} u \le C \inf_{B_r(x_0)} u.$$

Moser's proof is mostly measure-theoretical, stemming from the De Giorgi approach to the regularity of solutions of elliptic equations. Such a level of generality allowed to apply essentially the same technique to nonnegative solutions of general quasilinear equations of the form

ql
$$\operatorname{div} A(x, u, Du) = 0. \tag{2.5}$$

Indeed, in [59] [61], the same statement of the Harnack inequality has been proved for (3.11) instead of the linear equation (2.3), provided A satisfies for some $p > 1$ and $\Lambda \ge \lambda > 0$

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$$\begin{cases} A(x, s, z) \cdot z \ge \lambda |z|^p \\ |A(x, s, z)| \le \Lambda |z|^{p-1} \end{cases} \qquad x \in B_{2r}(x_0), s \in \mathbb{R}, z \in \mathbb{R}^N. \tag{2.6}$$

The power of the measure-theoretical approach was then fully exploited in [27], where the Harnack inequality has been deduced without any reference to an elliptic equation, proving that it is a consequence of very general energy estimates of Caccioppoli type, encoded in what are the nowadays called *De Giorgi classes*. For a comprehensive treatment of the latters see [18].

2.3. **Weak Harnack inequalities.** Moser's proof consists in showing that a weaker form of Harnack inequality holds for the much more larger class of *supersolutions* to (2.3), i.e. those functions satisfying $-L(u) \ge 0$ (subsolutions being defined through the opposite inequality). In modern terms the measure theoretic proof of the Harnack inequality is usually splitted in two parts:

(1) – $L^p - L^\infty$ *bound*:
Let u be a nonnegative subsolution of (2.3) in B_{2r}. For any $p > 0$ it holds

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$$\sup_{B_r} u \le C \left(\fint_{B_{2r}} |u|^p \, dx \right)^{\frac{1}{p}} \tag{2.7}$$

for some constant $C = C(N, \Lambda/\lambda, p)$.

(2) – *Weak Harnack inequality*:
Let u be a nonnegative supersolution of (2.3) in B_{2r}. For any $p \in]0, \frac{N}{N-2}[$ it holds

$$\left(\fint_{B_{2r}} u^p \, dx \right)^{\frac{1}{p}} \le C \inf_{B_r} u$$

for some constant $C = C(N, \Lambda/\lambda, p)$.

The range of exponents in the weak Harnack inequality is optimal , as the fundamental solution for the Laplacian shows.

Notice that the $L^\infty - L^p$ bound also implies a Liouville theorem for $L^p(\mathbb{R}^N)$ nonnegative subsolutions, while the weak Harnack inequality gives a lower asymptotic estimate for positive $L^p(\mathbb{R}^N)$ super solutions. From the local point of view, the latter is also sufficient for Hölder regularity and for strong comparison principles.

2.4. **Harnack inequality on minimal surfaces.** After considering the Harnack inequality for nonlinear operator, a very fruitful framework was to consider its validity for linear elliptic operators defined on *nonlinear* ambient spaces, such as Riemannian manifolds. One of the first examples of this approach was the Bombieri - De Giorgi - Miranda gradient bound [6] for solutions of the *minimal surface equation*

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0.$$

The approach of [6], later simplified in [63], consisted in showing that $w = \log\sqrt{1+|Du|^2}$ is a subsolution of the Laplace-Beltrami operator naturally defined on the graph of u considered as a Riemannian manifold. Since a Sobolev-Poincaré inequality can be proved for minimal graphs (see [?] for a refinement to smooth minimal submanifolds), the Moser iteration yelds an $L^\infty - L^1$ bound on w which is the core of the proof.

This approach was pushed forward in [7], where a pure Harnack inequality was shown for general linear operators on minimal graphs, with applications to Bernstein-type theorems. See also [12] for other applications of the Harnack inequality on minimal graphs.

2.5. **Differential Harnack inequality.** It is a classical fact that Harmonic functions in $B_r(x_0)$ satisfy the gradient estimate

$$|Du(x_0)| \le C(N)\frac{\sup_{B_{r/2}(x_0)}|u|}{r},$$

therefore Harnack's inequality implies that

$$u \ge 0 \text{ in } B_r(x_0) \quad \Rightarrow \quad |Du(x_0)| \le C(N)\frac{u(x_0)}{r}.$$

This can be rewritten in the following form:

Theorem 2.4 (Differential Harnack inequality)**.** *Let u be positive and harmonic in $B_r(x_0) \subseteq \mathbb{R}^N$. Then*

DH (2.8)
$$|D\log u(x_0)| \le \frac{C(N)}{r}.$$

The latter inequality can also be integrated back along segments, to give the original Harnack inequality. The differential form (2.8) of the Harnack inequality clearly requires much more regularity than the Moser's one, however, it was proved to hold in the Riemannian setting for the Laplace-Beltrami equation in the ground-breaking works [12, 64], under the assumption of non-negative Ricci curvature for the manifold. To appreciate the result, notice that all proofs of the Harnack inequality known at the time required a global Sobolev inequality, which is known to be false in general under the Ric ≥ 0 assumption.

The elliptic Harnack inequality in the Riemannian setting proved in [64] (and, even more importantly, its parabolic version proved soon after in [46]) again implies the Liouville property for semi bounded harmonic functions and it was one of the stepping-stones to the rise of modern geometric analysis. See for example the survey article [45] for recent results on the relationship between Liouville-type theorems and geometric aspects of the underlying manifold. The book [52] gives an in-depth exposition of the technique of differential Harnack inequalities in the framework of Ricci flow, culminating in Perelman differential Harnack inequality.

2.6. **Beyond smooth manifolds.** Clearly, the differential approach to the Harnack inequality is restricted to the Laplace-Beltrami operator, due to its smoothness and its close relationship with Ricci curvature given by the Bochner identity

$$\Delta u = 0 \quad \Rightarrow \quad \Delta\frac{|Du|^2}{2} = |D^2u|^2 + \operatorname{Ric}(Du, Du).$$

It was only after the works [36, 56] that a different approach to Moser's Harnack inequality on manifolds was found.[1] Essentially, it was realized that in order to obtain the Harnack inequality, on a Riemannian manifold (M, g) with corresponding volume m and geodesic distance, two ingredients suffices:

$$\begin{aligned} &\text{–}\textit{Doubling condition}: && m(B_{2r}(x_0)) \le C m(B_r(x_0)) \\ &\text{–}\textit{Poincaré inequality}: && \int_{B_r(x_0)} \Big| u - ⨍_{B_r(x_0)} u\, dm \Big|^2 dm \le C \int_{B_r(x_0)} |Du|^2\, dm \end{aligned} \tag{2.9}$$

for any $x_0 \in M$ and $r > 0$. These two properties hold in any Riemannian manifold with nonnegative Ricci curvature, thus giving a Moser-theoretic approach to the Harnack inequality in this framework. What is relevant here is that Doubling&Poincaré are stable with respect to quasi-isometries (i.e. bilipschitz homeomorphisms) and thus can hold in non-smooth manifolds, manifolds where $\mathrm{Ric} \ge 0$ does not hold (since curvature is not preserved through quasi-isometries), and/or for merely measurable coefficients elliptic operators. It is worth mentioning that Doubling&Poincaré were also shown in [13] to be sufficient conditions for the solution of Yau's conjecture on the finite-dimensionality of the space of harmonic functions of polynomial growth.

It was a long standing problem to give geometric conditions which are actually *equivalent* to the validity of the Harnack inequality, and thus to establish the stability of the latter with respect to quasi (or even rough) isometries. This problem has recently been settled in [4], to which we refer the interested reader for bibliographic reference and discussion.

3. Parabolic Harnack inequality

3.1. **Original Parabolic Harnack.** Looking at the fundamental solution for the heat equation

$$u_t - \Delta u = 0,$$

one finds out that there is no hope to prove a straightforward generalization of the Harnack inequality (2.2). In the stationary case, ellipticity is preserved by spatial homotheties and traslations, thus the corresponding Harnack inequality turns out to be scale and traslation invariant. For the heat equation, the natural scaling $(x, t) \mapsto (\lambda x, \lambda^2 t)$ preserves the equation and one expects a parabolic Harnack inequality to obey this invariance. Actually, an explicit calculation shows that it cannot hold for *fixed* times $t_0 > 0$ and corresponding space balls $B(x_0, R_0)$, even assuming that $t_0 \ge 1$. However, a similar argument rules out the possibility of a Harnack inequality in parabolic cylinders as well. The correct parabolic form of the Harnack inequality was found and proved independently by Pini and Hadamard in [37, 54] and reads as follows.

Theorem 3.1. *Let $u \ge 0$ be a solution of the heat equation in $B_{2\rho}(x_0) \times \{t_0 - 4\rho^2, t_0 + 4\rho^2\}$. Then there exists a constant γ depending only upon the dimension N, such that*

$$\sup_{B_\rho(x_0)} u(\cdot, t_0 - \rho^2) \le C(N) \inf_{B_\rho(x_0)} u(\cdot, t_0 + \rho^2). \tag{3.1}$$

As expected, this form of Harnack's inequality respects the scaling of the equations and introduces the notion of *waiting time* for a pointwise control to hold. It represents a quantitative bound from below on how much the positivity of $u(x_0, t_0)$ (physically, the temperature of a body at a point) propagates forward in time: in order to have such a bound in a whole ball of radius r we have to wait a time proportional to r^2.

[1] Actually, to a *parabolic version* of the Harnack inequality, which readily implies the elliptic one. For further details see the discussion on the parabolic Harnack inequality below and for a nice historical overview on the subject see [58], section 5.5.

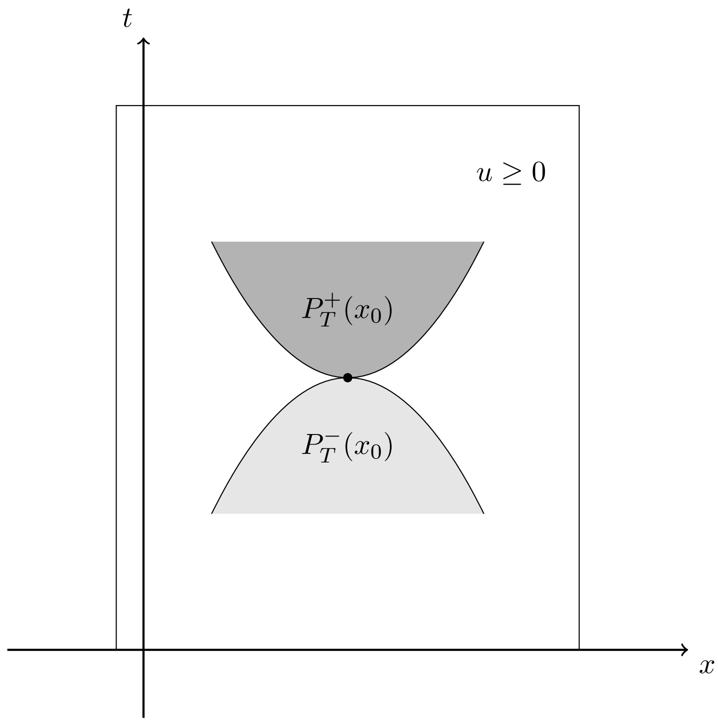

FIGURE 1. Assuming $u > 0$ in the boxed region $B_{2\sqrt{T}}(x_0) \times [0, 4T]$, the dark grey area is $P_T^+(x_0)$ where u is bounded below by $u(x_0, 2T)$, while the light grey is $P_+^T(x_0)$ where u is bounded above by $u(x_0, 2T)$.

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Another way of expressing this propagation for a nonnegative solution on $B_{2\sqrt{T}}(x_0) \times [0, 4T]$ is the following, which, up to numerical factors is equivalent to (3.1),

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$$C \inf_{P_T^+(x_0)} u \geq u(x_0, 2T) \geq C^{-1} \sup_{P_T^-(x_0)} u, \tag{3.2}$$

where $P_T^{\pm}(x_0)$ are the part of the forward (resp. backward) space-time paraboloid with vertex $(x_0, 2T)$ in $B_{\sqrt{T}}(x_0) \times [T, 3T]$ (see Figure 2):

$$P_T^+(x_0) = \{(x,t) : T - t_0 \geq t - t_0 \geq |x - x_0|^2\}, \qquad P_T^-(x_0) = \{(x,t) : t_0 - T \geq t_0 - t \geq |x - x_0|^2\}.$$

A consequence of the parabolic Harnack inequality is the following form of the strong maximum principle. We sketch a proof here since this argument will play a rôle in the discussion of the Harnack inequality for nonlinear equations.

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Corollary 3.2 (Parabolic Strong Minimum Principle)**.** *Let $u \geq 0$ be a solution of the heat equation in $\Omega \times [0, T]$, where Ω is connected, and suppose $u(x_0, t_0) = 0$. Then $u \equiv 0$ in $\Omega \times [0, t_0]$.*

Proof. (sketch) Pick $P_1 := (x_1, t_1) \in \Omega \times]0, t_0[$ and join P_1 and $P_0 = (x_0, t_0)$ with a smooth curve $\gamma : [0, 1] \to \Omega \times]0, t_0]$ such that γ' has always a positive t-component. By compactness there is $\delta > 0$ and a small forward parabolic sector $P_\varepsilon^+ = \{\varepsilon \geq t \geq |x|^2\}$ such that: *1)* $\gamma(\sigma) \in \gamma(\tau) + P_\varepsilon^+$ for all $\sigma \in [\tau, \tau + \delta]$ and *2)* the parabolic Harnack inequality holds in the form (3.2) for all $s \in [0, 1]$, i.e.

$$u(\gamma(s)) \leq \inf_{\gamma(s) + P_\varepsilon^+} u.$$

These two properties and $u(\gamma(1)) = 0$ readily imply $u(\gamma) \equiv 0$. $\square$

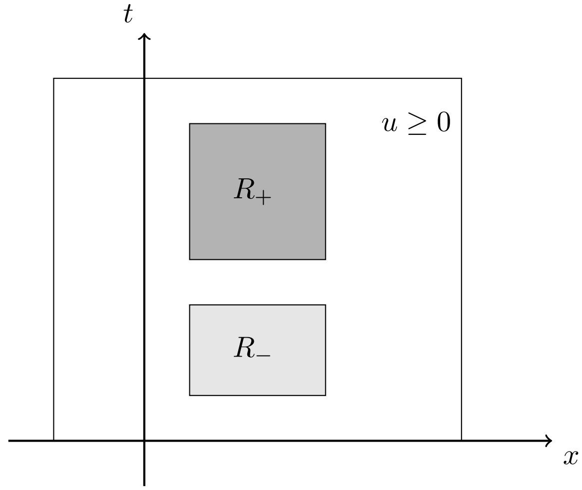

FIGURE 2. The regions R_+ and R_- where the Harnack inequality is stated.

3.2. **The linear case with coefficients.** In the seminal paper [53] on the Hölder regularity of solutions to elliptic parabolic equations with measurable coefficients, Nash already mentioned the possibility to obtain a parabolic Harnack inequality through his techniques. However, the first one to actually prove it was again Moser, who in [50] extended the Harnack inequality to linear parabolic equations of the form

$$u_t = \sum_{j,i=1}^{N} D_i(a_{ij}(x,t) D_j u). \tag{3.3}$$

Theorem 3.3 (Moser)**.** *Let u be a positive weak solution of (3.3) in $B_{2r} \times [0, T]$, where a_{ij} are measurable and satisfy the ellipticity condition (2.4). For any $0 < t_1^- < t_2^- < t_1^+ < t_2^+ < T$ define*

$$R_- := B_r \times [t_1^-, t_2^-], \quad R_+ := B_r \times [t_1^+, t_2^+].$$

Then it holds

$$\sup_{R_-} u \le C(N, \Lambda, \lambda, t_{1,2}^{\pm}) \inf_{R_-} u. \tag{3.4}$$

Using the natural scaling of the equation, the previous form the parabolic Harnack inequality can be reduced to (3.1). Later, in [51], Moser himself simplified his proof avoiding the use of a difficult parabolic versions of the John-Nirenberg BMO estimate. Nash program was later established in [31].

As in the elliptic case, the parabolic Harnack inequality provides an oscillation estimate giving the Hölder continuity of solutions to (3.3) subjected to (2.4). Moreover, (3.4) readily yelds a strong minimum principle for nonnegative solutions of (3.3).

On other hand, Liouville theorems in the parabolic setting are more subtle and do not follow from the parabolic version of the Harnack inequality. In fact, the Liouville property is false in general since, for example, the function $u(x,t) = e^{x+t}$ is clearly a nontrivial positive eternal (i.e., defined on $\mathbb{R}^N \times \mathbb{R}$) solution of the heat equation. A fruitful setting where to state Liouville properties in the one of *ancient* solutions, i.e. those defined on an unbounded interval $]-\infty, T_0[$.

Theorem 3.4 (Widder)**.** *Let $u > 0$ solve the heat equation in $\mathbb{R}^N \times]-\infty, T_0[$. Suppose for some $t_0 < T_0$ it holds*

$$u(x, t_0) \le C e^{o(|x|)}, \qquad |x| >> 1.$$

Then, u is constant.

The latter has been proved for $N = 1$ in [67], and we sketch the proof in the general case. By the Widder representation for ancient solutions (see [47]) it holds

$$u(x,t) = \int_{\mathbb{R}^N} e^{x\cdot\xi + t|\xi|^2}\, d\mu_\xi \tag{3.5}$$

for some nonnegative Borel measure μ. Call $\nu = e^{t_0|\xi|^2}\mu$ and observe that the Hölder inequality with respect to the measure ν implies that $x \mapsto \log u(x, t_0)$ is convex. The assumption then implies that $\log u(x, t_0)$ is constant and thus $x \mapsto u(x, t_0)$ is constant as well. Differentiating under the integral sign we obtain

$$0 = P(D_x)u(x,t_0)|_{x=0} = \int_{\mathbb{R}^N} P(\xi)\, d\nu$$

for any polinomial P such that $P(0) = 0$. By a classical Fourier transform argument, this implies that $\nu = c\delta_0$ and thus $u(x,t) \equiv c$ due to the representation (3.5).

Compare with [60] where it is proved that under the growth condition $0 \le u \le Ce^{o(|x|+\sqrt{|t|})}$ for $t \le 0$, there are no ancient non-constant solutions to the heat equation on a complete Riemannian manifold with $\mathrm{Ric} \ge 0$.

3.3. **Heat Kernel estimates.** Using Moser's Harnack inequality, Aronsson proved in [1] a *two sided* bound on the fundamental solution of (3.3), which reads

$$\frac{1}{C(t-s)^{N/2}} e^{-C\frac{|x-y|^2}{t-s}} \le \Gamma(t,x;s,y) \le \frac{C}{(t-s)^{N/2}} e^{-\frac{1}{C}\frac{|x-y|^2}{t-s}} \tag{3.6}$$

for some $C = C(N, \Lambda, \lambda)$ and $t > s > 0$, where the fundamental solution (or *heat kernel*) is defined as the solution of

$$\begin{cases} \partial_t \Gamma = \sum_{j,i=1}^N D_{x_i}(a_{ij}(x,t)D_{x_j}\Gamma) & \text{in } \mathbb{R}^N \times\,]s, +\infty[, \\ \Gamma(x,t;\cdot,s) \rightharpoonup^* \delta_x, & \text{as } t \downarrow s, \text{ in the measure sense.} \end{cases}$$

In [31], the previous kernel estimate was proved through Nash approach, and was shown to be equivalent to the parabolic Harnack inequality.

A *global* Harnack inequality also follows from (3.6): if $u \ge 0$ is a solution to (3.3) on $\mathbb{R}^N \times \mathbb{R}_+$ and $t > s > \tau \ge 0$, then using the representation

$$u(x,t) = \int_{\mathbb{R}^N} \Gamma(x,t;\xi,\tau)u(\xi,\tau)\, d\xi, \qquad t > \tau,$$

and the analogous one for (y, s), we get

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}^N} \Gamma(x,t;\xi,\tau)\Gamma^{-1}(y,s;\xi,\tau)\Gamma(y,s;\xi,\tau)u(\xi,\tau)\, d\xi \\ &\ge u(y,s)\frac{1}{C^2}\left(\frac{s-\tau}{t-\tau}\right)^{\frac{N}{2}} \inf_\xi e^{-C\frac{|x-\xi|^2}{t-\tau} + \frac{1}{C}\frac{|y-\xi|^2}{s-\tau}}. \end{aligned}$$

Now if $s/t \le 1/(2C^2)$ we choose $\tau = 0$ and compute

$$-C\frac{|x-\xi|^2}{t-\tau} + \frac{|y-\xi|^2}{s-\tau} \ge -\frac{|x-y|^2}{\lambda(t-s)}, \qquad \lambda = \frac{1}{2C}.$$

while if $s/t > 1/(2C^2)$, we set $\tau = s - (t-s)/(4C^2) > 0$ obtaining

$$-C\frac{|x-\xi|^2}{t-\tau} + \frac{|y-\xi|^2}{s-\tau} \ge -\frac{|x-y|^2}{\lambda(t-s)}, \qquad \lambda = \frac{1}{2C} - \frac{1}{4C^3}$$

while

$$\frac{s-\tau}{t-\tau} \geq \frac{1}{4C^2}.$$

Therefore the kernel bounds (3.6) imply a Harnack inequality *at large*, often called *sub-potential lower bound*, for positive solutions u of (3.3) on $\mathbb{R}^N \times]0, T[$: there exists a constant $C = C(N, \Lambda, \lambda) > 1$ such that

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$$u(x,t) \geq \frac{1}{C} u(y,s) \left(\frac{s}{t}\right)^{\frac{N}{2}} e^{-C\frac{|x-y|^2}{t-s}} \qquad \text{for all } T > t > s > 0. \tag{3.7}$$

A similar global estimate, with a non-optimal exponent $\alpha = \alpha(N, \Lambda, \lambda) > N/2$ was already derived through the so-called *Harnack chain* technique by Moser in [50].

3.4. **Riemannian manifolds and beyond.** Following the differential approach of [?], Li and Yau proved in [46] their celebrated parabolic differential Harnack inequality.

Theorem 3.5. *Let M be a complete Riemannian manifold of dimension $N \geq 2$ and $\mathrm{Ric} \geq 0$, and let $u > 0$ solve the heat equation on $M \times \mathbb{R}_+$. Then it holds*

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$$|D \log u|^2 - \partial_t(\log u) \leq \frac{N}{2t}. \tag{3.8}$$

In the same paper, many variants of the previous inequality are considered, including one for local solutions in $B_R(x_0) \times]t_0 - T, t_0[$ much in the spirit of [12] and several consequences are also derived. Integrating inequality (3.8) along geodesics provides, for any positive solution of the heat equation of $M \times \mathbb{R}_+$

LYp

$$u(x,t) \geq u(y,s) \left(\frac{s}{t}\right)^{\frac{N}{2}} e^{-\frac{d^2(x,y)}{4(t-s)}}, \qquad t > s > 0, \tag{3.9}$$

where $d(x, y)$ is the geodesic distance between two points $x, y \in M$. This, in turn, gives the heat kernel estimate (see [58, Ch 5])

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$$\frac{1}{CV(x, \sqrt{t-s})}\, e^{-C\frac{d^2(x,y)}{t-s}} \leq \Gamma(x,t;y,s) \leq \frac{C}{V(x, \sqrt{t-s})}\, e^{-\frac{1}{C}\frac{d^2(x,y)}{t-s}}, \tag{3.10}$$

where $V(x, r)$ is the Riemannian volume of a geodesic ball $B(x, r)$. Notice that, in a general Riemannian manifold of dimension $N \geq 2$,

$$V(x,r) \simeq r^N \qquad \text{for small } r > 0,$$

but, under the sole assumption $\mathrm{Ric} \geq 0$, the best one can say is

$$\frac{r}{C} \leq V(x,r) \leq C r^N, \qquad \text{for large } r > 0.$$

Therefore, while Li-Yau estimate on the heat kernel coincides with Aronsson's one locally, it is genuinely different at the global level.

Other parabolic differential Harnack inequalities were then found by Hamilton in [38] for *compact* Riemannian manifolds with $\mathrm{Ric} \geq 0$, and were later extended in [43, 60] to complete, non-compact manifolds. Actually, far more general differential Harnack inequalities are available under suitable conditions on the Riemannian manifold, see the book [52] for the history and applications of the latters.

Again, the differential Harnack inequality (3.8) requires a good deal of smoothness both on the operator and on the ambient manifold. Yet, the corresponding pointwise inequality (3.9) doesn't depend on the smoothness of the metric g_{ij} but only on its induced distance and the dimension, hence one is lead to think that a smoothness-free proof exists. Indeed, the papers [36, 56] showed that the parabolic Harnack inequality (and the corresponding heat kernel estimates) can still be obtained through a Moser-type approach based solely on the Doubling & Poincaré condition (2.9). In fact, [36, 56] independently proved the following *equivalence*.

Theorem 3.6 (Parabolic Harnack principle)**.** *For any Riemannian manifold the following are equivalent:*

(1) The parabolic Harnack inequality (3.1)*.*
(2) The heat kernel estimate (3.10)*.*
(3) The Doubling & Poincaré condition (2.9)*.*

Since Doubling & Poincaré are stable with respect to quasi-isometries, the previous theorem ensures the stability of the parabolic Harnack inequality with respect to the latters, and thus its validity in a much wider class of Riemannian manifolds than those with $\mathrm{Ric} \geq 0$. Condition *(3)* also ensures that the parabolic Harnack inequality holds for general parabolic equations with elliptic and merely measurable coefficients, see [57]. Actually, under local regularity conditions, it can be proved for *metric spaces* which are roughly isometric to a Riemannian manifold with $\mathrm{Ric} \geq 0$, such as suitable graphs or singular limits or Riemannian manifolds.

3.5. **The nonlinear setting.** A deep analysis of Moser's proofs reveals that the linearity of the second order operator is immaterial, and that essentially the same arguments apply as well to nonnegative weak solutions to a wide family of quasilinear equations. In [2, 62], the Harnack inequality in the form (3.4) was proved to hold for nonnegative solutions of

$$u_t = \mathrm{div} A(x, u, Du) \tag{3.11}$$

where the function $A : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is only assumed to be measurable and satisfying

$$\begin{cases} A(x, s, z) \cdot z \geq \lambda |z|^2, \\ |A(x, s, z)| \leq \Lambda |z|, \end{cases}$$

for some given positive constants λ and Λ. Trudinger noted that the Harnack inequality for the case of general p-growth conditions (2.6) with $p \neq 2$ seemed instead a difficult task. He stated the validity of the Harnack inequality (3.4) for positive solutions of the doubly nonlinear equation

$$(u^{p-1})_t = \mathrm{div} A(x, t, u, Du)$$

where A obeys (2.6) with the same p as the one appearing on the right hand side, thus recovering a form of homogeneity in the equation which is lacking in (3.11). The doubly nonlinear result has later been proved in [35, 41], but it took around forty years to obtain the right form of the Harnack inequality for solutions of (3.11) under the general p-growth condition (2.6) on the principal part. The next chapter will deal with the latter problem.

It is worth noting that another parabolic equation which presented the same kind of difficulties is the *porous medium equation*, namely

$$u_t = \Delta u^m, \qquad m > 0.$$

In fact, most of the results in the following sections have analogue statements and proofs for positive solutions of the porous medium equation. To keep things as simple as possible, we chose not to treat this equation, limiting the exposition to (3.11). The interested reader may consult the monographs [25, 66] for the corresponding results for porous media.

4. Singular and degenerate parabolic equations

4.1. **The prototype equation.** Let us consider the parabolic p-Laplace equation

$$u_t = \operatorname{div}(|Du|^{p-2}Du), \quad p > 1, \tag{4.1}$$

which can be seen as a parabolic elliptic equation with $|Du|^{p-2}$ as (intrinsic) isotropic coefficient. The coefficient vanishes near a point where $Du = 0$ when $p > 2$, while it blows up near such a point when $p < 2$. For this reasons we call (4.1) *degenerate* when $p > 2$ and *singular* if $p < 2$.

In the fifties, the seminal paper by Barenblatt was the starting point of the study of the p-Laplacian equation (4.1). In [3] Barenblatt found explicit solutions to (4.1), playing the rôle of the fundamental solution.

Theorem 4.1. *For any $p > \frac{2N}{N+1}$ and $M > 0$, there exist constants $a, b > 0$ depending only on N and p such that the function*

$$\mathcal{B}_{p,M}(x,t) := \begin{cases} t^{-\frac{N}{\lambda}}\left[aM^{\frac{p}{\lambda}\frac{p-2}{p-1}} - b\left(\frac{|x|}{t^{\frac{1}{\lambda}}}\right)^{\frac{p}{p-1}}\right]_+^{\frac{p-1}{p-2}}, & \text{if } p > 2, \\ t^{-\frac{N}{\lambda}}\left[aM^{\frac{p}{\lambda}\frac{p-2}{p-1}} + b\left(\frac{|x|}{t^{\frac{1}{\lambda}}}\right)^{\frac{p}{p-1}}\right]^{\frac{p-1}{p-2}} & \text{if } 2 > p, \end{cases} \tag{4.2}$$

where

$$\lambda = N(p-2) + p$$

solves the problem

$$\begin{cases} u_t = \operatorname{div}(|Du|^{p-2}Du) & \text{in } \mathbb{R}^N \times]0, +\infty[, \\ u(\cdot, t) \rightharpoonup^* M\delta_0 & \text{as } t \downarrow 0. \end{cases}$$

The functions $\mathcal{B}_{p,M}$ are also called *fundamental solution of mass M*, or simply fundamental solution when $M = 1$, in which case one briefly writes $\mathcal{B}_{p,1} = \mathcal{B}_p$. Uniqueness of the fundamental solution for the prototype equation was proved by Kamin and Vázquez in [40] (the uniqueness for general monotone operators is still not known).

The Barenblatt solutions show that when (4.1) is degenerate the diffusion is very slow and the speed of the propagation of the support is finite, while in the singular case the diffusion is very fast and the solution may become extinct in finite time. These two phenomena are incompatible with a parabolic Harnack inequality of the form (3.1) or (3.4), (suitably modified taking account of the natural scaling) such as

$$C^{-1} \sup_{B_\rho(x_0)} u(\cdot, t_0 - \rho^p) \le u(x_0, t_0) \le C \inf_{B_\rho(x_0)} u(\cdot, t_0 + \rho^p) \tag{4.3}$$

with a constant C depending only on N. Indeed, in the degenerate case the Barenblatt solution has compact support for any positive time, violating the strong minimum principle dictated by (4.3) (the proof of Corollary 3.2 still works). Regarding the singular case this incompatibility is not immediately apparent from the Barenblatt profile itself and in fact the strong minimum principle still holds for solutions defined in $\mathbb{R}^N \times]0, T[$ when $p > \frac{2N}{N+1}$. However, consider the solution of the Cauchy problem (4.1) in a cylindrical domain $\Omega \times \mathbb{R}_+$ with $u(0, x) = u_0 \in C_c^\infty(\Omega)$ and Dirichlet boundary condition on $\partial\Omega \times \mathbb{R}_+$, with Ω *bounded*. An elementary energetic argument (see [**?**, Ch VII]) gives a suitable *extinction time* $T^*(\Omega, u_0)$ such that $u(\cdot, t) \equiv 0$ for $t > T^*$, again violating the strong minimum principle.

Let us remark here that for $1 < p \le \frac{2N}{N+1} =: p_*$ the Barenblatt profile ceases to exists. As it will be widely discussed in the following, the exponent p_* plays a fundamental, and in some aspects still unclear, rôle in the whole theory of singular parabolic equations.

4.2. **Regularity.** Let us consider equations of the type

$$u_t = \mathrm{div} A(x, u, Du) \tag{4.4}$$

with general measurable coefficients obeying

$$\begin{cases} A(x, s, z) \cdot z \ge \Lambda^{-1} |z|^p, \\ |A(x, s, z)| \le \Lambda |z|^{p-1}. \end{cases} \tag{4.5}$$

We are concerned with weak solutions in $\Omega \times [0, T]$, namely those satisfying

$$\int u\varphi \, dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_\Omega [-u\varphi_t + A(x, u, Du) \cdot \varphi] \, dx \, dt = 0$$

where φ is an arbitrary function such that

$$\varphi \in W^{1,2}_{\mathrm{loc}}(0, T; L^2(\Omega)) \cap L^p(0, T; W^{1,p}_0(\Omega)).$$

This readily implies that

$$u \in C_{\mathrm{loc}}(0, T; L^2_{\mathrm{loc}}(\Omega)) \cap L^p_{\mathrm{loc}}(0, T; W^{1,p}_{\mathrm{loc}}(\Omega)).$$

In the case $p = 2$, the local Hölder continuity of solutions to (4.4) has been proved by Ladyzhenskaya and Ural'tseva in [**?**] through a parabolic De Giorgi approach. The case $p \neq 2$ was considered a major open problem in the theory of quasilinear parabolic equation for over two decades. The main obstacle to its solution was that the energy and logathmic estimates for (4.4) are non-homogeneous when $p \neq 2$. It was solved by DiBenedetto [**?**] in the degenerate case and Chen & DiBenedetto in [**?**, **?**] for the singular case through an approach nowadays called *method of instrinsic scaling*. Roughly speaking, in order to recover from the lack of homogeneity in the integral estimates one works in cylinders whose natural scaling is modified by the oscillation of the solution itself. In the original proof, these rescaled cylinder are then sectioned in smaller sub-cylinders and the so-called *alternative* occurs: either there exists a sub-cylinder where u is sizeably (in a measure-theoretic sense) away from its infimum or in each sub-cylinder it is sizeably away from its supremum. In both cases a reduction in oscillation can be proved, giving the claimed Hölder continuity.

Stemming from recent techniques built to deal with the Harnack inequality for (4.4), simpler proofs are nowadays available, avoiding the analysis of said alternative. The main idea is to use a measure-theoretical Lemma proved in [**?**] called *clustering of positivity* coupled with the *expansion of positivity* effect. In the last section we will provide such a simplified proof.

As it turned out, Hölder continuity of *bounded* solutions to (4.5) (in fact, to much more general equations) always holds. In the degenerate case $p \ge 2$, a-priori boundedness follows from the natural notion of weak solution given above, but in the singular case there is a precise threshold: local boundedness is guaranteed only for $p > p_{**} := \frac{2N}{N+2}$, which is therefore another critical exponent for the singular equation, smaller than p_*. However, when $1 < p < p_{**}$, weak solutions may be unbounded: for example, a suitable multiple of

$$v(x, t) = \frac{(T - t)_+^{\frac{1}{2-p}}}{|x|^{\frac{p}{2-p}}}$$

solves the model equation (4.1) in the whole $\mathbb{R}^N \times \mathbb{R}$.

The critical exponents $p_* > p_{**}$ arise from the so-called $L^r - L^\infty$-estimates for sub-solutions, which are parabolic analogues of (2.7). Namely, when $p > p^*$, a $L^1 - L^\infty$ estimate holds true, eventually giving the intrinsic parabolic Harnack inequality. If only $p > p_{**}$ is assumed, one can still obtain a weaker $L^r - L^\infty$ estimate with $r > 1$ being the optimal exponent in the parabolic embedding

$$L^\infty(0,T; L^2(B_R)) \cap L^p(0,T; W^{1,p}(B_R)) \hookrightarrow L^r(0,T; L^r(B_R)), \qquad r = p\frac{N+2}{N}$$

which is ensured by the notion of weak solution.

4.3. **Intrinsic Harnack inequalities.** DiBenedetto and DiBenedetto & Kwong in [16] and [26] found and proved the suitable form of the parabolic Harnack inequality for the prototype equation (4.1). Comparison theorems where essential tools of the proof. A similar statement was later proved to hold for general parabolic quasilinear equations of p-growth in [], thus avoiding any use of the comparison principle (which may not hold under the sole assumption (2.6)).

Hdeg **Theorem 4.2** (Intrinsic Harnack inequality, degenerate case)**.** *Let $p \geq 2$ and u be a positive weak solution of*

$$u_t = \mathrm{div} A(x, u, Du)$$

in $\Omega \times [0,T]$ where A satisfies (4.5). Suppose

$$\theta_u = u^{2-p}(x_0, t_0) > 0$$

for some $(x_0, t_0) \in \Omega\times]0,T[$. Then, there exists $C = C(N,p,\Lambda) > 0$ and $\varepsilon = \varepsilon(N,p,\Lambda) > 0$ such that if

condn
$$B_{2\rho}(x_0) \times [t_0 - \varepsilon\,\theta_u\,(2\rho)^p, t_0 + \varepsilon\,\theta_u\,(2\rho)^p] \subseteq \Omega\times]0,T[, \tag{4.6}$$

then

pharnack
$$C^{-1} \sup_{B_\rho(x_0)} u(\cdot, t_0 - \varepsilon\,\theta_u\,\rho^p) \leq u(x_0,t_0) \leq C \inf_{B_\rho(x_0)} u(\cdot, t_0 + \varepsilon\,\theta_u\,\rho^p). \tag{4.7}$$

Clearly, for $p = 2$ we recover (3.1). For $p > 2$, the waiting time is larger the smaller $u(x_0,t_0)$ which means that $u(x_0,t_0)$ bounds from below u on thinner and thinner "p-paraboloids" the smaller $u(x_0,t_0)$ is.

hsing **Theorem 4.3** (Intrinsic Harnack inequality, singular supercritical case)**.** *Let $2 > p > \frac{2N}{N+1}$ and u be a positive weak solution of*

$$u_t = \mathrm{div} A(x, u, Du)$$

in $\Omega \times [0,T]$ where A satisfies (4.5). Suppose

$$\theta_u = u^{2-p}(x_0, t_0) > 0$$

for some (x_0,t_0) and let

$$\mathcal{M} = \sup_{B_\rho(x_0)\times\{t_0\}} u^{2-p}.$$

Then, there exists $C = C(N,p,\Lambda) > 0$ and $\varepsilon = \varepsilon(N,p,\Lambda) > 0$ such that if

condM
$$B_{2\rho} \times [\mathcal{M}\,(2\rho)^p, \mathcal{M}\,(2\rho)^p] \subseteq \Omega\times]0,T[, \tag{4.8}$$

then

fbh
$$C^{-1} \sup_{B_\rho(x_0)} u(\cdot, t_0 - \varepsilon\,\theta_u\,\rho^p) \leq u(x_0,t_0) \leq C \inf_{B_\rho(x_0)} u(\cdot, t_0 + \varepsilon\,\theta_u\,\rho^p). \tag{4.9}$$

We thus see that in the singular case, $u(x_0,t_0)$ bounds from below u in wider and wider paraboloids the smaller it is. The main issue in the singular supercritical case lies in the qualitative requirement (4.8), which prevents a direct application of the Harnack inequality to sub-potential lower bounds which we'll discuss later. A form of the Harnack inequality only requiring the natural condition (4.6) is much desirable, but it is not known under the sole structural conditions (4.5). If a monotonicity assumption holds, one actually has the optimal statement.

Theorem 4.4 (Intrinsic Harnack inequality, singular supercritical monotone case)**.** *Suppose A satisfies (4.5) and the monotonicty assumption*

$$(A(x,s,z)-A(x,s,w))\cdot(z-w)\geq 0 \quad \forall z,w\in\mathbb{R}^N. \tag{4.10}$$

Then, Theorem 4.2 holds for all $p>\frac{2N}{N+1}$.

In the singular supercritical case another, unexpected form of the Harnack inequality holds, which is essentialy different from (4.7).

Theorem 4.5 (Elliptic Harnack inequality for singular supercritical parabolic equations)**.** *Let the assumptions of Theorem 4.3 hold. Then there exists $C=C(N,p,\Lambda)>0$ and $\varepsilon=\varepsilon(N,p,\Lambda)$ such that*

$$\sup_{Q_{\rho,\theta_u}(x_0,t_0)} u \leq C \inf_{Q_{\rho,\theta_u}(x_0,t_0)} u, \tag{4.11}$$

where $Q_{\rho,\theta_u}(x_0,t_0)$ is the forward-backward intrinsic cylinder

$$Q_{\rho,\theta_u}(x_0,t_0)=B_\rho(x_0)\times\{|t-t_0|\leq\varepsilon\,\theta_u\,\rho^p\}.$$

Moreover, if (4.10) is also assumed, the claim holds under assumption (4.6) instead of (4.8).

Recall that an elliptic form of the Harnack inequality such as (4.11) does not hold for the classical heat equation. This forces the constants appearing in the previous theorem to blow-up as $p\uparrow 2$. The previous examples also show that they must blow-up also for $p\downarrow p_*$.

In the critical and subcritical case $1<p\leq p_*$ a pointwise Harnack inequality such as (4.7) cannot hold. This is not only due to the boundedness issues described in the previous section when $p<p_{**}$, as for example

$$u(x,t)=(T-t)_+^{\frac{N+2}{2}}\left(a+b|x|^{\frac{2N}{N-2}}\right)^{-\frac{N}{2}} \tag{4.12}$$

is a bounded solution in $\mathbb{R}^N\times\mathbb{R}$ of the prototype equation (4.1) for any $p\in\,]1,p_*[$, $N>2$ and suitably chosen $a,b>0$, violating (4.7). To see this, note that for bounded solutions, the opening of the paraboloids derived from (4.7) where u is controlled by $u(x_0,t_0)$ is bounded from below, thus the proof of the strong minimum principle Corollary 3.2 carries over. Since the previous function violates the latter, (4.1) cannot hold. In passing, this argument also shows that, despite extinction in finite time always holds for singular equations in *bounded* domains, it never holds for *spatially entire* solutions of (3.11). In the critical case

$$u(x,t)=\left(e^{ct}+|x|^{\frac{2N}{N-1}}\right)^{-\frac{N-1}{2}} \tag{4.13}$$

again solves (4.1) in $\mathbb{R}^N\times\mathbb{R}$ when $p=p_*$, $N\geq 2$ and suitable $c>0$ and a direct computation. It doesn't violate the stong minimum principle, however notice that (4.7) implies a weak form of monotonicity for space-time entire nonnegative solutions, namely $u(x,t)\geq Cu(x,s)$ for all $t\geq s$, which is violated by (4.13).

In the subcritical case, different forms of the Harnack inequality have been considered. Here we mention the one obtained in [?] generalizing to monotone operators a result of Bonforte and Vazquez [8], [9] on the prototype equation.

Theorem 4.6 (Subcritical case)**.** *Let $1<p<2$ and u be a locally bounded positive weak solution of*

$$u_t = \mathrm{div} A(x,u,Du)$$

in $\Omega\times[0,T]$ where A satisfies the general growth condition (4.5) and the monotonicty assumption (4.10). For $(x_0,t_0)\in\Omega\times[0,T]$ and for any fixed $r\geq 1$ such that $\lambda_r := N(p-2)+pr>0$ set

$$\theta_u = \left(\fint_{B_\rho(x_0)} u(x,t_0)\,dx\right)^{2-p}, \qquad A_u = \left[\frac{\fint_{B_\rho(x_0)} u(x,t_0)\,dx}{\left(\fint_{B_\rho(x_0)} u^r(x_0,t_0)\,dx\right)^{\frac{1}{r}}}\right]^{\frac{pr}{\lambda_r}}.$$

There exists constants $C(N,p,\Lambda)$, $\varepsilon=\varepsilon(N,p,\Lambda)$ and $\delta=\delta(N,p,\Lambda)>0$ such that if

$$B_{4\rho}(x_0)\times[t_0,t_0+\varepsilon\,\theta(4\rho)^p]\subseteq\Omega\times\,]0,T[\,,$$

then

$$\sup_{\widetilde{Q}_{\rho,\theta_u}(x_0,t_0)} u \leq C A_u^{\delta} \inf_{\widetilde{Q}_{\rho,\theta_u}(x_0,t_0)} u \tag{4.14}$$

where $\widetilde{Q}_{\rho,\theta_u}(x_0,t_0)$ is the space-time cylinder

$$\widetilde{Q}_{\rho,\theta_u}(x_0,t_0) := B_{\rho/2}\times[t_0+\varepsilon\,\theta_u\rho^p, t_0+\varepsilon\,\theta_u(2\rho)^p].$$

Notice that in the singular supercritical case one can take $r=1$ and thus $A_u\equiv 1$ in the previous statement to recover Theorem 4.4. The main point of the (elliptic) harnack inequality (4.14) lies in the dependence of the constant from the solution itself. In general, a constant depending on u won't allow to deduce Hölder continuity, but as noted in [**?**] the peculiar structure of A_u permits such a deduction.

Other weaker forms not requiring the monotonicity assumption (4.10) are available, (see [23] or [**?**][Theorem 11.1]), however the picture is not completely clear up to now.

4.4. **Liouville theorems.** As for the classical heat equation, a one sided bound is not sufficient to ensure triviality of the solutions of the prototype equation (4.1). Indeed, a suitable positive multiple of the function

$$u(x,t) = (1-x+ct)_+^{\frac{p-1}{p-2}}$$

solves (4.1) on $\mathbb{R}\times\mathbb{R}$ whenever $c>0$ and $p>2$.

Theorem 4.7. *Let $p>2$ and u be a non-negative solution of*

$$u_t = \mathrm{div}(A(x,u,Du)) \qquad \text{on } \mathbb{R}^N\times\,]-\infty,T[\tag{4.15}$$

under the growth condition (4.5). If for some $t_0<T$, $u(\cdot,t_0)$ is bounded above, then u is constant.

An optimal Liouville condition such as the one of Theorem 3.4 is unknown and clearly the previous example shows that it must involve polynomial growth condition instead of a sub-exponential one. For the prototype equation, a polynomial growth condition on both x and t more in the spirit of [60] is considered in [**?**].

On the complementary side, boundedness for *fixed* x_0 can also be considered, yielding:

Theorem 4.8. *Let $p>2$ and u be a nonnegative solution in $\mathbb{R}^N\times\mathbb{R}$ of (4.15) under condition (4.5). If*

$$\limsup_{t\to+\infty} u(x_0,t) < +\infty \qquad \text{for some } x_0\in\mathbb{R}^N,$$

u is constant.

In the singular, supercritical case, the elliptic form (4.11) of the Harnack inequality directly ensures that, contrary to what happens for classical heat equation, a one-sided bound suffices to obtain a Liouville theorem. This is no longer true in the critical and subcritical case, as the functions (4.13) and (4.12) show. However, a two sided bound suffices.

Theorem 4.9. *Let $1 < p < 2$ and u be a weak solution on $\mathbb{R}^N \times]-\infty, T[$ of (4.15) under condition (4.5). If u is bounded below and above, it is constant.*

4.5. **Harnack estimates at large.** By Harnack estimates at large, we mean *global* results such as the sub-potential lower bound (3.7) or the two-sided Kernel estimate (3.6). For the quasilinear equation

$$u_t = \operatorname{div} A(x, u, Du) \tag{4.16}$$

with p-growth assumptions (4.5), the natural candidates to state analogous inequalities are the Barenblatt profiles $\mathcal{B}_{p,M}$ given in (4.2). When A satisfies smoothness and monotonicity assumptions such as

$$\begin{cases} (A(x,s,z) - A(x,s,w)) \cdot (z-w) \geq 0 & \forall s \in \mathbb{R},\ x, z, w \in \mathbb{R}^N, \\ |A(x,s,z) - A(x,r,z)| \leq \Lambda(1+|z|)^{p-1}|s-r| & \forall s, r \in \mathbb{R},\ x, z \in \mathbb{R}^N. \end{cases} \tag{4.17}$$

Such set of assumptions imply a comparison principle for weak solutions and guarantee the existence of the solution of a Cauchy problem with L^1 initial datum.

We start by considering the singular supercritical case, since the diffusion is fast and positivity spreads instantly on the whole $\mathbb{R}^N$, giving a behaviour similar to the one of the heat equation.

Theorem 4.10 ([10]). *Let $\frac{2N}{N+1} < p < 2$ and u be a nonnegative solution of (4.16) in $\mathbb{R}^N \times]0, +\infty[$ under assumptions (4.5), (4.17). There is $C = C(N, p, \Lambda) > 0$, $\delta = \delta(N, p, \Lambda) > 0$ such that if $u(x_0, t_0) > 0$, then*

$$u(x,t) \geq \gamma u(x_0,t_0) \mathcal{B}_p \left(u(x_0,t_0)^{\frac{p-2}{p}} \frac{(x-x_0)}{t_0^{\frac{1}{p}}}, \frac{t}{t_0} \right), \qquad (x,t) \in \mathbb{R}^N \times [t_0(1-\delta), +\infty[. \tag{4.18}$$

As an example, assume $x_0 = 0$, $t_0 = 1$ and $u(P_0) = 1$. Then, the previous sub-potential lower bound becomes

$$u(x,t) \geq \gamma \mathcal{B}_p(x,t)$$

for any $(x,t) \in \mathbb{R}^N \times [1-\delta, \infty[$. Such a result is obviously sharp, because it is sharp for the Barenblatt solution. As a corollary, for any fundamental solution of (4.16), one obtains the two-sided kernel bounds (proved in [55] for the first time)

$$C^{-1} \mathcal{B}_{p,M_1}(x,t) \leq \Gamma(x,t) \leq C \mathcal{B}_{p,M_2}(x,t)$$

for some $C, M_1, M_2 > 0$ depending on the data.

Notice how the elliptic nature of (4.16) for $p \in]p_*, 2[$, as expressed by the forward-backward Harnack inequality (4.9), allows to obtain the bound (4.18) also for some $t < t_0$. As shown in [**?**], this phenomenon not only happens for past times near t_0 but, as long as the spatial diffusion has had enough room to happen, it also hold for arbitrarily remote past times. More precisely, in [**?**] it is proved that (4.18) holds for all

$$(x,t) \in \mathcal{P}^c := \left\{ t > 0, |x - x_0|^p u(x_0, t_0)^{2-p} > 1 - \frac{t}{t_0} \right\},$$

while a weaker, but still optimal, lower bound holds in $\mathcal{P}$.

In the degenerate case $p > 2$, the finite speed of propagation implies that if the initial datum u_0 has compact support, then any solution of (4.16) keeps having compact support for any time $t > 0$.

The finite speed of propagation has been quantified in [**?**], under the sôle p-growth assumption (4.5).

Theorem 4.11. *Let $p > 2$ and u be a weak solution of the Cauchy problem*

$$\begin{cases} u_t = \operatorname{div} A(x, u, Du) & \text{in } \mathbb{R}^N \times]0, +\infty[, \\ u(x, 0) = u_0 \end{cases}$$

under assumption (4.5). If $R_0 = \operatorname{diam}(\operatorname{supp} u_0) < +\infty$, then

$$\operatorname{diam}(\operatorname{supp} u(\cdot, t)) \leq 2R_0 + Ct^{1/\lambda} \|u_0\|_{L^1(\mathbb{R}^N)}^{\frac{p-2}{\lambda}},$$

where $\lambda = N(p-2) + p$ and $C = C(N, p, \Lambda)$.

Such an estimate actually holds for a suitable class of degenerate systems, see [**?**].

Theorem 4.12. *Let $p > 2$ and u be a nonnegative solution of (4.16) in $\mathbb{R}^N \times]0, +\infty[$ under assumptions (4.5), (4.17). There is $C = C(N, p, \Lambda) > 0$, $\varepsilon = \delta(N, p, \Lambda) > 0$ such that if $u(x_0, t_0) > 0$, then (4.18) holds for*

$$t > t_0, \qquad |x - x_0|^p \leq \varepsilon u(x_0, t_0)^{p-2} t_0 \min \left\{ \frac{t - t_0}{t_0}, \left(\frac{t - t_0}{t_0} \right)^{p/\lambda} \right) ,$$

with $\lambda = N(p-2) + p$.

The last condition on the region of validity of (4.18) is sharp, especially when $t \simeq t_0$ and the minimum is the first one (see [**?**, Remark 1.3] for details).

Again, the sub-potential lower bound implies the two-sided Kernel estimate.

Theorem 4.13. *Let $p > 2$. Under assumption (4.5) and (4.17), equation (4.16) possesses at least a fundamental solution, and any fundamental solution satisfies (**??**) for suitable M_1, M_2, C depending only on N, p and Λ.*

5. Expansion of positivity approach to regularity

5.1. **Elliptic regularity.** An alternative and self contained proof of the Hölder continuity of locally bounded solutions to some elliptic equations, including the equation associated with the $p-$Laplacian operator, was given in [30]. By combining the techniques introduced by De Giorgi and Moser, this method can be seen more geometric and relatively intuitive.

It is known that Poincaré estimates are necessary to obtain the Harnack inequality and regularity estimates. Since the approach here uses only a $1-$dimensional Poincaré inequality, i.e., Poincaré inequalities direction by direction, it could be useful for anisotropic as well as non-homogeneous operators, such as *Hörmander operators* and the *subelliptic Laplacians*.

Assume that u is a locally bounded weak solution of the following equation:

equation (5.1)
$$\operatorname{div} A_p (x, u, Du) = 0 \,,$$

where A_p is Carathéodory vector field defined in $\Omega \times \mathbb{R} \times \mathbb{R}^n$; i.e., $A_p (x, s, \xi)$ is measurable with respect to $x \in \Omega$ and continuous in $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$ for almost every $x \in \Omega$. We assume that A_p satisfies the following structural conditions (*"natural growth conditions"*) for an exponent $p > 1$ and for some positive constants m, M

_conditions (5.2)
$$\begin{cases} A_p (x, s, \xi) \, \xi \geq m \, |\xi|^p \\ |A_p (x, s, \xi)| \leq M \, |\xi|^{p-1} \end{cases} .$$

Theorem 5.1. *Let us assume that the growth conditions (5.2) hold. Let u be a locally bounded weak solution of (5.1). Then u is locally Hölder continuous in Ω and there exist two constants α, c depending only upon the data, with $0<\alpha<1$ and $c>0$, such that, for every open set Ω' whose closure is contained in Ω and for every $x_1, x_2 \in \Omega'$,*

$$|u(x_1)-u(x_2)| \le c\,\|u\|_{L^\infty(\Omega')}\left(\frac{|x_1-x_2|}{\operatorname{dist}(\Omega',\partial\Omega)}\right)^{\alpha}. \tag{5.3}$$

Here the constant depends only upon n, m, M, p.

Let $R>r>0$ and $B_r(x_0)=\{x\in\mathbb{R}^n : |x-x_o|<r\}$ is a ball of center $x_0\in\Omega$, and radius r and $B_R(x_0)$ is the similar ball of radius $R>r$. Assume that the closure of $B_R(x_0)$ is contained in Ω.

Moreover, let $\mu_-=\mu_-(R)$ (resp. $\mu_+=\mu_+(R)$) the *essential infimum* (resp. the *essential supremum*) of u in $B_R(x_0)$ and $\omega=\omega(R):=\mu_+-\mu_-$ the *oscillation* of u in $B_R(x_0)$.

To get the result in Theorem 5.1, here are some auxilry lemmas:

Lemma 5.2 (Logarithmic lemma). *Let u be a locally bounded weak solution of equation (5.1) and assume that the growth conditions (5.2) are satisfied. Then there exists a constant c_1 (depending only upon the data) such that*

$$\int\limits_{B_r(x_0)}\left|D\log_+\frac{H\omega}{u-\mu_-+a\omega H}\right|^p dx \le c_1\int\limits_{B_R(x_0)}|D\varphi|^p\,dx\,. \tag{5.4}$$

Here $\log_+(\cdot)=[\log(\cdot)]_+$ is the positive part of $\log(\cdot)$; moreover $r<R$ and $\varphi\in C^1$ is a nonnegative test function such that $\varphi=1$ in $B_r(x_0)$, $\varphi=0$ in $\Omega-B_R(x_0)$ and a, H with $0<a<1$, $0<H<1$ are given numbers.

Lemma 5.3 (Caccioppoli's type inequality). *Let u be a locally bounded weak solution of (5.1) and assume that the growth conditions (5.2) are satisfied. Then there exists a constant c_2 (depending only upon the data) such that*

$$\int_\Omega \varphi^p\left|D(u-k)_-\right|^p dx \le c_2\int_\Omega |D\varphi|^p\left|(u-k)_-\right|^p dx$$

for every constant k and for every test function $\varphi\in C_0^1(\Omega)$, $\varphi\ge 0$. An analogous result holds if we deal with the positive part $(u-k)_+$.

Lemma 5.4 (Algebraic lemma). *Let $\{Y_j\}$, $j=0,1,2,\dots,$ be a sequence of positive numbers satisfying the recursive inequalities*

$$Y_{j+1}\le c\,b^j\,Y_j^{1+\beta}\,,$$

for some positive constants c and β and some $b>1$. If $Y_0\le c^{-1/\beta}b^{-1/\beta^2}$, then $Y_j\le Y_0\,b^{-j/\beta}$. Thus, in particular, $\{Y_j\}$ converges to zero as $j\to\infty$.

Now let introduce some notations which will be used in the next lemma. Let $R_0, k_0>0$ be given real numbers and

$$R_j=\frac{R_0}{2}+\frac{R_0}{2^{j+1}}. \tag{5.5}$$

We assume that the closure of the ball $B_{R_0}(x_0)$ is contained in Ω. Let us define

$$k_j=\frac{k_0}{2}+\frac{k_0}{2^{j+1}}+\mu_-,\qquad A_j=\{x\in B_{R_j}(x_0): u(x)\le k_j\}, \tag{5.6}$$

where, as above, μ_- is the essential infimum of u in $B_{R_0}(x_0)$. We also denote by $R_\infty = \frac{R_0}{2}$ and by A_∞ the intersection of the sets A_j ; i.e.,

$$A_\infty = \underset{j}{\cap} A_j = \{x \in B_{R_\infty}(x_0) : \ u(x) \leq \frac{k_0}{2} + \mu_-\}.$$

Next lemma concerns with the sequence $\{Y_j\}$, $j = 0, 1, 2, \ldots$, of real numbers defined by

$$Y_j = \frac{|A_j|}{\left|B_{R_j}(x_0)\right|} \tag{5.7}$$

where $|\cdot|$ denotes the Lebesgue *measure* of the corresponding set.

Lemma 5.5 (De Giorgi type lemma)**.** *There is a number $\nu > 0$, depending only upon the data (and not depending on u, R_0, and k_0), such that, if $Y_0 < \nu$, then the sequence $\{Y_j\}$ in (5.7) converges to zero as j goes to infinity.*

Finally the following lemma has a very important part for the proof of the main theorem. Its proof can be found in [**?**]. In [**?**] the result is proved using the $n-$dimensional Poincaré inequality. In the appendix of [28] the same result is proved by induction, using time by time the $1-$dimensional Poincaré inequality to increase the space dimension.

Lemma 5.6 (Measure theoretical lemma)**.** *Let B_r be a ball in $\mathbb{R}^n$ and let $u \in W^{1,1}(B_r)$ be a nonnegative function that satisfies*

$$\|u\|_{W^{1,1}(B_r)} \leq \gamma\chi\, r^{n-1}, \qquad |\{x \in B_r : u(x) \geq \chi\}| \geq \alpha\, |B_r|\,,$$

for some $\gamma, \chi > 0$ and $\alpha \in (0,1)$. Then for any $\delta \in (0,1)$ and $0 < \beta < 1$ there exist $x_0 \in B_r$ and $\eta = \eta(\alpha, \delta, \gamma, \beta, n) \in (0,1)$ such that

$$|\{x \in B_{\eta r}(x_0) : u(x) \geq \beta\chi\}| \geq (1-\delta)\,|B_{\eta r}|\,.$$

Proof of Theorem 5.1

Let give a simplified proof of this known algebraic lemma, from which the Hölder continuity estimate (5.3) will follow.

Lemma 5.7 (Criterion for Hölder continuity)**.** *For $r \leq R$ let $\omega(r) = \mu_+(r) - \mu_-(r)$ be the oscillation of $u(x)$ in a ball $B_r(x_0)$. Assume that $\overline{B}_R(x_0) \subset \Omega$.*

If

$$\omega(ar) \leq b\,\omega(r)\,, \qquad \forall\, r \leq R, \tag{5.8}$$

for some real numbers $a, b \in (0,1)$, then for $\alpha := \frac{\log b}{\log a}$ we have

$$\omega(r) \leq \frac{\omega(R)}{b}\left(\frac{r}{R}\right)^\alpha, \quad \forall\, r \leq R. \tag{5.9}$$

Remark 5.8. *The Hölder estimate in (5.9) also gives*

$$\omega(r) \leq \frac{2}{b}\,\|u\|_{L^\infty(B_R(x_0))}\left(\frac{r}{R}\right)^\alpha, \quad \forall\, r \leq R,$$

which implies (5.3). Note in particular that $\alpha = \frac{\log b}{\log a} > 0$ and that $\alpha \in (0,1)$ if $0 < a < b < 1$. Lemma 5.7 will be applied at the end of this section, precisely by choosing $a = 1/4$ and $b = 1 - \frac{1}{16e^s}$, s is a positive real parameter to be fixed later.

Let fix $x_0 \in \Omega$ and $R > 0$. In order to avoid cumbersome notations below, set $r = 4R$ and assume that the closure of the ball $B_{4R}(x_0)$ is contained in Ω.

Let consider the *oscillation* $\omega(4R) := \omega$ of $u(x)$ in the ball $B_{4R}(x_0)$:

$$\begin{aligned} \omega &= \sup\{u(x) : x \in B_{4R}(x_0)\} - \inf\{u(x) : x \in B_{4R}(x_0)\} \\ &= \mu_+(4R) - \mu_-(4R) := \mu_+ - \mu_- . \end{aligned} \tag{5.10}$$

The aim is to prove that the oscillation of u is reduced by a fixed quantity (a fixed factor) in the ball $B_R(x_0)$. To obtain this fact one can observe that there are two possibilities (both being possible at the same time): either

$$\left|\left\{x \in B_R(x_0) : u(x) \geq \mu_- + \frac{1}{2}\omega\right\}\right| \geq \frac{1}{2}|B_R(x_0)| \tag{5.11}$$

or

$$\left|\left\{x \in B_R(x_0) : u(x) \leq \mu_- + \frac{1}{2}\omega\right\}\right| \geq \frac{1}{2}|B_R(x_0)| . \tag{5.12}$$

Let assume that the first alternative occurs (the second one can be studied analogously). We know that $u(x) \geq \mu_-$. The first goal is to find a small ball inside $B_R(x_0)$ where $u(x) \geq \mu_- + \frac{1}{8}\omega$ almost everywhere in the ball, which is called *clustering the positivity.*

Lemma 5.9 (Clustering the positivity)**.** *If (5.11) holds, then there exist a constant $\varepsilon_0 \in (0,1)$ (depending only upon the data) and a point $x_1 \in B_R(x_0)$ such that $u(x) > \mu_- + \frac{1}{8}\omega$ for almost every $x \in B_{\varepsilon_0 R}(x_1)$.*

Proof. In order to apply Lemma 5.6 its needed to estimate the $W^{1,1}$ norm of u in $B_R(x_0)$. To this aim, the Caccioppoli inequality of Lemma 5.3 with $k = \mu_-$ is used and let consider a test function $\varphi \in C_0^1(B_{2R}(x_0))$ such that $\varphi = 1$ in $B_R(x_0)$, $\varphi \geq 0$, $|D\varphi| \leq 2/R$ in $B_{2R}(x_0)$, $\varphi = 0$ outside $B_{2R}(x_0)$. Then,

$$\begin{aligned} \int_{B_R(x_0)} |Du|\, dx &\leq \left(\int_{B_R(x_0)} |Du|^p\, dx\right)^{\frac{1}{p}} |B_R(x_0)|^{\frac{p-1}{p}} \\ &\leq c_{10}\left(\int_{B_{2R}(x_0)} |D\varphi|^p\, |(u-\mu_-)_+|^p\, dx\right)^{\frac{1}{p}} |B_R(x_0)|^{\frac{p-1}{p}} . \end{aligned}$$

Since $\mu_- \leq u \leq \mu_- + \omega$,

$$\begin{aligned} \int_{B_R(x_0)} |Du|\, dx &\leq 2c_{10}\omega\left(\int_{B_{2R}(x_0)} |D\varphi|^p\, dx\right)^{\frac{1}{p}} |B_R(x_0)|^{\frac{p-1}{p}} \\ &\leq \frac{4c_{10}}{R}\omega\,|B_R(x_0)| = c_{11}\omega\, R^{n-1}. \end{aligned}$$

Thus Lemma 5.6 to the function $u - \mu_-$ with $\beta = \frac{1}{2}$, $\chi = \frac{1}{2}\omega$ and $\delta \in (0,1)$ to be chosen can be applied. Then the existence of $x_1 \in B_R(x_0)$ and $\eta \in (0,1)$ is obtained such that

$$\left|\left\{x \in B_{\eta R}(x_1) : u(x) > \mu_- + \tfrac{1}{4}\omega\right\}\right| \geq (1-\delta)\,|B_{\eta R}(x_1)| ,$$

(note the strict inequality $u(x) > \mu_- + \frac{1}{4}\omega$, possible by changing $\beta = \frac{1}{2}$ with $\beta = \frac{1}{2} + \varepsilon$) that is equivalent to

$$\left|\left\{x \in B_{\eta R}(x_1) : u(x) < \mu_- + \tfrac{1}{4}\omega\right\}\right| < \delta\,|B_{\eta R}(x_1)| .$$

Let denote by $R_0 = \eta R$ and let define $R_j = \frac{R_0}{2} + \frac{R_0}{2^{j+1}}$ as in (5.5) and k_j, A_j, Y_j as in (5.6), (5.7). Then, if by choosing $\delta = \nu$, where ν is the constant defined in Lemma 5.5 (note that ν does not depend on $R_0 = \eta R$), it is obtained that

$$Y_0 = \frac{|A_0|}{|B_{R_0}(x_1)|} = \frac{\left|\left\{x \in B_{\eta R}(x_1) : u(x) < \mu_- + \frac{1}{4}\omega\right\}\right|}{|B_{\eta R}(x_1)|} < \delta = \nu.$$

Thus, by applying Lemma 5.5,

$$0 = \lim_{j\to+\infty} Y_j = \frac{|A_\infty|}{|B_{R_\infty}(x_1)|} = \frac{|\cap_j A_j|}{|B_{R_\infty}(x_1)|} = \frac{\left|\left\{x \in B_{\frac{\eta R}{2}}(x_1) : u(x) \le \mu_- + \frac{1}{8}\omega\right\}\right|}{\left|B_{\frac{\eta R}{2}}(x_1)\right|}.$$

Therefore $u(x) > \mu_- + \frac{1}{8}\omega$ for almost every $x \in B_{\frac{\eta R}{2}}(x_1)$, which corresponds to the conclusion with $\varepsilon_0 = \frac{\eta}{2}$. □

Let x_1 be the point in the statement of Lemma 5.9 and let use the notation $x_1 = (\overline{x}, \overline{y}) \in \mathbb{R}^n$, with $\overline{x} \in \mathbb{R}$ and $\overline{y} \in \mathbb{R}^{n-1}$. Let denote by $B_{n-1,R}(\overline{y})$ a ball in $\mathbb{R}^{n-1}$ of center in $\overline{y}$ and radius $R > 0$; i.e., $B_{n-1,R}(\overline{y}) = \{\overline{y} \in \mathbb{R}^{n-1} : |y - \overline{y}| < R\}$. Given a radius $R > 0$ and a real number $L > 0$ let define the cylinder $C_{R,L}(x_1)$ in $\Omega \subset \mathbb{R}^n$

$$\begin{aligned} C_{R,L}(x_1) &= \left\{(x,y) \in \mathbb{R} \times \mathbb{R}^{n-1} : |x - \overline{x}| < RL,\ y \in B_{n-1,R}(\overline{y})\right\} \\ &= (\overline{x} - RL, \overline{x} + RL) \times B_{n-1,R}(\overline{y}). \end{aligned} \tag{5.13}$$

Note that $C_{\varepsilon_0 R,3}(x_1)$ is contained in the original ball $B_{4R}(x_0)$.

Let recall the radius $\varepsilon_0 R$ of the ball $B_{\varepsilon_0 R}(x_1) \subset B_R(x_0)$ in the statement of Lemma 5.9. By using the half radius $\varepsilon_0 R/2$, let define a set $D_{n-1,s} \subset \mathbb{R}^{n-1}$, depending on a positive real parameter s to be fixed later,

$$D_{n-1,s} = \{y \in B_{n-1,\varepsilon_0 R/2}(\overline{y}) : \exists\, x \in \left[\overline{x} - \tfrac{5}{2}R, \overline{x} + \tfrac{5}{2}R\right],\ u(x,y) \le \mu_- + \frac{1}{8e^s}\omega\}. \tag{5.14}$$

Lemma 5.10 (Expansion of positivity)**.** *For every positive constant $\nu_0 \in (0,1)$ there exists a $s \in \mathbb{R}^+$ such that*

$$|D_{n-1,s}| \le \nu_0 \left|B_{n-1,\varepsilon_0 R/2}(\overline{y})\right|.$$

Proof. This result is proved in the cylinder $[\overline{x}, \overline{x} + \frac{5}{2}R] \times B_{n-1,\varepsilon_0 R/2}(\overline{y})$, the case $[\overline{x} - \frac{5}{2}R, \overline{x}] \times B_{n-1,\varepsilon_0 R/2}(\overline{y})$ being analogous.

Let $y \in D_{n-1,s}$. Then, by the definition of D in (5.14), there exists $x \in [\overline{x}, \overline{x} + \frac{5}{2}R]$ such that $u(x,y) \le \mu_- + \frac{1}{8e^s}\omega$. By Lemma 5.9, $u(\overline{x}, y) > \mu_- + \frac{1}{8}\omega$ for every $y \in B_{n-1,\varepsilon_0 R/2}(\overline{y})$. Therefore, for this x value,

$$\frac{u(\overline{x},y) - \mu_- + \frac{1}{8e^s}\omega}{u(x,y) - \mu_- + \frac{1}{8e^s}\omega} \ge \frac{\frac{1}{8}\omega + \frac{1}{8e^s}\omega}{2\frac{1}{8e^s}\omega} = \frac{e^s + 1}{2}, \qquad \text{if } s \ge \log 3.$$

Thus if $s \ge \log 3$, the logaritmic function coincides with the $\log_+(\cdot)$ function for both sides of the above inequality. For such s values we obtain

$$s - 1 \le \log \frac{e^s + 1}{2} \le \log_+ \frac{u(\overline{x},y) - \mu_- + \frac{1}{8e^s}\omega}{u(x,y) - \mu_- + \frac{1}{8e^s}\omega}$$

$$= \log_+ \frac{\frac{1}{8}\omega}{u(x,y) - \mu_- + \frac{1}{8e^s}\omega} - \log_+ \frac{\frac{1}{8}\omega}{u(\overline{x},y) - \mu_- + \frac{1}{8e^s}\omega}$$

$$= \int_{\overline{x}}^{x} \frac{d}{dt} \log_+ \frac{\frac{1}{8}\omega}{u(t,y) - \mu_- + \frac{1}{8e^s}\omega}\, dt \leq \int_{\overline{x}}^{\overline{x}+\frac{5}{2}R} \left| \frac{d}{dt} \log_+ \frac{\frac{1}{8}\omega}{u(t,y) - \mu_- + \frac{1}{8e^s}\omega} \right| dt\, .$$

Let integrate the left and the right hand side of the above inequality over the set $D_{n-1,s}$ and we obtain

$$(s-1)\, |D_{n-1,s}| \leq \int_{B_{n-1,\varepsilon_0 R/2}(\overline{y})} \int_{\overline{x}}^{\overline{x}+\frac{5}{2}R} \left| D \log_+ \frac{\frac{1}{8}\omega}{u - \mu_- + \frac{1}{8e^s}\omega} \right| dx\, .$$

By the Hölder inequality

$$(s-1)\, |D_{n-1,s}|$$

$$\leq \left\{ \int_{B_{n-1,\varepsilon_0 R/2}(\overline{y})} \int_{\overline{x}}^{\overline{x}+\frac{5}{2}R} \left| D \log_+ \frac{\frac{1}{8}\omega}{u - \mu_- + \frac{1}{8e^s}\omega} \right|^p dx \right\}^{\frac{1}{p}} \left\{ \frac{5}{2} R \left| B_{n-1,\varepsilon_0 R/2}(\overline{y}) \right| \right\}^{\frac{p-1}{p}}\, .$$

Logarithmic lemma

Now consider Lemma 5.2 and we apply it when the balls $B_r(x_0)$ and $B_R(x_0)$ are replaced respectively by two cylinders, one compactly contained in the other. Now recall that the test function φ must be identically equal to one on the smaller cylinder and equal to zero outside of the larger one. Precisely, consider $\varphi = 1$ in $\left[\overline{x}, \overline{x} + \frac{5}{2}R\right] \times B_{n-1,\varepsilon_0 R/2}(\overline{y})$ and $\varphi = 0$ out of $[\overline{x}, \overline{x} + 3R] \times B_{n-1,\varepsilon_0 R}(\overline{y})$. With $H\omega = \frac{1}{8}\omega$ and $a = e^{-s}$ we get

$$(s-1)\, |D_{n-1,s}| \leq \left\{ c_1 \int_{B_{n-1,\varepsilon_0 R}(\overline{y})} \int_{\overline{x}}^{\overline{x}+3R} |D\varphi|^p\, dx \right\}^{\frac{1}{p}} \left\{ \frac{5}{2} R \left| B_{n-1,\varepsilon_0 R/2}(\overline{y}) \right| \right\}^{\frac{p-1}{p}}\, .$$

Therefore, if $|D\varphi| \leq 2/(\varepsilon_0 R)$,

$$(s-1)\, |D_{n-1,s}| \leq \frac{2}{\varepsilon_0 R} \left\{ c_1 \left| B_{n-1,\varepsilon_0 R}(\overline{y}) \right| R \right\}^{\frac{1}{p}} \left\{ \frac{5}{2} R \left| B_{n-1,\varepsilon_0 R/2}(\overline{y}) \right| \right\}^{\frac{p-1}{p}}$$

$$\leq \frac{c_{12}}{\varepsilon_0} \left| B_{n-1,\varepsilon_0 R/2}(\overline{y}) \right|\, .$$

The statement follows by choosing s large enough so that

$$\frac{c_{12}}{\varepsilon_0 (s-1)} \leq \nu_0\, .$$

∎

ll cylinder **Lemma 5.11** (Positivity almost everywhere in a small cylinder)**.** *Under the previous notations and assumptions, in* $C_{\varepsilon_0 \frac{R}{4}, 2}(x_1)$ *we have*

allcylinder

$$u \geq \mu_- + \frac{1}{16e^s}\omega\, . \tag{5.15}$$

Proof. The intention is to apply Lemma 5.5 to the cylinder $C_{\varepsilon_0 \frac{R}{2},\frac{5}{2}}(x_1)$ where

$$u \geq \mu_- + \frac{1}{8e^s}\omega$$

except a set with small measure. The Lemma 5.5 is proved for a ball and here we have to consider a possible long cylinder. Thus try to adapt the lemma to this case. For this aim Lemma 5.10 is applied by choosing

$$\nu_0 = \delta_0 \nu \,, \tag{5.16}$$

where ν is the constant defined in Lemma 5.5 and (here the classical symbol ω_n to denote the measure of the unit ball in $\mathbb{R}^n$)

$$\delta_0 = \frac{|B_{n,1}|}{2\,|B_{n-1,1}|} = \frac{\omega_n}{2\omega_{n-1}}\,. \tag{5.17}$$

For any $z \in [\overline{x} - 2R, \overline{x} + 2R]$ let consider the ball $B_{\varepsilon_0 R/2}(z, \overline{y})$ in $\mathbb{R}^n$. Being $\varepsilon < 1$ we have that $B_{\varepsilon_0 R/2}(z, \overline{y})$ is contained in $C_{\varepsilon_0 \frac{R}{2},\frac{5}{2}}(x_1)$, so Lemma 5.10 can be applied. Recalling that $B_{n-1,\varepsilon_0 R/2}(\overline{y})$ is a ball in $\mathbb{R}^{n-1}$ centered at $\overline{y}$ and using the following inclusion of sets

$$B_{\varepsilon_0 R/2}(x, \overline{y}) \subset \left[x - \tfrac{\varepsilon_0}{2}R, x + \tfrac{\varepsilon_0}{2}R\right] \times B_{n-1,\varepsilon_0 R/2}(\overline{y}) \,,$$

then

$$\left|\left\{(x,y) \in B_{\varepsilon_0 R/2}(z, \overline{y}) :\ u\,(x,y) \leq \mu_- + \frac{1}{8e^s}\omega\right\}\right|$$

$$\leq \left|\left\{(x,y) \in \left[z - \tfrac{\varepsilon_0}{2}R, z + \tfrac{\varepsilon_0}{2}R\right] \times B_{n-1,\varepsilon_0 R/2}(\overline{y}) :\ u\,(x,y) \leq \mu_- + \frac{1}{8e^s}\omega\right\}\right|\,.$$

By Lemma 5.10 there exists a positive number s such that

$$\left|\{y \in B_{n-1,\varepsilon_0 R/2}(\overline{y}) :\ \exists\, x \in \left[z - \tfrac{\varepsilon_0}{2}R, z + \tfrac{\varepsilon_0}{2}R\right],\ u(x,y) \leq \mu_- + \frac{1}{8e^s}\omega\}\right|$$

$$\leq |D_{n-1,s}| = \left|\{y \in B_{n-1,\varepsilon_0 R/2}(\overline{y}) :\ \exists\, x \in \left[\overline{x} - \tfrac{5}{2}R, \overline{x} + \tfrac{5}{2}R\right],\ u(x,y) \leq \mu_- + \frac{1}{8e^s}\omega\}\right|$$

$$\leq\ \nu_0 \left|B_{n-1,\varepsilon_0 r/2}(\overline{y})\right|\,.$$

By combining the previous inequalities

$$\left|\left\{(x,y) \in B_{\varepsilon_0 R/2}(z, \overline{y}) :\ u\,(x,y) \leq \mu_- + \frac{1}{8e^s}\omega\right\}\right|$$

$$\leq \varepsilon_0 R\, |D_{n-1,s}| \leq\ \nu_0\, \varepsilon_0 R \left|B_{n-1,\varepsilon_0 r/2}(\overline{y})\right|\,.$$

Therefore, by (5.16) and (5.17),

$$\left|\left\{(x,y) \in B_{\varepsilon_0 R/2}(z, \overline{y}) :\ u\,(x,y) \leq \mu_- + \frac{1}{8e^s}\omega\right\}\right|$$

$$\leq \nu \frac{|B_{n,1}|}{|B_{n-1,1}|} \tfrac{\varepsilon_0}{2} R \left|B_{n-1,\varepsilon_0 R/2}\right| = \nu \left|B_{n,\varepsilon_0 R/2}\right|\,.$$

Hence Lemma 5.5 can be applied, as in the proof of Lemma 5.9, it is obtained that $u\,(x,y) > \mu_- + \frac{1}{16e^s}\omega$ for every $(x,y) \in B_{\varepsilon_0 R/4}(z, \overline{y})$. In particular, since z is a generic point in $[\overline{x} - 2R, \overline{x} + 2R]$, we get $u \geq \mu_- + \frac{1}{16e^s}\omega$ in $C_{\varepsilon_0 \frac{R}{4},2}(x_1)\,.$ $\square$

Conclusion of the proof of Theorem 5.1. Let ζ be a versor of $\mathbb{R}^n$ and Let make the orthogonal change of variables φ that maps the vector ζ on the first axis. The function $v = u(\varphi)$ is a solution to an equation of the same type of (5.1)

$$\operatorname{div}\widetilde{A}_p\left(x, u, Du\right) = 0\,,$$

satisfying the structure conditions (5.2) and defined in $B_{4R}(x_0)$. Therefore, reasoning as in lemma 5.11,

$$v \geq \mu_- + \frac{1}{16e^s}\omega$$

in $C_{\varepsilon_0 \frac{R}{4},2}\left(\varphi(x_1)\right)$. This means that $u \geq \mu_- + \frac{1}{16e^s}\omega$ a.e. in a cylinder centered in x_1, with length $2R$, with radius $\frac{\varepsilon_0}{4}R$ and having direction ζ. Let consider the family of versors of $\mathbb{R}^n$ with rational coordinates ζ_i , $i \in \mathbb{N}$. Repeating the previous argument one can find that $u \geq \mu_- + \frac{1}{16e^s}\omega$ a.e. in all the cylinders centered in x_1, with length $2R$, with radius $\frac{\varepsilon_0}{4}R$ and having direction ζ_i. Since this is a countable covering of the unitary ball $B_R(x_0)$, we have that

$$\inf\left\{u\left(x\right) : x \in B_R\left(x_0\right)\right\} \geq \mu_- + \frac{1}{16e^s}\omega\,. \tag{5.18}$$

Now the condition (5.8) follows. In fact, with the notation $r = 4R$ and recalling (5.10), it is obtained that

$$\omega\left(r\right) := \omega = \sup\left\{u\left(x\right) : x \in B_{4R}\left(x_0\right)\right\} - \inf\left\{u\left(x\right) : x \in B_{4R}\left(x_0\right)\right\} = \mu_+ - \mu_-$$

while, for $a = \frac{1}{4}$, from the estimate (5.18) for the infimum the following is obtained:

$$\begin{aligned}\omega\left(ar\right) = \omega\left(R\right) &= \sup\left\{u\left(x\right) : x \in B_R\left(x_0\right)\right\} - \inf\left\{u\left(x\right) : x \in B_R\left(x_0\right)\right\}\\ &\leq \mu_+ - \mu_- - \frac{1}{16e^s}\omega = \left(1 - \frac{1}{16e^s}\right)\omega\,.\end{aligned}$$

Therefore, by Lemma 5.7, one can get the conclusion

$$\omega\left(ar\right) \leq b\,\omega\left(r\right), \qquad \forall\, r,$$

with $a = \frac{1}{4}$ and $b = 1 - \frac{1}{16e^s}$. As described in the Remark 5.8, this completes the proof of Theorem 5.1. □

5.2. **Parabolic regularity: the degenerate case.**

Theorem 5.12. *Let u be a locally bounded weak solution of (**??**) in Ω_T. Then, up to modification on a set of measure zero, u is locally Hölder continuous in $\Omega \times (0, T]$. The Hölder constants can be determined a priori only in terms of the data.*

Lemma 5.13 (A Measure Theory Lemma, [**?**])**.** *Let $v \in W^{1,1}(B_\rho)$ satisfy*

$$\|v\|_{W^{1,1}(B_\rho)} \leq \gamma\rho^{N-1}, \quad |\{v > \beta\}| \geq \alpha|B_\rho|,$$

for some $\gamma \geq 0$, $\beta \in \mathbb{R}$ and $\alpha \in (0, 1)$. Then $\forall\delta \in (0, 1)$ and $\lambda < \beta$, $\exists x_0 \in B_\rho$ and $\eta = \eta(\alpha, \beta, \gamma, \delta, \lambda, N)$ such that

$$|B_{\rho\eta}(x_0) \cap \{v \geq \lambda\beta\}| \geq (1 - \delta)|B_{\rho\eta}(x_0)|. \tag{5.19}$$

We consider now a solution $u(x, t)$ of (**??**). We introduce the set

$$Q_{\rho,\tau}(x_0, t_0) := B_\rho(x_0) \times (t_0, t_0 + \tau),$$

with $Q_{\rho,\tau} \subset \Omega \times (t > 0)$ and a piecewise smooth cutoff function ζ, $0 \leq \zeta \leq 1$, such that $|\nabla\zeta| < +\infty$ and $\zeta(x, t) = 0$ if $x \notin B_\rho(x_0)$. We recall now the energy estimate with $(x_0, t_0) = (0, 0)$.

For the proof of the following result we refer the reader to the monograph [**?**] pag.24.

Lm:2.2 **Lemma 5.14** (Local energy estimates)**.** *Let u be a local weak solution of* (Eq1 ??). *For all $t \in (0,\tau)$, $\exists C = C(p) > 0$ such that for all cylinders $Q_{\rho,\tau}(0,0) \subset \Omega \times (0,\tau)$ and $\forall k \in \mathbb{R}$*

$$\sup_{(0,\tau)} \int_{B_\rho \times \{t\}} (u-k)_\pm^2 \, \zeta^p dx + \int_0^\tau \int_{B_\rho} |\nabla (u-k)_\pm|^p \, \zeta^p dxdt$$

Eq:2.2 (5.20)
$$\leq \int_{B_\rho \times \{0\}} (u-k)_\pm^2 \, \zeta^p dx + C \int_0^\tau \int_{B_\rho} (u-k)_\pm^p \, |\nabla \zeta|^p dxdt +$$

$$+p \int_0^\tau \int_{B_\rho} (u-k)_\pm^2 \, \zeta^{p-1} \, \zeta_t \; dxdt.$$

Moreover we will use a variant of a DeGiorgi-like Lemma (see [**?**]). We have assumed that u_0 is non-negative. For a fixed cylinder

$$Q_{2\rho,\theta(2\rho)^p}(x_0,t_0) := B_{2\rho}(x_0) \times (t_0, t_0 + \theta\omega^{2-p}(2\rho)^p) \subset \Omega \times (t>0),$$

with $\theta > 0$, let $\mu_\pm$ and ω be non-negative numbers such that

$$\mu_+ \geq \sup_{Q_{2\rho,\theta(2\rho)^p}} u, \quad \mu_- \leq \inf_{Q_{2\rho,\theta(2\rho)^p}} u, \quad \omega \geq \mu_+ - \mu_-.$$

Denote by λ and a fixed numbers in (0,1) and by $Q_\rho(\theta) = B_\rho \times (0, \theta\omega^{2-p}\rho^p)$.

Lm:2.3 **Lemma 5.15** ([**?**], see also [**?**] pag. 49)**.** *Let u be a local weak solution of* (Eq1 ??) *and let $0 < a < 1$. There exists a number $\nu > 0$ depending upon θ, a and the data such that if*

$$|(u \leq \mu_- + \lambda\omega) \cap Q_{2\rho}(\theta)| \leq \nu \; |Q_{2\rho}(\theta)|,$$

then

Eq:2.3 (5.21)
$$u \;\geq\; \mu_- + a \; \lambda \; \omega, \quad a.e. \;\; in \;\; Q_\rho(\theta).$$

We remark that in [**?**] the result of Lemma Lm:2.3 5.15 is stated in a more general form; here we simplify it, according to our hypotheses.

Assume $\mu_- = 0$, then Lemma Lm:2.3 5.15 can be formulated in this way:

Lm:2.4 **Lemma 5.16** ([**?**])**.** *Let u be a local weak solution of* (Eq1 ??) *and let λ and β be two positive numbers, with $0 < \lambda < 1$ such that*

$$u(x,t_0) \geq \lambda \; \beta, \qquad a.e. \; x \in B_{2\rho};$$

then $\forall \, a \in (0,1)$

Eq:2.4 (5.22)
$$u(x,t) \geq a \; \lambda \; \beta, \qquad a.e. \;\; in \; B_\rho \times (t_0, t_0 + \theta\omega^{2-p}(2\rho)^p),$$

with $\theta = \dfrac{\delta}{(\lambda\beta)^{p-2}}$, and $\delta \in (0,1)$ is a quantity that depends only on a, and the data.

Lm:2.5 **Lemma 5.17** ([**?**], sec.5)**.** *Let u be a local weak solution of* (Eq1 ??) *and let λ and β be two positive numbers, with $0 < \lambda < 1$ such that*

$$u(x,0) \geq \lambda \; \beta, \qquad a.e. \; x \in B_{2\rho};$$

then $\forall t_0 > 0$ and $\forall t \geq t_0 > 0$

Eq:2.5 (5.23)
$$u(x,t) \geq \gamma_0 \; \frac{\lambda\beta\rho^{\frac{p}{p-2}}}{(t + \rho^p\omega^{p-2})^{\frac{1}{p-2}}}, \;\; a.e. \; x \in B_\rho,$$

where γ_0 depends upon the data and t_0.

Lastly let us state the so-called "shrinking lemma"

Lm:2.5 **Lemma 5.18** ([?]). *Let u be a local weak solution of*

Eq12
$$u_t = div\mathbf{A}(x,t,u,Du) - cu. \tag{5.24}$$

where c is a non-negative cylinder. Let Q be a cylinder $B_R \times (0,T)$ and let ω be the oscillation of u and assume $T \geq R^p\omega^{2-p}$ .Let $\eta < 1$ be a strictly positive number. For any level $t \in (0,T)$ define $A(t) \equiv \{x \in B_R : u(x,t) \geq \mu_- + \eta\omega\}$. If there exists a $\delta_0 > 0$ such that for any $t \in (0,T)$ we have that $|A(t)| \geq \delta_0 R^n$ and if $T \geq R^p(a\omega^{2-p})$, then in the cylinder $Q_{\frac{1}{2}}$ $equivB_{\frac{R}{2}} \times (\frac{T}{2}, T)$ we have

Eq:2.55
$$|A_k(x,t) : \{(x,t) \in Q_{\frac{1}{2}} : u(x,t) \leq \mu_- + 2^{-k}a\omega\}| \leq C\frac{1}{\sqrt{k}}|Q_{\frac{1}{2}}| \tag{5.25}$$

where k is any positive integer and C depends upon δ_0 and the data but does not depend upon k, η, ω.

First STEP: Setting the geometry of the problem.

The classical approach of the regularity comes by proving a quantitative reduction of the oscillation of the solutions: i.e if the oscillation of the solution u in a cylinder is ω, we have to prove that there is a constant $\eta < 1$, depending only upon the data, the oscillation of u in the half cylinder is $\eta\omega$.

WLOG, we may consider a cylinder $Q \equiv \{(x,t) \in \mathbb{R}^{n+1} : |x| < 1, -A \leq t \leq 0\}$, where A is a natural number to be fixed later. We also assume that the oscillation of u in Q is 1, and $0 \leq u(x,t) \leq 1$. We have proved the regularity result if we are able that there are positive constants η, η_1 such that the oscilllation of u in $Q_1 \equiv \{(x,t) \in \mathbb{R}^{n+1} : |x| < \eta_1, -\eta_1 A \leq t \leq 0\}$ is lesser than η.

Second STEP: Clustering the positivity of the solution at the bottom of the cylinder Q.

Let
$$Q_0 \equiv \{(x,t) \in \mathbb{R}^{n+1} : |x| < 1, -A \leq t \leq -A+1\}$$
and
$$Q_1 \equiv \{(x,t) \in \mathbb{R}^{n+1} : |x| < \frac{1}{2}, -A + \frac{1}{4} \leq t \leq -A + \frac{3}{4}\}.$$
Let S be a set and define with $|S|$ the Lebesgue measure of the set S.

Let $\mathcal{A} \equiv \{(x,t) \in u \geq \frac{1}{2}\}$. WLOG we may assume $|\mathcal{A}| \geq |Q_1|$ (otherwise we would consider $v = 1-u$).

Apply Lemma Lm:2.2 5.14 choosing $k = 0$ and a cut-off functions ζ such that $\zeta = 1$ in Q_1 and $\zeta = 0$ outside Q_0 to get
$$\int\int_{Q_1} |Du|^p dxdt \leq C.$$
Let
$$\mathcal{B} \equiv \{-A + \frac{1}{4} \leq \tau \leq -A + \frac{3}{4} : \int_{B_{\frac{1}{2}}} |Du|^p(\tau)dx \leq 16C \text{ such that the measure of } (x,\tau)$$
with $x \in B_{\frac{1}{2}} \geq \frac{1}{8}|B_{\frac{1}{2}}|\}$.

We have that $|\mathcal{B}| \geq \frac{1}{16}$. Choose $\tau_0 \in \mathcal{B}$. By Lemma 5.13, there exists $x_0 \in B_{\frac{1}{2}}$ and $\rho > 0$ such that

$$|B_\rho(x_0) \cap \{u \geq \frac{1}{4}\}| \geq (1 - \frac{\nu}{2})|B_\rho(x_0)|$$

where ν is the constant defined in Lemma 5.15 choosing $a = \frac{1}{2}$ and $\theta = 1$.

Third STEP: Expansion of positivity in time.

Apply energy estimastes in the cylinder with basis $B_\rho(x_0)$ at time level τ_0 and height τ_1. Choose as cut-off function, a function ζ independent of time t with $\zeta = 1$ on $B_{(1-\varepsilon)\rho}(x_0)$, $\varepsilon \in (0,1)$ to be chosen later and $\zeta = 0$ outside $B_\rho(x_0)$ and get

$$\sup_{t\in(\tau_0,\tau_0+\tau_1)} \int_{B_{(1-\varepsilon)\rho}(x_0)} (u(t) - \frac{1}{4})^2_- dx$$
$$\leq \int_{B_\rho(x_0)} (u(\tau_0) - \frac{1}{4})^2_- dx + C\varepsilon^{-p} \int\int_{B_\rho(x_0)\times(\tau_0,\tau_0+\tau_1)} (u(t) - \frac{1}{4})^p_- dxdt$$

Let $\xi \in (0, \frac{1}{4})$. Denote with $A(t)$ the set in $B_{(1-\varepsilon)\rho}(x_0)$ where $u(x,t) \leq \xi$.
The previous inequality says that for any $t \in (\tau_0, \tau_0 + \tau_1)$

$$\sup_{t\in(\tau_0,\tau_0+\tau_1)} |A(t)|(\frac{1}{4} - \xi)^2 \leq \frac{\nu}{2}|B_\rho|(\frac{1}{4})^2 + C\varepsilon^{-p}\tau_1|B_\rho|(\frac{1}{4})^2$$

Hence

$$\sup_{t\in(\tau_0,\tau_0+\tau_1)} \frac{|A(t)|}{|B_{(1-\varepsilon)\rho}|} \leq (1-\varepsilon)^{-n}\frac{(\frac{1}{4})^2}{(\frac{1}{4} - \xi)^2}(\frac{\nu}{2} + C\varepsilon^{-p}\tau_1)$$

Choose ε and ξ so close to 0 such that $(1-\varepsilon)^{-n}\dfrac{(\frac{1}{4})^2}{(\frac{1}{4} - \xi)^2} = \dfrac{4}{3}$ and $\tau_1 = (\dfrac{\nu\varepsilon^p}{4C})$ so that

$$\sup_{t\in(\tau_0,\tau_0+\tau_1)} \frac{|A(t)|}{|B_{(1-\varepsilon)\rho}|} \leq \nu.$$

Let $\rho_1 = (\tau_1\xi^{p-2})^{\frac{1}{p}}$. In the cylinder with basis $B_{(1-\varepsilon)\rho}(x_0)$ at time level τ_0 and height τ_1, there is at least a subcylinder centered in x_1, Q_{x_1}, having basis $B_{\rho_1}(x_1)$ at time level τ_0 and height τ_1 such that $|(x,t) \in Q_{x_1} : u(x,t) \leq \xi| \leq \nu|Q_{x_1}$. By Lemma 5.15, in the half-cylinder we have that $u \geq \frac{\xi}{2}$. In particular for any $x \in B_{\frac{\rho_1}{2}}(x_1)$ we have that $u(x, \tau_0 + \tau_1) \geq \frac{\xi}{2}$.

Therefore by Lemma 5.18, we have that for any $t > 0$, for any $x \in B_{\frac{\rho_1}{4}}(x_1)$

$$u(x, t + \tau_0 + \tau_1) \geq \gamma\frac{\xi}{(t+1)^{\frac{1}{p-2}}}$$

where γ is a constant depending on ξ and ρ_1.

Fourth STEP: Change of variables.

For sake of semplicity change the origin from $(0,0)$ to $(x_1, \tau_0 + \tau_1)$. Reasoning as in [**?**] introduce the function

$$w(t,x) = u(x,t)t^{\frac{1}{p-2}}.$$

By the change of variable $t = e^\tau$ The function $w(x,\tau)$ is a non negative solution of the equation

Eq:5.2
$$w_\tau = div\ e^{\tau(\frac{p-1}{p-2})}\mathbf{A}(x,t,u,\nabla u) - \frac{1}{p-2}w, \tag{5.26}$$

that is

Eq:5.3
$$w_\tau = div\ \tilde{\mathbf{A}}(x,\tau,w,\nabla w) - \frac{1}{p-2}w, \tag{5.27}$$

Note that $\tilde{\mathbf{A}}$ satisfies the same structure conditions of $\mathbf{A}$.

By the results of the previous step, there are two positive constants r_0 and ε_0, such that for any $x \in B_{r_0}$ and for any $t > 0$, $w(x,t) \geq \varepsilon_0$. Choose k_0 such that

Fifth STEP: Proof concluded.

Apply the shrinking lemma to w with $R = 8$, $\omega = 1$, $\mu_- = 0$, $\eta = \varepsilon_0$, $T = 8^p\varepsilon_0^{2-p}2^{k_0+1}$, where k_0 is an integer to be chosen. Let k_0 such that $C\frac{1}{\sqrt{k}} \leq \nu$ where C is the constant appearing in Eq:2.55 5.25 and ν is the constant appearing in Lemma Lm:2.3 5.15 choosing $a = \frac{1}{2}$. Therefore, by the shrinking lemma and Lemma Lm:2.3 5.15 we have that $w \geq \frac{\varepsilon_0}{2^{k_0+1}}$ in the cylinder $B_2 \times (\frac{3}{4}T, T)$. recalling that $w(t,x) = u(x,t)t^{\frac{1}{p-2}}$ and $t = e^\tau$, we have reduced the oscillation of u choosing, in the First Step, the length of the cylinder equal to $A = e^{8^p\varepsilon_0^{2-p}2^{k_0+1}} + 1$

5.3. **Parabolic regularity: the singular case.** In this section we want to prove Theorem ThHold 5.12 in the singular case $1 < p < 2$. We give a proof of the based on a result proved in [**?**].

Geom
Lemma 5.19. *Let u be a non-negative, local, weak solution to equation (Eq1 **??**)*

Assume that $1 < p < 2$ (i.e. we are considering the singular case)

$$[u(\cdot,t) > M] \cap B_\rho(y)\ > \alpha|B_\rho|$$

for all times

$$s - \varepsilon M^{2p}\rho^p \leq t \leq s$$

for some $M > 0$, and some α and ε in $(0,1)$, and assume that $B_{16\rho} \times [s - \varepsilon M^{2p}\rho^p, s]$ is contained in the domain.

There exist $\sigma \in (0,1)$ and $\eta \in (0, \frac{1}{2}\varepsilon)$, that can be determined a priori, quantitatively only in terms of the data, and the numbers α and ε, and independent of M, such that

Eq:sing.3
$$u(x,t) \geq \sigma M \quad \forall x \in B_{2\rho}(y) \tag{5.28}$$

for all times

$$s - \eta M^{2p}\rho^p < t \leq s$$

Thus, this measure-theoretical information on the measure of the "positivity set" in $B_\rho(y)$ for all times in implies that such a positivity set actually "expands" to $B_{2\rho}(y)$ almost for comparable times

The second argument we will use is a L^1 form of the Harnack inequality

Thm:2.2
Theorem 5.20 (An L^1_{loc} Form of the Harnack Inequality for all $1 < p < 2$, [**?**])**.** *Let u be a non–negative, weak solution to (Eq1 **??**) and let $1 < p < 2$. There exists a positive constant C depending only upon the data, such that for all cylinders $\Omega_{2\rho}(y) \times [s,t]$*

Eq:2.4
$$\sup_{s<\tau<t}\int_{\Omega_\rho(y)} u(x,\tau)dx \leq C \inf_{s<\tau<t}\int_{\Omega_{2\rho}(y)} u(x,\tau)dx + C\left(\frac{t-s}{\rho^\lambda}\right)^{\frac{1}{2-p}} \tag{5.29}$$

where $\lambda = N(p-2)+p$. *The constant* $C = C(N,p,\Lambda) \to \infty$ *as either* $p \to 2$ *or as* $p \to 1$.

We will prove this result following an alternative argument

Setting the geometry

Up to translations, local parabolic Hölder continuity follows from a decay in oscillation in intrinsic cylinders of the form

$$Q_\rho(\omega) := B_\rho \times [-\varepsilon_0 \omega^{2-p}, 0],$$

as long as one can find $\delta, \eta \in]0,1[$ such that inductively defining the cylinders

$$\begin{cases} Q_n = Q_{\delta^n}(\omega_{n-1}) \\ \omega_n = \operatorname{osc}(u, Q_{n-1}) \end{cases}$$

for $n \geq 1$, it holds the oscillation reduction

$$\omega_n \leq \eta\, \omega_{n-1}.$$

Assuming without loss of generality $\omega_0 = 1$, at each step, the function

$$Q_1(1) \ni (x,t) \mapsto \omega_{n-1}^{-1} u(\delta^n x, \varepsilon_0 \omega_{n-1}^{2-p} \delta^{np} t)$$

satisfies the same type of equation as u (as long as ε_0 is universally chosen) and $\operatorname{osc}(u, Q_1(1)) \leq 1$. A further, non intrinsic, change of variable allows to suppose that u solves the equation in $Q_{2\theta} = B_{2\theta} \times [-1, 0]$ and $\operatorname{osc}(u, R_{2\theta}) \leq 1$, where $\theta = \varepsilon_0^{-1/p}$ is a suitable large constant to be determined depending only on the data. Translating u we can furthermore suppose $0 \leq u \leq 1$ in $Q_{2\theta}$, and we want to prove a quantitative reduction in oscillation in a suitable subcylinder. More precisely that

$$\operatorname{osc}(u, B_\theta \times [-\eta, 0]) < 1 - \sigma$$

for suitably large θ and small σ, η depending on the data.

Assume without loss of generality that

alternative (5.30)
$$\left|\left\{(x,t) \in Q_\theta : u(x,t) \geq \frac{1}{2}\right\}\right| \geq \frac{1}{2}|Q_\theta|$$

(otherwise we can consider $1-u$). Then, there exists $\tau \in (-1, 0)$ such that

$$\left|\left\{x \in B_\theta : u(x,\tau) \geq \frac{1}{2}\right\}\right| \geq \frac{1}{2}|B_\theta|$$

and by (5.29) and Chebyshev's inequality

Eq:2.4bis (5.31)
$$\theta^N \leq C\left|\left\{x \in B_\theta : u(x,\tau) \geq \frac{1}{2}\right\}\right| \leq C\int_{B_\theta} u(x,\tau)\,dx \leq C \inf_{-1<s<0} \int_{B_{2\theta}} u(x,s)\,dx + C\left(\frac{1}{\theta^\lambda}\right)^{\frac{1}{2-p}}$$

Noting that $-\frac{\lambda}{2-p} < N$, choose θ so large that

$$C\left(\frac{1}{\theta^\lambda}\right)^{\frac{1}{2-p}} \leq \frac{1}{2}\theta^N$$

so that (5.31) gives

$$\theta^N \leq C\int_{B_{2\theta}} u(x,s)\,dx$$

for any $s \in (-1, 0)$. Since $0 \le u \le 1$ in $Q_{2\theta}$,

$$\int_{B_{2\theta}} u(x,s)\,dx \le \varepsilon |B_{2\theta}| + |\{x \in B_{2\theta} : u(x,s) \ge \varepsilon\}|$$

so that for ε such that

$$\varepsilon C |B_{2\theta}| \le \frac{1}{2}\theta^N$$

it holds

$$\alpha |B_\theta| := \frac{1}{2C}\theta^N \le |\{x \in B_\theta : u(x,s) \ge \varepsilon\}| \qquad \text{for all } s \in [-1, 0].$$

Therefore, by Lemma Geom 5.19, we have that there exists $\sigma, \eta > 0$ depending only on the data such that

$$u(x,s) \ge \sigma \qquad \forall (x,s) \in B_\theta \times [-\eta, 0],$$

which is the claimed reduction in oscillation.

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