Today

* finish checking hypotheses, apply Th. 3.1
* prove domain monotonicity by Rayleigh and Poincaré Principles.
* reaction-diffusion PDE: steady state and linearization

Compact imbedding $H^1_0 \hookrightarrow L^2$:
see Rellich's Theorem A.4 in Appendix of online notes (lovely Fourier based proof)
i.e. L^2 bound on deriv. $\Rightarrow$ L^2 convergence of some subseq.

Bilinear form: define

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega u v \, dx$$

$$= \langle u, v \rangle_{H^1}, \qquad u, v \in H^1_0(\Omega)$$

Clearly $a(u,v)$ is linear, symmetric, bounded by Cauchy-Schwarz.

Coercivity: $a(u,u) = \|u\|^2_{H^1}$. ✓

$\therefore$ Discrete Spectral Theorem applies.
Gives ONB of weak eigenfns., eigenvalues γ_j, which we write as $\gamma_j = \lambda_j + 1$, satisfying

$$a(u,v) = \langle u_j, v\rangle_{H^1} = (\lambda_j + \cancel{1}) \langle u_j, v\rangle_{L^2}$$
$$\langle \nabla u_j, \nabla v\rangle_{L^2} + \cancel{\langle u_j, v\rangle_{L^2}} = \quad \forall v \in H^1_0(\Omega)$$

$\Longrightarrow$

$$\int_\Omega \nabla u_j \cdot \nabla v \, dx = \lambda_j \int_\Omega u_j v \, dx \qquad (W)$$
$$\forall v \in H^1_0(\Omega)$$

which means

$$-\Delta u_j = \lambda_j u_j \quad \text{weakly}.$$

Next we show u_j satisfies this eq. classically. Indeed, elliptic regularity theory (Gilbarg & Trudinger, Cor. 8.11) implies that the weak eigenfn. u_j is actually not just in $H^1_0(\Omega)$, but is actually a smooth function. Formula (W) gives with

Green's Th:
$$\int_\Omega (-\Delta u_j) v \, dx = \int_\Omega (\lambda_j u_j) v \, dx$$
$$\forall v \in C^\infty_0(\Omega)$$
$$\int_\Omega (\Delta u_j + \lambda u_j) v \, dx = 0$$

If $\Delta u_j + \lambda u_j > 0$ at any point then choose $v > 0$ near that point to get a contradiction.

$\therefore$ $\Delta u_j + \lambda_j u = 0$ at every point.

$-\Delta u_j = \lambda_j u$ classically.

Dirichlet boundary condition:

$u_j = 0$ on $\partial\Omega$ in the sense of Sobolev spaces (ie trace theorem), since H_0^1 = closure of C_0^∞.

The boundary condition holds classically on any smooth portion of $\partial\Omega$.

Positivity of eigenvalues:

choose $v = u_j$ in (W)

get

$$\lambda_j = \frac{\int_\Omega |\nabla u_j|^2 \, dx}{\int_\Omega u_j^2 \, dx} \geq 0.$$

If $\lambda_j = 0$ then $\nabla u_j \equiv 0$ in Ω

so $u_j \equiv$ const., hence $u_j \equiv 0$ by Dirichlet BC, impossible since $\|u_j\|_{L^2} = 1$.

$\therefore \lambda_j > 0$

$$\therefore \boxed{0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to \infty}$$

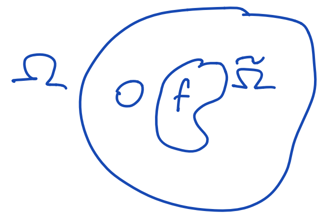
Domain monotonicity (special to Dirichlet)

"Making a drum smaller makes its frequencies bigger"

Theorem

$$\boxed{\lambda_j \le \tilde{\lambda}_j}$$

for all j



Proof:

① First eigenvalue λ_1.

Rayleigh Principle on $\tilde{\Omega}$:

$$\tilde{\lambda}_1 = \min_{\substack{f \in H_0^1(\tilde{\Omega}) \\ f \not\equiv 0}} \frac{\int_{\tilde{\Omega}} |\nabla f|^2 \, dx}{\int_{\tilde{\Omega}} f^2 \, dx}$$

Extend f to equal 0 outside $\tilde{\Omega}$. The value of Rayleigh quotient is unchanged by the extension. The extended f belongs to $H_0^1(\Omega)$.

Every f that works for $\tilde{\Omega}$ also works for Ω.

$\therefore \quad \lambda_1 \le \tilde{\lambda}_1$ because in the Rayleigh Principle for λ_1 we are taking a minimum over a larger class of functions.

② λ_j, $j \ge 1$:

use Poincaré minimax instead of Rayleigh Principle.

Exercise — regular Sturm-Liouville problem

$$-(pu')' + qu = \lambda w u$$
$$u = 0 \text{ at } x = 0, L$$

coefficient fns. p, q, w smooth on $[0, L]$
p, w positive on $[0, L]$

(i) Formulate the eigenvalue problem in terms of appropriate Hilbert spaces H and K and bilinear form $a(u,v)$.

(ii) Verify hypotheses of Discrete Spectral Theorem

(iii) Find lower bound on λ_1 in terms of the coefficient functions. For simplicity assume $q \geq 0$.

Chapter 6 — stability of steady states for reaction-diffusion PDE

Goal:
linearize a nonlinear reaction diffusion PDE

$$\boxed{u_t = u_{xx} + f(u)}$$

Competing effects can yield nonconstant steady states [diffusion: stabilizing   reaction term: can be destabilizing when $f > 0$]

around a steady state, and then determine its stability by spectral theory of linearized operator.

Assumptions: $f: \mathbb{R} \to \mathbb{R}$ smooth with f' bounded above.

Physically: $u(x,t)$ = temperature
f = rate of heat generation

$u(x,t)$ = chemical concentration
f = rate of chemical creation

Dirichlet BC $\quad u(0,t)=0 \quad u(X,t)=0$ for all t

Next time:
* steady state eq.
* linearize the PDE
* eigenvalue problem for stability
* instability of sign-changing steady states

Steady state: $U = U(x)$, no t-dependence

nonlinear oscillator equation

$$U'' + f(U) = 0, \qquad 0 < x < X$$

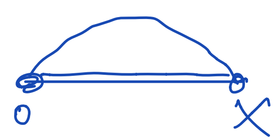
More than one steady state can exist.

eg $\quad f(y) = y, \quad X = \pi$

$$U'' + U = 0$$

$U(x) \equiv 0$ zero steady state

$U(x) = \sin x$ nonzero steady state



Linearizing the PDE

Perturb a steady state by considering

$$u = \underset{\text{steady state}}{U} + \underbrace{\varepsilon\phi}_{\substack{\text{small perturbation}\\ \phi(x,t)}}$$

$\phi = 0$ at $x = 0$ and $x = X$

Substituting into reaction-diffusion eq:

$$u_t = u_{xx} + f(u)$$

$$\underset{U_t = 0}{0} + \varepsilon\phi_t = U_{xx} + \varepsilon\phi_{xx} + f(U + \varepsilon\phi)$$

$$= \underbrace{U_{xx} + f(U)}_{=0} + \varepsilon\phi_{xx} + \underbrace{\varepsilon\phi f'(U) + O(\varepsilon^2)}_{\text{by Taylor expansion}}$$

Discard terms of order ε^2 and higher.

Terms of order ε:

$$\phi_t = \underbrace{\phi_{xx} + \overbrace{f'(U)}^{\text{function of } x}\phi}_{\equiv L\phi}$$

Separation of variables:

$$\phi(x,t) = \sum_j c_j e^{-\tau_j t} w_j(x)$$

decays if $\tau_j > 0$ $\Rightarrow$ perturbation decays

grows if $\tau_j < 0$ $\Rightarrow$ pert. grows

where

$$\boxed{-Lw_j = \tau_j w_j}$$

τ_j: eigenvalue of $-L$

w_j: eigenfn. of $-L$

with $w_j(0) = w_j(X) = 0$.

Pattern formation in animal coats: see Turing instability, eg in PDE textbook by Evans.

The steady state U is:

* linearly (asymptotically) stable if $\tau_1 > 0$ since then $\tau_j > 0$ $\forall j \Rightarrow$ perturbation decays

* linearly unstable if $\tau_1 < 0$
since then $e^{-\tau_1 t}$ grows $\Rightarrow$ perturbation grows

Task: understand $\tau_1 \ldots$

Apply Discrete Spectral Th. for $\boxed{Lw = w'' + f'(U)w}$

$\Omega = (0, X)$

$K = H_0^1(0,X)$ $\quad K \hookrightarrow H$ compactly by Rellich

$H = L^2(0,X)$

Bilinear form

$$a(u,v) = \int_0^X (u'v' \underbrace{- f'(U)uv + Cuv}_{-f'+C \geq 1})dx$$

where

$$C = 1 + \sup f'.$$

So coercivity holds: $a(u,u) \geq \int_0^X (u')^2 + u^2\, dx$

$\Rightarrow$ get ONB of eigenfns with eigenvalues γ_j.

Write $\gamma_j = \tau_j + C$.

Get

$$-Lw_j = \tau_j w_j$$

weakly. The w_j are smooth by elliptic regularity (not hard in 1-dim.). So

$$-Lw_j = \tau w_j$$

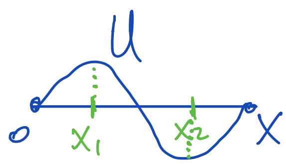
holds classically.

Linear instability of sign-changing steady states for Dirichlet BC

Theorem (R. Schaaf)

If the steady state U changes sign on $(0, X)$ then $\boxed{\tau_1 < 0.}$

Example: $f(y) = y$, $U'' + U = 0$, $X = (0, 2\pi)$



Exercise: compute eigenvalues of L, show $\tau_1 < 0$.

Proof:

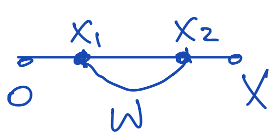
Find points $0 < x_1 < x_2 < X$

s.t. $U' = 0$ at x_1, x_2

$U' \neq 0$ between x_1 and x_2

Define trial fn.

$$w = \begin{cases} U' & \text{on } (x_1, x_2) \\ 0 & \text{elsewhere} \end{cases}$$



Then w is piecewise smooth, and continuous since

$$w(x_1) = U'(x_1) = 0$$
$$w(x_2) = U'(x_2) = 0$$

The numerator of the Rayleigh quotient for w is:

$$\int_0^X \left((w')^2 - f'(U)\, w^2 \right) dx$$

$$= \int_{x_1}^{x_2} \left(-w'' - f'(U)\, w \right) dx \quad \text{by parts}$$

using $w = 0$ at x_1, x_2

$= 0$

by steady state equation

since $\quad -w'' = -U'' = (f(U))' = f'(U)U'$
$$= f'(U)w.$$

$\therefore$ Rayleigh quotient $= 0$

So $\quad \tau_1 \le 0 \quad$ by Rayleigh Principle.

Suppose $\tau_1 = 0$. Then w is an eigenfn. with eigenvalue 0, by the condition for equality in the Rayleigh Principle. Since eigenfns. are smooth, we see $w'(x_2) = 0$ from the right and hence from the left also. But $w(x_2) = 0$, $w'(x_2) = 0$
$\Rightarrow w \equiv 0$ by ODE uniqueness.
Contradiction, since $w \not\equiv 0$ on (x_1, x_2).

$\therefore \tau_1 < 0$, giving instability.

Motivation for choice of trial fn:

U' lies in nullspace of L,
since
$$LU = (U')'' + f'(U)U'$$
$$= (U'' + f(U))' = 0.$$

"translational perturbation
$\Rightarrow$ zero mode".