Mini-courses in Mathematical Analysis
Padova 2025

Topics of the lectures and talks

Lectures

Generic properties of eigenvalues and other geometric functionals

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Many results of the type "*good properties are generic*" — that is, they hold on a dense and residual set — are based on the transversality theorem. This theorem is a fundamental tool in differential topology and differential geometry, with applications in numerous areas of mathematics. Among its main applications we will focus on

- **Simplicity of eigenvalues in spectral problems:**
  For example, the eigenvalues of the Dirichlet problem (or of other elliptic operators) are simple for generic perturbations of the domain or potential.
- **Morse functions:**
  Generic smooth functions on a manifold are Morse functions, meaning that all their critical points are non-degenerate.

Norm convergence of operators acting in different spaces

Olaf Post

Universität Trier, Germany

This minicourse is about operator norm convergence of operators acting in different Hilbert spaces. We start with classical results about operator norm convergence and the consequences such as convergence of spectra and operator functions. We also comment on the fact that norm convergence is better suited for spectral convergence than strong (pointwise) operator convergence.

Next, we introduce a concept where one uses identification operators between Hilbert spaces depending on the convergence parameter. This concept generalises at the same time norm convergence and unitary equivalence. It is perfectly suited in situations where operators act on spaces with collapsing dimension in the limit. We illustrate this in an example of differential operators on a tubular neighbourhood shrinking to an underlying metric graph (so-called "fat graphs").

Another concept due to Weidmann and suited mostly for domain perturbations is to embed all operators in a common Hilbert spaces and consider norm convergence in the common space. We show that both concepts are basically equivalent.

Finally, we explain that both convergence concepts are based on a notion of a distance between two operators acting in different spaces. We comment on this fact and show also, how this distance is related with a distance on the spectra of the operators.

Basecamp for ergodic theorems along subsequences

Máté Wierdl

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This course gives the ergodic theoretical foundations to subsequence ergodic theorems. The aim of the course is not to list the (quite formidable) techniques used in today's research but to give all the background material from ergodic theory that is needed in this area of mathematics. The goal is to develop a *feeling* for convergence along subsequences that may lead to the discovery of new results and to reasonable working hypotheses. To reach this goal, we will see the basic methods and tools that are used every day in this subject: transference, Rohlin's interval, baby spectral theorem and, if time permits, the circle method. We'll see lots of examples, counterexamples, pictures, problems to work on, and we'll also see some unsolved problems. The course will be a good preparation to successfully learn techniques that are used in current research. (It is not to say that one cannot do research just by using what is presented during the course.)

Talks

Convexity on manifolds without focal points and spectrum of the Laplacian

Sivashankar Balakrishnan

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The effect of convexity on the spectrum of the Laplacian has typically been studied under constraints on the sign of the sectional curvature. We shall discuss the geometric construction of a strictly convex function with bounded gradient on a large class of noncompact manifolds without focal points, making assumptions only on the Ricci curvature. This class includes Damek-Ricci spaces or more general asymptotically harmonic manifolds and symmetric spaces of noncompact type. We deduce that the spectrum of the Laplacian is purely absolutely continuous from strict convexity. For asymptotically harmonic manifolds, we show that equality holds in Cheeger's inequality for the bottom of the spectrum. In fact, we completely determine the spectrum of the Laplacian on harmonic manifolds where the unitarity of Fourier transform is not known.

Based on joint work with Dr. Aprameyan Parthasarathy (Indian Institute of Technology Madras).

Keywords: convexity, spectrum of the Laplacian, manifolds without focal points.

Capacitary inradius and Poincare-Sobolev inequalities

Francesco Bozzola

DIME, Università di Genova

For a general euclidean open set, the inradius, i.e. the radius of the largest inscribed ball, plays a fundamental role in the investigation of the geometry of its principal frequencies, namely its sharp Poincaré-Sobolev constants. A universal upper bound for principal frequencies in terms of the inradius holds true for every open set. On the other hand the lower bound is not always possible for every dimension of the ambient space: capacitary effects may be held responsible. Inspired by the pioneering works of Vladimir Maz'ya, we take into account a capacitary variant of the concept of inradius, the capacitary inradius, which will turn out to be the right tool to overcome these difficulties. We will present some new and classical results in this direction.

Based on joint work with Lorenzo Brasco.

Keywords: Poincaré-Sobolev inequalities, principal frequencies, capacitary inradius.

Partially concentrating solutions for systems with Lotka-Volterra type interactions

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Sabrina Caputo

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In this talk we consider the existence of standing waves for a coupled system of k equations with Lotka-Volterra type interaction. We prove the existence of a standing wave solution with all nontrivial components satisfying a prescribed asymptotic profile. In particular, the $k-1$-last components of such solution exhibits a concentrating behavior, while the first one keeps a quantum nature. We analyze first in detail the result with three equations since this is the first case in which the coupling has a role contrary to what happens when only two densities appear. We also discuss the existence of solutions of this form for systems with other kind of couplings making a comparison with Lotka-Volterra type systems.

Based on joint work with Giusi Vaira, Department of Mathematics, University of Bari, Italy.

Keywords: nonlinear systems with Lotka-Volterra type interactions, singularly perturbed problems, Lyapunov-Schmidt reduction.

The optimal constant for the critical Sobolev inequality of higher order on manifolds

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Lorenzo Carletti

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Let (M, g) be a closed Riemannian manifold of dimension n, and $k \geq 1$ an integer such that $n > 2k$. We show that the optimal constant for the critical Sobolev embedding of higher-order $H^k(M) \subset L^{2^\sharp}(M)$ is the same as in the Euclidean setting, where $2^\sharp = \frac{2n}{n-2k}$ is the critical Sobolev exponent. Precisely, we prove that there exists $B_0 > 0$ such that $\|u\|^2_{L^{2^\sharp}(M)} \leq K_0^2 \int_M |\nabla^k u|^2 + B_0 \|u\|^2_{H^{k-1}(M)}$ for all $u \in H^k(M)$, here $K_0(n,k)$ is the smallest possible constant such that $\|u\|^2_{L^{2^\sharp}(\mathbb{R}^n)} \leq K_0^2 \int_{\mathbb{R}^n} |\nabla^k u|^2$ for all $u \in C_c^\infty(\mathbb{R}^n)$. This result extends the cases $k = 1$ showed by Hebey-Vaugon in 1996, and $k = 2$ showed by Hebey in 2003. For $k \geq 3$, we propose a new strategy of proof : We study the pointwise blow-up of a sequence of positive functions $(u_\alpha)_{\alpha \geq 1}$ satisfying $(\Delta_g + \alpha)^k u_\alpha = u_\alpha^{2^\sharp - 1}$ in M, as $\alpha \to \infty$. We are able to obtain a precise description of the solutions near the blow-up point, allowing to overcome the difficulties that come with the higher-order setting.

Keywords: Sobolev embeddings of higher order, blow-up analysis, critical non-linear elliptic equation.

Some extensions of the Leray-Trudinger inequality

Giuseppina Di Blasio

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In this talk, we will present some recent results on Leray-Trudinger type inequalities that are closely related to Trudinger-Moser and Hardy inequalities. The aim of the talk is to present the origin and the history of the problem and to present an optimal analogous of the Trudinger inequality. Moreover, an extension of the Lerey- Trudinger inequality to the anisotropic setting induced by a strongly convex Finsler norm F will be discussed.

The presentation is based on joint works with G. Pisante and G. Psaradakis.

Rectifiability in Carnot groups

Daniela Di Donato

Department of Mathematics, University of Pavia

Intrinsic regular surfaces in Carnot groups play the same role as C^1 surfaces in Euclidean spaces. As in Euclidean spaces, intrinsic regular surfaces can be locally defined in different ways: e.g. as non critical level sets or as continuously intrinsic differentiable graphs. The equivalence of these natural definitions is the problem that we are studying. Precisely our aim is to generalize some results proved by Ambrosio, Serra Cassano, Vittone valid in Heisenberg groups to the more general setting of Carnot groups.

Based on joint work with Antonelli, Don and Le Donne.

Keywords: Carnot groups, C^1 surfaces, intrinsic Lipschitz maps.

Nemytzkij operators on weighted function spaces

Douadi Drihem

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Let $G : \mathbb{R} \to \mathbb{R}$ be a continuous function. The corresponding composition operator T_G is defined by $T_G : f \longmapsto T_G(f) = G(f)$. In the first part of this talk, we present a necessary and sufficient conditions on G such that

$$\{G(f) : f \in W_p^m(\mathbb{R}^n, |\cdot|^\alpha)\} \subset W_p^m(\mathbb{R}^n, |\cdot|^\alpha)$$

holds, with some suitable assumptions on m, α and p. Under some assumptions on G, s, α, p and q we have that

$$\{G(f) : f \in A_{p,q}^s(\mathbb{R}^n, |\cdot|^\alpha)\} \subset A_{p,q}^s(\mathbb{R}^n, |\cdot|^\alpha) \tag{1}$$

implies that G is a linear function. Here $A_{p,q}^s(\mathbb{R}^n, |\cdot|^\alpha)$ stands either for the Besov space $B_{p,q}^s(\mathbb{R}^n, |\cdot|^\alpha)$ or for the Triebel-Lizorkin space $F_{p,q}^s(\ \mathbb{R}^n, |\cdot|^\alpha)$. Some sufficient conditions on G such that (1) holds are give.

Let ω, ϖ be two nonnegative locally integrable functions on $\mathbb{R}^n$. In the second part of this talk and under some suitable assumptions on ω and ϖ a necessary conditions on G such that

$$\{G(f) : f \in \mathcal{D}(\mathbb{R}^n)\} \subset B_{p,q}^s(\mathbb{R}^n, \omega)$$

or

$$\{G(f) : f \in B_{p,p_1}^s(\mathbb{R}^n, \omega)\} \subset B_{q,\infty}^r(\mathbb{R}^n, \varpi),$$

hold are given.

Keywords: Besov-Triebel-Lizorkin spaces, Muckenhoupt weights, Nemytzkij operators.

Steklov dual eigenvalues and the Babuška paradox

Francesco Ferraresso

Università degli Studi di Verona, Italy

The Dirichlet Biharmonic Steklov (DBS) and the Modified Dirichlet Biharmonic Steklov (MDBS) consist in looking for biharmonic functions in Ω satisfying $u = 0$ on $\partial\Omega$ and a Steklov type boundary condition $\Delta u = \lambda \partial_\nu u$ or $\partial_{\nu\nu}^2 u = \lambda \partial_\nu u$ on $\partial\Omega$, where $\lambda \in \mathbb{R}$ is the eigenvalue. I will give a new take on the appearance of Babuška paradoxes in the approximation of the Steklov problems on convex domains via sequences of convex polygons. The new approach is based on the analysis of the dual Steklov eigenvalues, spectral points associated with a non-local problem for harmonic functions, that turn out to be in a one-to-one correspondence with the (DBS) eigenvalues. By means of this technique, the Babuška paradox for the (DBS) problem is shown to appear even in a low-regularity setting, where smooth functions are not dense in the domain of the operator.

Based on joint work with Pier Domenico Lamberti (University of Padova).

Keywords: eigenvalue problem, biharmonic operator, Babuška paradox.

Invertibility of a modified Stokes operator and of its layer potential operators on domains with cylindrical ends

Mirela Kohr

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We prove that a modified Stokes operator on a manifold with cylindrical ends is invertible and give a description of the inverse operator. This invertibility result allows us to introduce the Stokes layer potentials, and their properties are established using a pseudodifferential calculus. The invertibility of the main boundary integral operators then leads to well-posedness results for the modified Stokes system with Dirichlet boundary condition on domains with cylindrical ends. A modified Navier-Stokes operator is also considered.

Joint work with Victor Nistor (Metz) and Wolfgang L. Wendland (Stuttgart).

Blow-up and unique continuation for the Grushin Laplacian

Paolo Luzzini

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The Grushin Laplacian $-\Delta_\alpha$ is a degenerate elliptic operator in $\mathbb{R}^{h+k}$ that degenerates on $\{0\} \times \mathbb{R}^k$. In this talk we study weak solutions of

$$-\Delta_\alpha u = Vu$$

under suitable regularity assumptions on the potential V. By means of an Almgren-type monotonicity formula we identify the exact asymptotic blow-up profile of solutions on degenerate points. As a main consequence, we derive strong unique continuation properties for such solutions.

Based on joint work with L. Abatangelo (Polimi) and A. Ferrero (Upo).

Keywords: Grushin Laplacian, blow-up analysis, unique continuation principles.

An analog of the Hille theorem for functions given in a finite-dimensional commutative algebra

Sergiy Plaska

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E. Hille (1944) proved the theorem that a locally bounded Gâteaux differentiable mapping in a complex Banach space is also Fréchet differentiable.

We proved that a locally bounded and differentiable in the sense of Gâteaux function given in a finite-dimensional commutative Banach algebra over the complex field is also differentiable in the sense of Lorch.

Note that it is impossible to establish this result using the mentioned Hille theorem because the Fréchet differentiability does not imply the existence of Lorch derivative.

Based on joint work with Vitalii Shpakivskyi and Maksym Tkachuk.

Keywords: Gâteaux derivative, Fréchet derivative, Lorch derivative.

Uniqueness of the critical point of semi-stable solutions on model spaces

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We consider positive, semi-stable solutions of $-\Delta u = f(u)$ on domains of the model spaces of constant curvature $\mathbb{S}^2$, $\mathbb{R}^2$ and $\mathbb{H}^2$. We provide geometric conditions on the domains guaranteeing the uniqueness and the non-degeneracy of the critical point, and we discuss their sharpness.

Based on joint work with F. Gladiali and M. Grossi.

Keywords: semilinear elliptic equations, critical points, model spaces.

Multi-bubble solutions for the Brezis-Nirenberg problem in four dimensions

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Giuseppe Mario Rago

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The paper addresses the existence of multi-bubble solutions for the well-known Brezis–Nirenberg problem. Although there is extensive literature on the subject, the existence of solutions that blow up at multiple points in a 4D bounded domain remains an open problem. The goal of the present paper is to resolve this longstanding issue. In particular, we exhibit examples of domains where a large number of multi-bubble solutions exist. Our result can also be seen as the counterpart of the asymptotic analysis carried out by König and Laurin in *Ann. Inst. H. Poincaré C Anal. Non Linéaire, 2024*.

Based on joint work with Angela Pistoia and Giusi Vaira.

Keywords: Brezis-Nirenberg problem, blowing-up solutions, Lyapunov-Schmidt reduction.

On analytic properties of a generalization of the Mittag-Leffler and Le Roy functions

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Sergei Rogosin

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We discuss the analytic properties of certain generalization of special functions whose contain powers, logarithms and derivatives of the Gamma-functions in the coefficients of their Taylor series. Common property of these functions is their representability by integrals of Mellin-Barnes type. The expressions of the partial derivatives of Mittag-Leffler- and Le Roy-type functions, with respect to the involved parameters, allow to determine a second-order partial differential equation for these functions. This equation is transformed into a second-order hyperbolic partial differential equation. We also derive the Laplace transforms of the derivatives with respect to parameters of certain special functions. These formulas show interconnection of these functions and lead to better understanding of their behaviour on the real line. These results could serve to improve our understanding about the local and global behaviour of the considered special functions. They can be useful for the solution of differential equations. The formulas of the Laplace transforms of the derivatives are reconstructed using Efros theorem.

Based on joint work with M. Dubatovskaya, F. Giraldi and F. Mainradi.

Keywords: Mittag-Leffler function, differentiation with respect to parameters, Laplace transforms.

Counting nodal domains of eigenfunctions in non-smooth setting

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Ivan Yuri Violo

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The Courant nodal domain theorem states that the k-th Laplacian eigenfunction with Dirichlet boundary condition has at most k nodal domains. This fact admits an asymptotic version due to Pleijel, which in particular says that this estimate is strict for all but a finite number of eigenfunctions. We extend this result to Neumann Laplacian eigenfunctions on domains with low boundary regularity and allowing the ambient space to be a non-smooth metric space.

Based on joint work with Nicolò De Ponti and Sara Farinelli.

Keywords: nodal domains, eigenfunctions, metric Spaces.

Bending and twisting of electromagnetic waveguides

Michele Zaccaron

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Consider a reference homogeneous and isotropic electromagnetic waveguide with a simply connected cross-section embedded in a perfect conductor. When the waveguide is straight, the spectrum of the associated Maxwell operator with a *constant twist* is entirely essential, lies on the real line and is symmetric with respect to zero forming a gap around it.
In this talk, we present new results on the effects of geometric deformations on the spectrum of the Maxwell operator. More precisely, we provide sufficient conditions on the asymptotic behaviour of curvature and torsion ensuring the preservation of the essential spectrum. Our approach relies on a Birman–Schwinger-type principle. Moreover, we give sufficient conditions, involving the shape of the cross-section, so that the deformation creates discrete spectrum within the gap of the essential spectrum, and give some result on its localization. We finally show some numerical and theoretical results analyzing the sufficient condition involving the cross-section.

Based on joint work with Philippe Briet, Maxence Cassier and Thomas Ourmières-Bonafos.

Keywords: electromagnetic waveguides, geometric spectral theory, Birman-Schwinger principle.