Analysis on manifolds and applications to layer potentials

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Dedicated to the memory of Gabriela Kohr

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Introduction

These are the lecture notes that I will use for my five-hour mini-course at the conference. I plan to do “chalk and board” talks in the beginning, so I will not use these slides for all the lectures.

These notes contain more material then what I intend to present in my mini-course. In particular, the last section (the second part of the lecture notes) was not covered in the mini-course. (This second part contains an attempt to generalize the results of first part.)

At this time, these notes have not been revised carefully. In particular, not enough credit is given to other researchers, and I sincerely appologize for this omission. However, I include a very short (and extremely incomplete) reference section at the end of each of the two parts of the lecture notes.

I have **two main goals** for my mini-course:

- First, to give a very quick introduction to some of the main tools and concepts used in analysis on manifolds.
- Second, to use some of these tools to investigate the method of layer potentials on manifolds with cylindrical ends.

  Even if one is interested in domains in R^n, the method of layer potentials require us to work on a non-trivial manifold, namely the boundary of our domain.
- If time permits, I will state the well-posedness of the Poisson problem on manifolds with finite width.

The **table of contents** follows next:

- (Pseudo)differential operators on Lie manifolds
- Groupoids
- C^*-algebras
- Lie Groupoids
- (ISF) and the Fredholm conjecture
- Manifolds with cylindrical ends
- Manifolds with bounded geometry
- Well-posedness for the Poisson problem

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Motivation

- Even if one is interested in analysis on Euclidean spaces, one is lead to consider **(smooth) manifolds,** because they are the **boundaries** of smooth domains.
- Because certain domains are not smooth (i.e. do not have a smooth boundary), we are also lead to consider singular spaces. Example: domains with conical points (including polygons).
- A related example: **domains with conical points.** (Our method to study domains with conical points is to use domains with cylindrical ends.)
- More complicated domains, such as **polyhedral domains,** require the use of **Lie manifolds,** discussed in the last section.
- Very many researchers: Stein, Strichartz, Browder, ... (early works on analysis on non-compact manifolds), Dauge, Kondratiev, Melrose, Schulze, ...

Smooth manifolds

Definition

A **smooth manifold** M is a locally Euclidean, second countable Hausdorff space endowed with a smooth structure.

Thus $M = \cup_{\alpha \in I} U_\alpha$ is Hausdorff, second countable, with each U_α an open subset of M and there exist homeomorphisms $\phi_\alpha : U_\alpha \to W_\alpha \subset \mathbb{R}^n$, such that, for all $\alpha, \beta \in I$, the induced map

$$\phi_{\alpha\beta} := \phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is a **diffeomorphism.**

A pair (U_α, ϕ_α) is called a **coordinate chart** and the set $\{U_\alpha \mid \alpha \in I\}$ is called an **atlas** on M.

$f : M \to \mathbb{R}$ is **smooth** (i.e. in $\mathcal{C}^\infty(M)$) if all $f \circ \phi_\alpha^{-1} : W_\alpha \to \mathbb{R}$ are smooth. M **will always denote a smooth manifold.**

Remark on the definition of smooth manifolds

Some authors replace second countability by the weaker requirement that the manifold be a paracompact topological space. The second choice implies that every connected component is second countable.

Thus, a manifold in the second sense is a manifold in the first sense if, and only if, the set of connected components is countable. For instance, an **uncountable** set D with the discrete topology is not a manifold in the first sense, but it is a 0-dimensional manifold in the second sense.

For our results (and for most statements in differential geometry), it does not matter which definition we choose (second countable or paracompact).

Vector bundles

Let E, X be topological vector space together with a map $\pi : E \to X$.
For every $A \subset X$, we shall write $E|_A := \pi^{-1}(A) \subset E$.

Definition

(E, π) is a (real, topological) **vector bundle over X** if

- For every $p \in X$, the set $E_p := \pi^{-1}(p)$ is a real vector space;
- there exists an open covering $M = \cup_{\alpha \in I} U_\alpha$ and a **fiber-preserving*** homeomorphisms

$$\psi_\alpha : E|_{U_\alpha} \to U_\alpha \times \mathbb{R}^{n_\alpha}$$

  that is *linear* in each fiber.

* (i.e. $\psi_\alpha(E_q) = \{q\} \times \mathbb{R}^{n_\alpha}$, equivalently, $\pi_1(\psi_\alpha(x)) = \pi(x)$.)

Consequences

It follows that, for any two $\alpha, \beta \in X$, the resulting map

$$\psi_{\alpha\beta} := \psi_\alpha \circ \psi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^{n_\beta} \to (U_\alpha \cap U_\beta) \times \mathbb{R}^{n_\alpha}$$

is a homeomorphism linear in each fiber, thus of the form

$$\psi_{\alpha\beta}(x, v) = (x, A_{\alpha\beta}(x)v),$$

where

$$A_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(\mathbb{R}^{n_\beta}) := \mathrm{End}(\mathbb{R}^{n_\beta})^{-1}$$

is continuous (the famous "**transition functions**").

Alternatively, the continuous vector bundle can be defined starting from the transition functions, as long as they satisfy the **cocycle condition:**

$$A_{\alpha\beta}(x)A_{\beta\gamma}(x) = A_{\alpha\gamma}(x), \quad \text{for } x \in U_\alpha \cap U_\beta \cap U_\gamma .$$

Smooth vector bundles

Using the cocycle condition, we can identify

$$E \simeq \bigsqcup \Big(U_\alpha \times \mathbb{R}^{n_\alpha} \Big) / \sim$$

(disjoint union), where $(x, v) \sim (x, A_{\alpha\beta}(x)v)$ (with the first x in U_β and the second one in U_α and $v \in \mathbb{R}^{n_\beta}$).

Our vector bundles are thus **locally** trivial.

The simplest example of a vector bundle is the **trivial vector bundle**

$$E := X \times \mathbb{R}^n.$$

Then all $U_\alpha = X$ and the transition functions are the identity matrices.

Complex vector bundles are treated similarly.

Continuous sections of a vector bundle

Definition

Let (E, π) be a topological vector bundle over X. A (continuous) **section** of E is a continuous map $s : X \to E$ such that

$$\pi \circ s = id.$$

The last condition means that $s(x) \in E_x := \pi^{-1}(x)$, for all $x \in X$.

If $E = X \times \mathbb{R}^N$ (trivial of rank N), then a section $s : X \to E$ is simply an *n*-**uple** of continuous functions:

$$s(x) = (x, s_1(x), s_2(x), \ldots, s_n(x)),$$

$s_j : X \to \mathbb{R}$ continuous.

Smooth vector bundles

Definition

Let (E, π) be a vector bundle over X. Let us assume that X is a *smooth manifold* and that π and the transition functions $A_{\alpha\beta}$ are *smooth.* Then E is a **smooth vector bundle** (and a smooth manifold).

If $U \subset \mathbb{R}^n$ is an open subset, then it is a smooth manifold and

$$TU \simeq U \times \mathbb{R}^n .$$

In general, if M is a smooth manifold with coordinate charts (U_α, ϕ_α), then the differentials

$$D\phi_{\alpha\beta}(x) \in M_n(\mathbb{R})^{-1}$$

satisfy the cocycle condition and can be used to define a smooth (real) vector bundle $\pi : TM \to M$, called the **tangent bundle of M.**

The tangent bundle

Let M be a smooth manifold. Recall that a function $f : M \to \mathbb{R}$ is in $\mathcal{C}^\infty(M)$, iff $f \circ \phi_\alpha^{-1}$ is smooth on $W_\alpha := \phi_\alpha(U_\alpha)$.

An element $v \in T_pM = (TM)_p = \pi^{-1}(p)$ can be defined intrinsically as a **derivation** of $\mathcal{C}^\infty(M)$ at p, that is, a linear map $v : \mathcal{C}^\infty(M) \to \mathbb{R}$ s.t.

$$v(fg) \;=\; v(f)g(p) + f(p)v(g)\,.$$

Similarly, a smooth section X of TM can be identified with a derivation $X : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$. For instance, if $U \subset \mathbb{R}^n$ is an open subset, a section X of $TU = U \times \mathbb{R}^n$ is of the form $X = \sum_{k=1}^n X_k \frac{\partial}{\partial x_k}$, explicitly

$$Xu(x) \;=\; \sum_{k=1}^n X_k(x) \frac{\partial u}{\partial x_k}(x)\,.$$

Moreover, $[X, Y]u := X(Yu) - Y(Xu)$ **Lie bracket** is also a derivation (i.e. a vector field.)

Covariant derivatives

For reasons that will become apparent in applications, we want to work not just with smooth functions on M, but also with *smooth sections* $\mathcal{C}^\infty(E) = \mathcal{C}^\infty(M; E)$ of various smooth vector bundles $E \to M$.

Note that $\mathcal{C}^\infty(E)$ is a $\mathcal{C}^\infty(M)$-module.

The big question that we face then is how to **differentiate** the smooth sections of such a bundle **(not a unique solution).**

Definition

A **covariant derivative** $\nabla = \nabla^E$ on $E \to M$ is a bilinear map $\nabla = \nabla^E : \mathcal{C}^\infty(TM) \times \mathcal{C}^\infty(E) \to \mathcal{C}^\infty(E)$ s.t.

1. $\nabla_{fX}(\xi) = f\nabla_X(\xi)$
2. $\nabla_X(f\xi) = X(f)\xi + f\nabla_X\xi$ (Leibnitz's rule),

$\forall\, \xi \in \mathcal{C}^\infty(E), X \in \mathcal{C}^\infty(TM), f \in \mathcal{C}^\infty(M)$.

The local form of covariant derivatives I

Let us assume that $U \subset \mathbb{R}^n$ is an open subset and that $E = U \times \mathbb{R}^N$ is a trivial vector bundle.

A possible choice of a connection $\nabla : \mathcal{C}^\infty(TU) \times \mathcal{C}^\infty(E) \to \mathcal{C}^\infty(E)$ is

$$\nabla_X u \;=\; Xu \;:=\; \sum_{k=1}^{n} X_k \frac{\partial}{\partial x_k} u\,,$$

where $X = \sum_{k=1}^n X_k \frac{\partial}{\partial x_k}$ and the derivation is component-wise. That is

$$\nabla_{\frac{\partial}{\partial x_k}} u \;=\; \Big(\frac{\partial}{\partial x_k} u_1, \frac{\partial}{\partial x_k} u_2, \ldots, \frac{\partial}{\partial x_k} u_n \Big)\,,$$

where $u = (u_1, u_2, \ldots, u_n)$. **This is the trivial connection on the trivial vector bundle** $E = U \times \mathbb{R}^N$.

The local form of covariant derivatives II

The choice

$$\nabla_X u = Xu := \sum_{k=1}^{n} X_k \frac{\partial}{\partial x_k} u \,,$$

for the connection $\nabla : \mathcal{C}^\infty(TU) \times \mathcal{C}^\infty(E) \to \mathcal{C}^\infty(E)$ on the trivial bundle $E = U \times \mathbb{R}^N$ is, however, not unique, since, given any choice of smooth matrix functions $A_k \in \mathcal{C}^\infty(M; M_N)$, the formula

$$\nabla'_X u = Xu + \Big(\sum_{k=1}^{n} X_k A_k \Big) u := \sum_{k=1}^{n} X_k \Big(\frac{\partial}{\partial x_k} u + A_k u \Big) \,,$$

also defines a connection on $E \to U$.

These are **all connections** on E.

The local form of covariant derivatives III

We thus have

$$\nabla'_X u - \nabla_X u = \Big(\sum_{k=1}^{n} X_k A_k \Big) u =: \mathcal{A}(X) u \,,$$

In general, i.e., given **two covariant derivatives** ∇ and ∇' on any $E \to M$

$$\mathcal{A}(X) := \nabla'_X - \nabla_X = \mathcal{C}^\infty(M; \mathrm{End}(E)) \,,$$

We can regard then $\mathcal{A} \in \Omega^1(M; \mathrm{End}(E)) := \mathcal{C}^\infty(M; T^*M \otimes \mathrm{End}(E))$

and $\nabla = \nabla^E$ as a **differential operator**

$$\nabla : \mathcal{C}^\infty(E) \to \mathcal{C}^\infty(T^*M \otimes E) \,,$$

satisfying $\nabla_X u = \langle X, \nabla u \rangle$ and $\nabla(f\xi) = df \otimes \xi + f \nabla \xi$.

Operations with vector bundles

If $E, F \to M$ are two (smooth) vector bundles, we can define the (smooth) vector bundles

$$E^*, \ \mathrm{End}(E), \ \mathrm{Hom}(E; F), \ E \otimes F, \ \Lambda^k E \ \to \ M \,,$$

by defining their fibers

$$(E^*)_x \ := \ (E_x)^*, \ \mathrm{End}(E)_x \ := \ \mathrm{End}(E_x), \ \mathrm{Hom}(E; F)_x \ := \ \mathrm{Hom}(E_x; F_x), \ \ldots \,,$$

for any $x \in M$. Their topology (or smooth structure) is obtained from the cocycle construction.

Connections on E and F will define natural connections on these new (derived) bundles.

A **metric** g on E is a smooth section of $E^* \otimes E^* \simeq (E \otimes E)^*$ that is a *metric* on each fiber.

Connections on the new bundles

Recall: connection $\nabla = \nabla^E$ on $E \to M$ is a **differential operator**

$$\nabla : \mathcal{C}^\infty(E) \to \mathcal{C}^\infty(T^*M \otimes E) \quad \textbf{s.t.} \quad \nabla_X u = \langle X, \nabla u \rangle \,.$$

- **Connection on E^*:** For $\xi \in \mathcal{C}^\infty(E^*)$ and $\eta \in \mathcal{C}^\infty(E)$, we let

$$\langle \nabla_X \xi, \eta \rangle \;:=\; X\langle \xi, \eta \rangle - \langle \xi, \nabla_X \eta \rangle \,.$$

- **Connection on $E \otimes F$:** we consider the connection

$$\nabla^{E\otimes F} \;:=\; \nabla^E \otimes 1 + 1 \otimes \nabla^F ,$$

that is

$$\nabla_X^{E\otimes F}(\xi \otimes \eta) \;:=\; \nabla_X^E \xi \otimes \eta + \xi \otimes \nabla_X^F \eta \,.$$

Higher order covariant derivatives

We endow TM with the **Levi-Civita connection**

$$\nabla^M : \mathcal{C}^\infty(M; TM) \to \mathcal{C}^\infty(M; T^*M \otimes TM)\,.$$

This allows us to iterate:

$$\nabla^k : \mathcal{C}^\infty(E) \xrightarrow{\nabla} \mathcal{C}^\infty(T^*M \otimes E) \xrightarrow{\nabla} \mathcal{C}^\infty(T^*M \otimes T^*M \otimes E) \xrightarrow{\nabla} \mathcal{C}^\infty(T^{*\otimes k}M \otimes E)\,,$$

where endow each tensor bundle $T^{*\otimes k}M \otimes E$ with the induced (Levi-Civita) tensor product connection:

$$\nabla_X(\xi_1 \otimes \ldots \otimes \xi_k \otimes \eta) := \nabla^M_X(\xi_1) \otimes \ldots \otimes \eta + \xi_1 \otimes \nabla^M_X(\xi_2) \otimes \ldots \otimes \eta + \ldots + \xi_1 \otimes \xi_2 \otimes \ldots \otimes \xi_k \otimes \nabla^E_X(\eta)\,.$$

Metric preserving connections

These considerations will not be used in what follows.

Recall that a **metric** g on E is a smooth section of $E^* \otimes E^* \simeq (E \otimes E)^*$ that is a *metric* on each fiber. That is, g is a *smoothly varying* family of metrics g_x on $E_x := \pi^{-1}(x)$.

The connection ∇ is called **metric preserving** if, $\nabla g = 0$, explicitly

$$g(\nabla_X \xi, \eta) + g(\xi, \nabla_X \eta) \;=\; Xg(\xi, \eta)\,.$$

A **metric g on M** is a metric on TM, in which case M is called **Riemannian.**

(M, g) **will be a Riemannian manifold from now on.**

The Levi-Civita connection

These considerations will not be used in what follows.

The **Levi-Civita connection** ∇^M may be defined using **Koszul's formula:** If X, Y, Z are vector fields (i.e. in $\mathcal{C}^\infty(M; TM)$), then

$$\begin{aligned}2g(\nabla^M_X Y, Z) \,=\, & X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) \\ & - g([Y,X],Z) - g([X,Z],Y) - g([Y,Z],X)\,,\end{aligned}$$

It is the unique *metric preserving* connection on TM that is also *torsion-free,* in the sense that

$$\nabla^M_X Y - \nabla^M_Y X \,=\, [X,Y]$$

for all (smooth) vector fields $X, Y \in \mathcal{C}^\infty(M; TM)$.)

Sobolev spaces

Let $E \to M$ be a finite dimensional vector bundle with metric and metric preserving connection $\nabla = \nabla^E$. We can define then $L^p(M; E)$ with norm $\left(\int_M \|s\| dx \right)^{1/p}$ (ess-sup if $p = \infty$).

Definition

Let $k \in \mathbb{Z}_+$ and $p \in [1, \infty]$. Then

$$W^{k,p}(M; E) := \{u \in L^p(M; E) \mid \nabla^j u \in L^p(M; T^{*\otimes j} M \otimes E)\,,\ 1 \le j \le k\}/\equiv$$

is the *order* k, L^p*–type covariant Sobolev space* of sections of E.

$$\|u\|_{W^{k,p}(M;E)} := \begin{cases} \left(\sum_{j=0}^{k} \|\nabla^j(u)\|^p_{L^p(M;E\otimes T^{*\otimes j}M)} \right)^{1/p} & \text{if } p < \infty \\ \max_{j=0}^{k} \|\nabla^j(u)\|_{L^p(M;E\otimes T^{*\otimes j}M)} & \text{if } p = \infty \end{cases}$$

Newtonian potential

The goal is to solve the **Dirichlet boundary value problem**

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega\,. \end{cases}$$

To explain the layer potentials method, recall that the solution of

$$\Delta v = h \in \mathcal{C}_c(\mathbb{R}^n)$$

is given by the **Newtonian potential** (fundamental solution of Δ):

$$v(x) = \Delta^{-1} h(x) = c_n \int_{\mathbb{R}^n} |x - y|^{2-n} h(y) dy\,.$$

The method of layer potentials ($\mathcal{S}$)

The **method of the single layer potential** (for Δ) is to try to represent u in the form

$$u(x) := \Delta^{-1}(g \otimes \delta_{\partial\Omega}) = \int_{\partial\Omega} |x-y|^{2-n} g(y) d\sigma \,.$$

More precisely, let:

- $d\sigma$ is the **surface measure** on $\partial\Omega$.
- $\langle h, g \otimes \delta_{\partial\Omega} \rangle := \int_{\partial\Omega} h(x) g(x) d\sigma(x) \,.$

Then the **single layer potential** operator $\mathcal{S}$

$$\mathcal{S}g(x) := \Delta^{-1}(g \otimes \delta_{\partial\Omega}) = \int_{\partial\Omega} |x-y|^{2-n} g(y) d\sigma \,,$$

Harmonic function and boundary condition

The **first formula:**

$$\mathcal{S}g(x) := \Delta^{-1}(g \otimes \delta_{\partial\Omega})$$

gives that $\mathcal{S}g$ is **harmonic** inside Ω:

$$\Delta \mathcal{S}g(x) = \Delta(\Delta^{-1}(g \otimes \delta_{\partial\Omega}))(x) = g \otimes \delta_{\partial\Omega}(x) = 0 .$$

The **second formula:**

$$\mathcal{S}g(x) := c_n \int_{\partial\Omega} |x - y|^{2-n} g(y) d\sigma(y) , \quad x \in \Omega ,$$

for $x \to z \in \partial\Omega$ gives directly

$$\mathbf{S}g(z) := \mathcal{S}g|_{\partial\Omega}(z) := c_n \int_{\partial\Omega} |z - y|^{2-n} g(y) d\sigma(y) , \quad z \in \partial\Omega .$$

Boundary conditions

Let (as on the previous slide)

$$\mathbf{S}g(z) := \mathcal{S}g|_{\partial\Omega}(z) := c_n \int_{\partial\Omega} |z-y|^{2-n} g(y) d\sigma(y)\,, \quad z \in \partial\Omega\,,$$

(a “jump relation.”)

Summary of the single layer potential method

Let $g = \mathbf{S}^{-1} f$. Then $u := \mathcal{S}g$ solves the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega\,. \end{cases}$$

So we **it is enough to invert S**.

The second lecture

Double layer potential

We prefer the analogous **double layer potential** operator $\mathcal{D}$, namely:

$$\mathcal{D}h(x) := c_n \int_{\partial\Omega} \partial_{\nu_y} |x - y|^{2-n} h(y) d\sigma(y) = \Delta^{-1}(f \otimes \delta'_{\partial\Omega}),$$

Difference: if we pass to the limit $x \to z \in \partial\Omega$, we obtain a "jump relation" that has some additional terms:

$$\lim_{x \to z} \mathcal{D}h(x) = c_n \text{ p.v.} \int_{\partial\Omega} \partial_{\nu_y} |z - y|^{2-n} h(y) d\sigma(y) + \frac{h(z)}{2}$$
$$= \left(\mathbf{K} + \frac{1}{2}\right) h(z), \quad z \in \partial\Omega.$$

Summary of the double layer potential method

Our Dirichlet problem ($\Delta u = 0$ and $u = f$ on $\partial\Omega$) is reduced to

$$\left(\tfrac{1}{2} + \mathbf{K}\right) h = f.$$

Pseudodifferential operators

Recall that (up to a factor) **K** is the integral operator with kernel

$$\partial_{\nu_y} |z - y|^{2-n} = (2-n)\frac{(z-y)\cdot \nu_y}{|z-y|^n} .$$

Hence, if $\partial\Omega$ **is smooth**, then **K** is an **order** -1 **pseudodifferential operator.** (A similar fact holds for **S**, but will not be used.)

An order -1 pseudodifferential operator on a **compact** manifold is **compact.** Thus

Basic fact

If Ω is smooth and bounded, then **K** is **compact.**

Fredholm operators

Definition

Let $T : X \to Y$ be a continuous, linear map between *Banach spaces.* We say that T is **Fredholm** if $T^{-1}(0)$ and Y/TX are *finite dimensional.* Then its **index** is $\mathrm{ind}(T) := \dim\ker(T) - \dim(Y/TX)$.

Easy observation

An often used argument

If T is **Fredholm of index zero** and **injective,** then it is **an isomorphism** (invertible).

We obtain that $\frac{1}{2} + \mathbf{K}$ is a **Fredholm** operator.

Well-posedness of the Dirichlet problem

We can now prove the Hadamard well posedness (solvability) of our Dirichlet problem ($\Delta u = 0$ and $u|_{\partial\Omega} = f$) if Ω is **smooth and bounded.**

Recall that it is enough to invert $\frac{1}{2} + \mathbf{K}$, since our solution is

$$u = \mathcal{D}\left(\left(\frac{1}{2} + \mathbf{K}\right)^{-1} f\right).$$

The proof of the invertibility of $\frac{1}{2} + \mathbf{K}$ is as follows:

Invertibility of $\frac{1}{2} + \mathbf{K}$ and solvability for Ω smooth, bounded

- Since **K** is of order -1, we know that it is **compact**, and hence $P := \frac{1}{2} + \mathbf{K}$ is **Fredholm of index** $:= \dim T^{-1}0 - \dim T^{*-1}0 = 0.$
- Energy estimates ($\Delta u = 0$ and $u \in H_0^1(\Omega)$ imply $u = 0$) and the restriction to the boundary (or "jump") relations show that P has **zero kernel.**
- Hence P is **invertible** and the solution to our Dirichlet problem is
$$u(x) := \mathcal{D}\left(P^{-1}(f)\right) = \mathcal{D}\left(\left(\frac{1}{2} + \mathbf{K}\right)^{-1}(f)\right).$$

Indeed, we already noticed that u is **harmonic.** Then

$$u|_{\partial\Omega} := \mathcal{D}\left(\left(\frac{1}{2} + \mathbf{K}\right)^{-1}(f)\right)|_{\partial\Omega} = \left(\frac{1}{2} + \mathbf{K}\right)\left(\frac{1}{2} + \mathbf{K}\right)^{-1}(f) = f$$

Extensions

This completes our discussion of layer potentials on a **smooth, bounded domain** Ω.

If Ω is not smooth *or* not bounded, **K** may **no longer be compact.**

Many further results and extensions: **S. Chandler-Wilde, M. Costabel, M. Dauge, M. Kohr, M. Lanza de Cristoforis, V. Mazya, S.E. Mikhailov, M. Putinar, W.L. Wendland, . . .**

We want to extend these results to the case when Ω is a **domain with cylindrical ends** (inside some manifold). (Outlets, wave-guides, tubes **(David's talk :-)** ...)

Extensions to manifolds: D. & I. & M. Mitrea and M. Taylor.

The Mitrea-Taylor "trick"

Let us assume that $\Omega \subset M =$ a smooth manifold.

The first issue: $\Delta := d^*d$ may no longer be invertible on M.

The **Mitrea-Taylor "trick" is to replace** Δ **with** $\Delta + V$, where V is a suitable potential and then, to replace Δ^{-1} (which may not exist) with $(\Delta + V)^{-1}$ (after proving that the later exists).

Then

$$\mathcal{S}f(x) := (\Delta + V)^{-1}(f \otimes \delta_{\partial\Omega}) \quad \text{and}$$
$$\mathcal{D}f(x) := (\Delta + V)^{-1}(f \otimes \delta'_{\partial\Omega}) .$$

How does the "trick" work?

Let $V \in \mathcal{C}^\infty(M)$ satisfy:

- $V \geq 0$ on M and $V = 0$ on Ω,
- V is not identically 0 on M.

Assume M to be **smooth, compact, and connected**, then:

- $\Delta + V \geq 0$ is **Fredholm of index zero.**
- If $(\Delta + V)u = 0$, then

$$0 = ((\Delta + V)u, u) = (du, du) + (Vu, u) ,$$

  and hence $Vu = 0$ and $du = 0$. Since $V \neq 0$, $u = 0$ **(L^2-unique continuation property)**. We obtain that $\Delta + V$ is **injective.**
- Consequently, $\Delta + V$ is **invertible.**

The (proved) invertibility of $\Delta + V \geq 0$ allows us to define the layer potential operators

$$\mathcal{S}f(x) := (\Delta + V)^{-1}(f \otimes \delta_{\partial\Omega}) \quad \text{and}$$
$$\mathcal{D}f(x) := (\Delta + V)^{-1}(f \otimes \delta'_{\partial\Omega}) .$$

Let us notice that, because $V = 0$ **inside** Ω, we obtain

$$\Delta\mathcal{D}(f) = (\Delta + V)(\Delta + V)^{-1}(f \otimes \delta'_{\partial\Omega}) = f \otimes \delta'_{\partial\Omega} = 0 ,$$

and hence $\mathcal{D}(f)$ (and, similarly, $\mathcal{S}(f)$) are again **harmonic** in Ω, in spite of us having introduced the potential V.

The proof that $\frac{1}{2} + \mathbf{K}$ is invertible is then **the same.**

This then gives the **the well-posedness of the Dirichlet problem.**

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Having explained the Mitrea-Taylor trick for compact manifolds, we now want to extend it to **manifolds with cylindrical ends.** This will take most of the rest of the first part of the lecture notes.

More precisely,

- We want to **first invert** $\Delta + V$ (the Mitrea-Taylor trick).
- This allows us to define the layer potential operators. (For instance, recall $\mathcal{D}f := (\Delta + V)^{-1}f$.)
- The jump relations extend immediately (they are the same).
- We then want to show that $\frac{1}{2} + \mathbf{K}$ is **also invertible:** another use of the Mitrea-Taylor trick.

The M.-T. trick for cylindrical ends

In order to **invert** $\Delta + V$ and then $\frac{1}{2} + \mathbf{K}$, we will need:

- To establish that our layer potential operators **K** and **S** are "nice" pseudodifferential operators (essentially translation invariant at infinity).
- The **Fredholm property** for these "nice" pseudodifferential operators.
- The jump relations.
- The L^2-unique continuation property.

This will be done in the following (several) slides.

(Additional results were given in Mirela Kohr's talk.)

Outline of this section

Pseudodifferential operators and cylindrical ends

- In **this section**, our main interest is in:
  - closed manifolds (smooth, compact, without boundary) **(0)** and
  - manifolds with cylindrical ends **(1)**.
- In the **second part,** more general classes of manifolds, incl.:
  - conformally compact manifolds **(2)**;
  - Euclidean spaces **(3)**.

Motivation

- Let D be a **differential** operator some manifold M.
- Assume $D =$ invertible between certain Sobolev spaces.
- its inverse D^{-1} will **not be a differential operator.** (we assume D is not a multiplication operator).

$D^{-1} =$ a **pseudodifferential op.** of order $= -$order(D) (Beals).

The pseudodifferential operators generalize:

- the differential,
- the convolution, and
- the regularizing operators (order $-\infty$).

The main formula of pseudodifferential theory

The **MAIN FORMULA** of **pseudodifferential theory** is:

$$a(x,D)u(x) \;:=\; \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{\imath\langle x,\xi\rangle}\, a(x,\xi)\hat{u}(\xi)\, d\xi\,,$$

(**pseudodifferential operator with symbol** $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$.)

$a(x)$ is a multiplication operator and $a(D)$ is a convolution operator.

Here $\langle x,\xi\rangle := x_1\xi_1 + x_2\xi_2 + \ldots x_n\xi_n$ and

$$\hat{u}(\xi) \;=\; \mathcal{F}u(\xi) \;:=\; \int_{\mathbb{R}} e^{-\imath\langle y,\xi\rangle}\, u(y)\, dy\,.$$

is the **Fourier transform** of u.

Symbols and pseudodifferential operators on $\mathbb{R}^n$

Question: Which "symbols" $a(x,\xi)$ should we allow?

Order m, (1,0)-symbols, Hörmander:

$$S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n) := \{a : \mathbb{R}^{2n} \to \mathbb{C} \mid \ \forall \alpha, \beta, \ \exists C_{\alpha,\beta} \geq 0$$
$$\text{s.t.} \quad |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|}\}\,.$$

Theorem

If $a \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, then it defines a continuous map

$$a(x,D) : \mathcal{C}_c^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^n)\,.^a$$

[a] (Schwartz' kernel theorem)

The third lecture

The distribution kernel of $a(x, D)$

Theorem

The distribution kernel $k_{a(x,D)}$ of $a(x, D)$ is:

$$\text{“}k_{a(x,D)}(x, y)\text{”} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{\imath\langle x-y,\xi\rangle} a(x, \xi)\, d\xi = (\mathcal{F}_\xi^{-1} a)(x, x-y)\,.$$

Fourier inversion gives $a(x, \xi)$ in terms of $k_{a(x,D)}(x, y)$.

If $\partial\Omega$ is smooth, this proves that **S** and **K** are **pseudodifferential operators of order** $k = -1$ on $\partial\Omega$, because they have distribution kernels

$$k_{\mathbf{S}}(x, y) := \frac{c_n}{|x-y|^{2-n}} \quad \text{and} \quad k_{\mathbf{K}}(x, y) := \frac{c'_n\langle x-y, \nu_y\rangle}{|x-y|^{2-n}}\,,$$

which are (essentially) homogeneous of order

$$-(n-1) - (-1) = -\dim(\partial\Omega) - k\,.$$

Product and diffeomorphism invariance

Theorem

Let $a \in S^{m}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ *and* $b \in S^{m'}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$. *Then* $ab \in S^{m+m'}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ *and there exists* $c \in S^{m+m'-1}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ *s. t.*

$$a(x, D)b(x, D) = (ab)(x, D) + c(x, D)\,.$$

Localization: Let $U \subset \mathbb{R}^n$ be open and

$$\Psi^m_c(U) := \{a(x, D) \mid \operatorname{supp} k_{a(x,D)} \Subset U \times U\}\,.$$

Theorem

For $U \subset \mathbb{R}^n$ *open,* $\Psi^\infty_c(U) := \cup_{m \in \mathbb{Z}} \Psi^m_c(U)$ *is a filtered algebra invariant under diffeomorphisms.*

Principal symbol

Recall:

- $\Psi_c^m(U) := \{a(x,D) \mid \text{supp}\,(k_{a(x,D)})$ compact $\subset U \times U\}$ and
- $a(x,D)b(x,D) = ab(x,D) +$ **a lower order operator.**

Let $S_c^m(T^*U) := \{a \in S_{1,0}^m(\mathbb{R}^n \times \mathbb{R}^n) \mid \text{supp}(a) \subset K \times \mathbb{R}^n,\ K \Subset U\}$.

Theorem

If $a(x,D) \in \Psi_c^m(U)$, then the **principal symbol**

$$\sigma_m(a(x,D)) := a + S_c^{m-1}(T^*U) \in S_c^m(T^*U)/S^{m-1}(T^*U)$$

is ***well-defined, multiplicative,*** *and* ***diffeomorphism invariant.***

Pseudodifferential operators on manifolds

Let $\phi : U \simeq W \subset \mathbb{R}^n$, $U \subset M$ open (arbitrary $\phi =$coordinate chart).

Diffeomorphism invariance gives

$$\Psi_c^m(U) := \phi_*^{-1}\Psi_c^m(W)$$

Definition

A linear map $P : \mathcal{C}_c^\infty(M) \to \mathcal{C}^\infty(M)$ is a **pseudodifferential operator** of order $\leq m$ on M if, for any $U \subset M$ as above and $\eta \in \mathcal{C}_c^\infty(U)$, we have

$$\eta P \eta \in \Psi_c^m(U).$$

Let $\Psi^m(M)$ =set of order $\leq m$ **pseudodifferential operators** on M.

For instance $\Delta^{-1} = a(D) \in \Psi^{-2}(\mathbb{R}^n)$, although $a(\xi) := -|\xi|^{-1}$ is **not** in $S_{1,0}^{-2}(\mathbb{R}^n)$ (it has the *right decay* in ξ, but is *not smooth*).

Examples and properties

Recall $\Psi^m(M)$ =order m **pseudodifferential operators** on M.

If $\partial\Omega$ is smooth, them $\mathbf{S}, \mathbf{K} \in \Psi^{-1}(\partial\Omega)$ since their distribution kernels have the *right asymptotic developement* at the diagonal and are *smooth outside the diagonal.*

Theorem

The principal symbol gives ***surjective, diffeomorphism invariant maps***

$$\sigma_m : \Psi^m(M) \to S^m_{1,0}/S^{m-1}_{1,0}(T^*M)\,.$$

Definition

$P \in \Psi^m(M)$ is **elliptic** if $\sigma_m(P)$ is **invertible** (mod. lower order ops.).

Theorem

Let $a \in S^m(T^*M)$. The following are equivalent

1. *There exist $b \in S^{-m}(T^*M)$ such that $ab - 1 \in S^{-1}(T^*M)$*
2. *There are $C, R > 0$ such that $|a(\xi)| \geq C|\xi|^m$ for $\xi \in T^*M$, $|\xi| \geq R$.*

Then a is called **elliptic** *and this property is a property of its class in $S^m(T^*M)/S^{m-1}(T^*M)$.*

Properties of psdos on manifolds

1. If $P \in \Psi^m(M)$, then $P : \mathcal{C}_c^\infty(M) \to \mathcal{C}^\infty(M)$ **continuously,** if $P, Q \in \Psi^\infty(M)$, the product PQ may **not be defined.**

   (It is enough for them to have distribution kernels **supported "close" to the diagonal** i.e **properly supported**.)

2. $\Psi^m(M)$ contains all **differential operators** of order m.

3. $\Psi^m(M) \subset \Psi^{m'}(M)$ for $m < m'$ and $\Psi^{-\infty}(M) := \cap_m \Psi^m(M) =$all operators with **smooth kernel.**

4. k_P is **smooth away from the diagonal.**

Properties of psdos on closed manifolds

Assume $M =$ **closed, then we can compose any two pseudos!**

We can easily include **vector bundles.**

1. If $P \in \Psi^m(M; E)$, then $P : H^s(M; E) \to H^{s-m}(M; E)$ is **bounded.**
2. If $P \in \Psi^q(M; E)$, $q < 0$, then $P : H^s(M; E) \to H^s(M; E)$ is **compact.**
3. If $P \in \Psi^m(M; E)$ is **elliptic,** then, for any $s \in \mathbb{R}$,
$$P : H^s(M; E) \to H^{s-m}(M; E) \quad \text{is} \quad \textbf{Fredholm.}$$

The proof of the last result is based on Atkinson's theorem (see next slide).

Proof of 'elliptic $\Rightarrow$ Fredholm'

Theorem (Atkinson's theorem)

*Let X and Y be Banach spaces; $T \in \mathcal{B}(X; Y)$ is **Fredholm** iff there exists $Q \in \mathcal{B}(Y; X)$ such that $TQ - 1_Y$ and $QT - 1_X$ are **compact.***

Proof.

s[Proof of 'elliptic $\Rightarrow$ Fredholm'] • P is elliptic means that there is $b \in S^{-m}(T^*M)$ such that $\sigma_m(P)b - 1 \in S^{-1}(T^*M)$.

• The surjectivity of σ_{-m} and its multiplicativity give that there exists $Q \in \Psi^{-m}(M)$ (**parametrix**) such that $PQ - 1, QP - 1 \in \Psi^{-1}(M)$.

• Consequently, $PQ - 1$ and $QP - 1$ are compact operators. Atkinson's theorem then gives that P is Fredholm. □

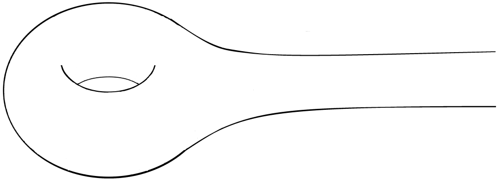
The converse is true (Fredholm for one s implies elliptic).

The last two lectures

“Pictorial” definition of cylindrical ends

We want to extend the Fredholmness result to **manifolds with cylindrical ends** M. (Additional ingredient **limit operator.**)

- Let $\overline{M}$ be a smooth manifold with **boundary** $\partial\overline{M}$.
- To $\overline{M}$ we attach **the cylindrical end** $\partial\overline{M} \times (-\infty, 0]$ along $\partial\overline{M}$.



- Thus $M := \overline{M} \cup (\partial\overline{M} \times (-\infty, 0])$.

Translation invariant operators near infinity

Consider the **partial translations** (isometries)

$$\Phi_s(x,t) := (x, t-s), \quad s \geq 0$$
$$\Phi_s : \partial\overline{M} \times (-\infty, 0] \to \partial\overline{M} \times (-\infty, -s].$$

Definition

$P : C_c^\infty(M) \to C_c^\infty(M)$ is called **translation invariant at infinity** if

- its distribution kernel is supported in a neighborhood of the diagonal $\{(x,y) \in M \times M : \mathrm{dist}(x,y) < \varepsilon\}$ for some $\varepsilon > 0$, and
- there is $R > 0$ s.t. if $\mathrm{supp}(f) \subset \partial\overline{M} \times (-\infty, -R)$ and $s > 0$,

$$P\Phi_s(f) = \Phi_s P(f).$$

Equivalently: $k_P(t, x, t', x') = k_P(t-s, x, t'-s, x')$, if $t, t' << 0$.

Psdos translation invariant near infinity

Definition

Let $\Psi^m_{\text{inv}}(M)$ be the space of (classical) pseudodifferential operators P of order $\leq m$ that are **translation invariant in a neighborhood of infinity.**

(It is contained in the b-calculus of Melrose and Schulze.)

As usual:

- $\Psi^\infty_{\text{inv}}(M) := \bigcup_{m\in\mathbb{Z}} \Psi^m_{\text{inv}}(M)$.
- We can also introduce *vector bundles* $E, F \to M$ to obtain

$$\Psi^m_{\text{inv}}(M; E, F), \ldots$$

Almost translation invariant operators at infinity

Drawback: $\Psi^{\infty}_{\mathrm{inv}}(M)$ **is not stable under inversion.**

Definition

$\Psi^{\infty}_{\mathrm{ess}}(M)$= *operators essentially translation invariant:*

- $\Psi^{m}_{\mathrm{ess}}(M) := \Psi^{m}_{\mathrm{inv}}(M) + \Psi^{-\infty}_{\mathrm{ess}}(M)$
- $\Psi^{-\infty}_{\mathrm{ess}}(M) =$ **suitable closure** of $\Psi^{-\infty}_{\mathrm{inv}}(M)$ (w.r.t the norms $\|\cdot\|_{m',m}$ of linear operators $H^{m'}(M) \to H^{m}(M)$, $m', m \in 2\mathbb{Z}$ **and** with respect to the norms $\|ad_x^k(P)\|_{m',m}$.)

$\Psi^{\infty}_{\mathrm{ess}}(M)$ is larger than $\Psi^{\infty}_{\mathrm{inv}}(M)$ and **stable under inversion** (next). (It is smaller than the c-calculus: Mazzeo-Melrose, Cipriana Anghel, ...)

Spectral invariance

- $\Psi^{\infty}_{\text{inv}}(M)$ is a subset of the **b-calculus**, but $\Psi^{\infty}_{\text{ess}}(M)$ is not.
- Our definition is simple and our algebra is (**stable under inversion**).

Theorem (Kohr-Mitrea-V.N.)

Let $T \in \Psi^{m}_{\text{ess}}(M; E, F)$ $m \geq 0$, be such that T is **invertible** *as a (possibly unbounded*[a]*) operator on $L^2(M; E)$. If $m > 0$, we assume also that T is* **elliptic**. *Then $T^{-1} \in \Psi^{-m}_{\text{ess}}(M; F, E)$.*

[a] T is **invertible** as an unbounded operator if T is injective and T^{-1} extends to a bounded operator.

Specific feature: limit operator

Definition

Let $P \in \Psi^{\infty}_{\rm inv}(M)$. The **limit (or normal) operator** associated to P is the unique pseudodifferential operator $\widetilde{P} : C_c^{\infty}(\partial\overline{M} \times \mathbb{R}) \to C_c^{\infty}(\partial\overline{M} \times \mathbb{R})$ s.t.

$$\widetilde{P}(f) \,:=\, \Phi_{-s} P \Phi_s(f)\,,$$

for s **large enough** and Φ_s is the translation by s on $\partial\overline{M} \times \mathbb{R}$.

$$k_{\widetilde{P}}(t, x, t', x') := \lim_{s\to\infty} k_P(t - s, x, t' - s, x') = k_P(t - s_{Lrg}, x, t' - s_{Lrg}, x')\,.$$

- $\widetilde{P}$ is **well-defined**: $\Phi_{-s} P \Phi_s(f)$ is independent of s as above.
- $\widetilde{P}$ is **pseudodifferential** and **translation invariant.**

Fredholm conditions

Next theorem: generalization of the result on closed manifolds.

Theorem (Kondratiev, Melrose and Mendoza, Kohr-Mitrea-Nistor)

Let $M = \overline{M} \cup (\partial\overline{M} \times (-\infty, 0])$ *be a manifold with cylindrical and* $P \in \Psi^m_{ess}(M; E, F)$. *Then* $P : H^s(M; E) \to H^{s-m}(M; F)$ *is a Fredholm operator if, and only if,*

- *it is* **elliptic** *and*
- *its* **limit operator**

$$\widetilde{P} : H^s(\partial\overline{M} \times \mathbb{R}; E) \to H^{s-m}(\partial\overline{M} \times \mathbb{R}; F)$$

is invertible.

Also, including many generalizations: Anghel, Kondratiev, Mazzeo-Melrose, Lauter-Moroianu, Schrohe, Schulze, Kohr-V.N., ...

“Pictorial” definition of domains with cylindrical ends

N has a boundary and M does not (both have straight cylindrical ends)

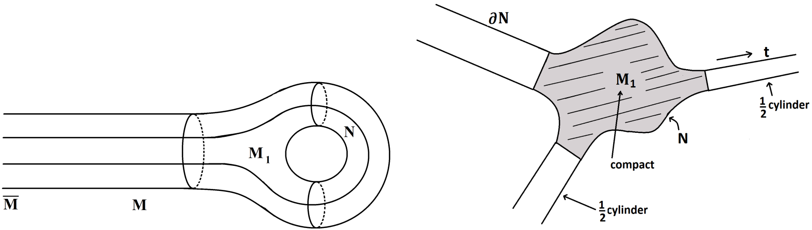


Figure: Manifolds with boundary (or domains) with cylindrical end

The domains with cylindrical ends in $\mathbb{R}^n$ are called **“with outlets.”**

Operators on manifolds with cylindrical ends

- $\Delta = d^*d =$ **Laplace-Beltrami op.** on $M = \overline{M} \cup \partial\overline{M} \times (-\infty, 0]$.
- Then

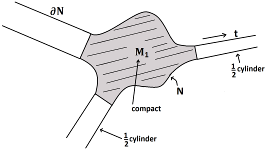
$$\Delta = \Delta_M = -\partial_t^2 + \Delta_{\partial\overline{M}} \text{ on cylindrical end } \partial\overline{M} \times (-\infty, 0]\,,$$

is translation invariant (in a nbhd of infinity) **v important.**

- To construct the **layer potential operators $\mathcal{S}$, $\mathcal{D}$, S, and K, in the case with boundary,** we need to **"invert"** Δ (or a suitable modification: Mitrea-Taylor "trick.")
- Recall that **inverting** Δ is a substitute for $|x-y|^{2-n}$ (i.e. the convolution with the fundamental solution $c_n|x|^{2-n}$ of the Laplacian, **which is not possible directly.**)

Doubling and “inverting” Δ

- To construct the layer potential operators on $\Omega = N$ **with boundary and cylindrical ends** (similar definition, see fig.) we include an additional function V (the Mitrea-Taylor ‘trick’).



- More precisely, we **double** Ω (to obtain M) by gluing two copies of Ω along the common boundary and we replace Δ with $\Delta + V$, where V vanishes on N, but not at infinity.
- The limit (jump) properties remain true (this is easy).
- We prove that $\Delta + V$ and then $\frac{1}{2} + \mathbf{K}$ are still Fredholm of index zero and injective, hence **isomorphisms** (using Fredholm and unique continuation properties, Kohr-Mitrea-V.N., discussed next).

Recall how does the “trick” work

Let $V \in \mathcal{C}^\infty(M)$ satisfy:

- $V \geq 0$ on M and $V = 0$ on Ω,
- V is not identically 0 on M (or at the “infinity” of M).

Assume M to be **smooth, compact, and connected**, then:

- $\Delta + V \geq 0$ is **Fredholm of index zero.**
- If $(\Delta + V)u = 0$, then

$$0 = ((\Delta + V)u, u) = (du, du) + (Vu, u),$$

  and hence $Vu = 0$ and $du = 0$. Since $V \not\equiv 0$, $u = 0$ **(L^2-unique continuation property)**. We obtain that $\Delta + V$ is **injective.**
- Consequently, $\Delta + V$ is **invertible** and $(\Delta + V)^{-1} \in \Psi_{ess}^{-2}(M)$

Domains with cylindrical ends: **delicate or different issues** in **red**.

Main result

Theorem (Mitrea-V.N., Kohr-V.N.)

Let $V \geq 0$ be non-zero and translation invariant at infinity and $\Omega \subset \mathbb{R}^n$ be a domain with outlets (cylindrical ends).

- *$\Delta + V$ is invertible.*
- *Let $\mathbf{K}$ be the double layer potential operator associated to the $\Delta + V$. Then $\frac{1}{2} + \mathbf{K}$ is invertible and hence the Dirichlet problem on Ω is well posed.*

We need to consider **suitable classes of (pseudodifferential) operators** that take into account the (translation invariant) structure of the Laplacian on manifolds with cylindrical ends.

Proof of the main result

The proof of our main result (the invertibility of $\Delta + V$ and of $\frac{1}{2} + \mathbf{K}$) is done then as in the compact case (outlined for $\Delta + V$)

- **Fredholm property**
- Properties of $\Psi^m_{ess}(M; E, F)$.
- Jump relations.

Moreover,

$$(\Delta + V)^{-1} \text{ and } \mathbf{K}$$

are **essentially translation invariant** (hence also $(\frac{1}{2} + \mathbf{K})^{-1}$).

Mirela Kohr: generalization to **Stokes** operator. (Joint w. Wendland.)

Similarities, but also **essential differences.** (The Neumann problem on domains with conical points behaves **quite differently** from the Dirichlet problem, Costabel, Dauge, Mazya, Nicaise, V.N., ...)

Conclusion

Although the double layer potential operator **K** is **no longer compact** on a manifold with cylindrical ends, we still obtain the invertibility of $\frac{1}{2} + \mathbf{K}$ and hence **the well-posedness of the Dirichlet problem.**

We also obtain the structure of the operators **K** and $(\frac{1}{2} + \mathbf{K})^{-1}$.

Thank you for your attention !

The material beginning with next slide were not covered in the Padova course

A very incomplete bibliography for the first part

- Dalla Riva, M., Lanza de Cristoforis, M., and Musolino, P. *Singularly Perturbed Boundary Value Problems. A Functional Analytic Approach*. 2021.
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- Kohr, M., and Nistor, V. Sobolev spaces and ∇-differential operators on manifolds I: basic properties and weighted spaces. *Annals of Global Analysis and Geometry 61* (2022), 721–758.
- Kohr, M., Nistor, V., and Wendland, W. Layer potentials and essentially translation invariant pseudodifferential operators on manifolds with cylindrical ends. ArXiv:2308.06308v3, to appear.
- Petersen, P. *Riemannian geometry*, 2nd ed., vol. 171 of *Grad. Texts Math.* New York, NY: Springer, 2006.

The second part of the lectures notes

(Not covered in the mini-course.)

More on Sobolev spaces II

- If M is **not compact** the Sobolev spaces $W^{m,p}$ **depend** on the choice of connection on E and on the choice of metric g on M, **global objects.** (Joint with Mirela Kohr.)
- If M is *complete,* we can define the space $H^s(M; E) := W^{s,2}(M; E)$ as the domain of $(1 - \Delta_g)^{s/2}$, where $\Delta_s := -\nabla^*\nabla$ (Bochner Laplacian).
- For $s > 0$ real, we define $H^s(M; E)$ by interpolation.
- The case $s < 0$, for **manifold with boundary and relative bounded geometry** (later).

Differential operators

Let $E, F \to M$ be vector bundles, with E endowed with a connection.

Definition

Let $a^{[k]}$ be measurable section of $\mathrm{Hom}(T^{*\otimes k} \otimes E; F)$ and $\nabla^0 = id$. A **∇–differential operator** is a map of the form

$$P = a \cdot \nabla^{tot} := \sum_{j=0}^{\mu} a^{[j]} \nabla^j : \mathcal{C}^\infty(M; E) \to \mathcal{C}^\infty(M; F) .$$

$\mathrm{ord}(P)$ is the least μ for which such a writing exists (the *order* of P).

Suitable extensions by continuity of P will also be called ∇–differential operators and will be denoted by the same letter.

Locally, there is no difference between the ∇–differential operators and the usual differential operators.

Mapping properties of differential operators

Let $E, F \to M$ be vector bundles with metrics and metric-compatible with connections.

Theorem

Let $\ell \in \mathbb{Z}_+$ and $a^{[k]} \in W^{\ell,\infty}(M; \mathrm{Hom}(T^{*\otimes k} \otimes E; F))$ and $\nabla^0 = id$. Then the ∇–differential operator

$$P = a \cdot \nabla^{tot} := \sum_{j=0}^{\mu} a^{[j]} \nabla^j : W^{s+\mu,p}(M; E) \to W^{s,p}(M; F) .$$

for $0 \leq s \leq \ell$. **(Operator with coefficients in $W^{\ell,\infty}$.)**

Better results (including $|s| \leq \ell$) for **manifolds with bounded geometry,** but first the simpler particular case of manifolds with *cylindrical ends.*

Curvature

The **curvature** R^E of (E, ∇) is the **anti-symmetric** part of ∇^2 (i.e. the composite map)

$$\nabla^2 : \mathcal{C}^\infty(E) \to \mathcal{C}^\infty(T^{*\otimes 2}M \otimes E) \to \mathcal{C}^\infty(\Lambda^2 T^* M \otimes E) .$$

That is,

$$R^E(X, Y)u := \langle \nabla^2 u, X \wedge Y \rangle .$$

The curvature of (M, g) is $R = R^{TM}$, the curvature of $E = TM$.

Curvature

It turns out that the curvature R^E is $C^\infty(M)$**–linear,** and hence

$$R^E \in \Omega^2(M; \mathrm{End}(TM)) = \mathcal{C}^\infty(\Lambda^2 T^* M \otimes \mathrm{End}(TM))$$

Explicitly,

$$\nabla^2 u(X, Y) = \nabla_X \nabla_Y u - \nabla_{\nabla_X Y} u \, ,$$

and hence the **curvature** R^E of (E, ∇) is given by

$$R^E(X, Y)u = \nabla_X \nabla_Y u - \nabla_Y \nabla_X u - \nabla_{[X,Y]} u \, .$$

Definition

We say that M has **totally bounded curvature** if its curvatures $R^M := (\nabla^M)^2$ and all its covariant derivatives $(\nabla^M)^k R^M$ are bounded.

Injectivity radius and geodesics

A C^1-curve $\gamma : (a, b) \to M$ is a **geodesic** if $\nabla^M_{\gamma'(t)}\gamma'(t) = 0$. It is locally distance minimizing and uniquely determined by any $\gamma'(t_0),\ t_0 \in I$. If $M = \mathbb{R}^d$ with the usual metric, then a geodesic is just a straight line.

Let

$$\exp^M(v) \ := \ \gamma_v(1),$$

where γ_v is the unique geodesic with $\gamma'(0) = v$, the **(geodesic) exponential map.**

Let $B_r^X(x) \ := \ \{y \in X \mid d(x, y) < r\}$, and

$$r_{\mathrm{inj}}^M(p) \ := \ \sup\{r \mid \exp^M\colon B_r^{T_pM}(0) \to B_r^M(p) \text{ is a diffeomorphism}\} \quad \text{and}$$

$$r_{\mathrm{inj}}(M) \ := \ \inf_{p\in M} r_{\mathrm{inj}}^M(p).$$

$r_{\mathrm{inj}}(M)$ =the **injectivity radius** of the Riemannian manifold (M, g).

Manifolds with bounded geometry

Recall the curvature $R^M(X,Y)u = \nabla^M_X \nabla^M_Y u - \nabla^M_Y \nabla^M_X u - \nabla^M_{[X,Y]} u$ and that M has **totally bounded curvature** if its curvatures $R^M := (\nabla^M)^2$ and all its covariant derivatives $(\nabla^M)^k R^M$ are bounded.

The following concept is crucial.

Definition

A smooth Riemannian manifold (M, g) is said to have **bounded geometry** if M has totally bounded curvature and $rinj(M) > 0$.

Then $H^{-s}(M) := H^{-s}(M)^*$. (**Negative order spaces.**)

Examples

The following are manifolds with bounded geometry:

(i) A closed manifold (i.e. a smooth, compact manifold without boundary).

(ii) $M = \mathbb{R}^d$ with the standard (Euclidean) metric $dx^2 := (dx_1)^2 + (dx_2)^2 + \ldots + (dx_d)^2$.

(iii) $D \times M$, where M is a manifold with bounded geometry and D is discrete set.

(iv) $M_1 \times M_2$, where M_1 and M_2 have bounded geometry.

(v) However, if M is a manifold with bounded geometry and $U \subset M$ is an open subset, then U has totally bounded curvature, but may not have positive injectivity radius (so no bounded geometry).

Boundary and bounded geometry

Let now M_0 be a Riemannian manifold *with boundary,* then $r_{\mathrm{inj}}(M_0) = 0,$ so a manifold with non-empty boundary will never have bounded geometry in the sense of the above definition. The way around this conundrum was found by Schick (2001), who has defined the concept of "manifold with boundary and bounded geometry," (we shall call these manifolds "manifolds with boundary and *relative* bounded geometry," to avoid confusions). We recall the equivalent definition of manifolds with boundary and relative bounded geometry in (Ammann-Grosse-V.N.)

The main point of that definition is to assume that the boundary ∂M_0 of M_0 is a suitable submanifold of a (boundaryless) manifold M with bounded geometry.

Second fundamental form

Let hence M be a (boundaryless) manifold with bounded geometry and let us consider a hypersurface $H \subset M$, that is, a submanifold H of M of codimension $\dim(M) - \dim(H) = 1$. We assume that H carries a globally defined unit normal vector field ν. We let

$$\exp^{\perp}(x, t) \ := \ \exp_x^M(t\nu_x)$$

be the exponential in the direction of the chosen unit normal vector. We shall need the **second fundamental form** II^H of H in M, which, we recall, is defined by

$$\mathrm{II}^H(X, Y)\nu \ := \ \nabla_X^M Y - \nabla_X^H Y,$$

where ∇^Z is the Levi-Civita connection of Z. Equivalently, since $g(\nu, \nabla_X^H Y) = 0$, we have $\mathrm{II}^H(X, Y) \ := \ g(\nu, \nabla_X^M Y)$.

Bounded geometry hypersurface

Definition

Let (M, g) be a Riemannian manifold of bounded geometry and $H \subset M$ be a hypersurface with unit normal vector field ν on H. We say that H is a **bounded geometry hypersurface in M** if:

(i) H is a closed subset of M;

(ii) all covariant derivatives $(\nabla^H)^k \mathrm{II}^H$, $k \geq 0$, are bounded;

(iii) $\exp^\perp : H \times (-\delta, \delta) \to M$ is a diffeomorphism onto its image for some $\delta > 0$.

Then H has bounded geometry.

If $H \subset M$ is a *compact* hypersurface, then H is a bounded geometry hypersurface in M.

Manifolds with boundary and relative bounded geometry

We are ready now to recall the definition of a central concept in analysis on manifolds.

Definition

We shall say that M_0 is a **manifold with boundary and relative bounded geometry** if M_0 is isometrically contained in a (boundaryless) Riemannian manifold M with bounded geometry such that ∂M_0 is a bounded geometry hypersurface in M.

Note that we use the term *"manifold with boundary and relative bounded geometry,"* which we think is more precise than the term *"manifold with boundary and bounded geometry"* used before.

Trace and 'negative' spaces

Let ∂_ν be the normal derivative at the boundary and let

$$H_0^k(M_0) := \cap_{j=0}^{k-1} \ker(\mathrm{res} \circ \partial_\nu^j)$$

denote the joint kernel of the restrictions maps $\mathrm{res} \circ \partial_\nu^j$, $0 \le j \le k-1$.

Theorem (Trace theorem: Grosse-Schneider, Triebel)

Let M_0 be a manifold with boundary and relative bounded geometry. Then, for every $s > 1/2$, the restriction $res: \mathcal{C}_c^\infty(M_0) \to \mathcal{C}_c^\infty(\partial M_0)$ extends by continuity to a surjective map

$$\mathrm{res} : H^s(M_0) \to H^{s-\frac{1}{2}}(\partial M_0).$$

Moreover, $\mathcal{C}_c^\infty(M_0 \smallsetminus \partial M_0)$ is dense in $H_0^k(M_0)$ and $H^{-k}(M_0)$ identifies with $H_0^k(M_0)^*$, $k \in \mathbb{N}$.

Finite width

Definition

Let (M_0, g) be a Riemannian manifold with boundary ∂M_0. We say that M_0 has *finite width* if:

(i) (M_0, g) is a manifold with boundary and relative bounded geometry and

(ii) The function $M_0 \ni x \to \mathrm{dist}_{M_0}(x, \partial M_0)$ is bounded on M_0.

The last condition is equivalent to

"$\exists\, R > 0$ such that $M_0 \subset \{x \in M \mid \exists y \in \partial M_0,\ \mathrm{dist}_{M_0}(x, y) < R\}$."

Poincaré inequality

Theorem (Ammann-Grosse-V.N., Sakurai)

Let (M_0, g) be a Riemannian manifold with finite width. Then there exists $0 < C_{M_0} < \infty$ such that, for all $f \in \mathcal{C}_c^\infty(M_0)$ (thus $f = 0$ on the boundary of M_0),

$$\|f\|_{L^2(M_0)} \leq C_{M_0} \|df\|_{L^2(M_0)} .$$

A regularity argument then yields the following result.

Theorem (Ammann-Grosse-V.N.)

Let M_0 be a smooth Riemannian manifold with smooth boundary ∂M_0 and finite width. Then Δ induces isomorphisms

$$\Delta_D = \Delta\colon H^{m+1}(M_0) \cap H_0^1(M_0) \to H^{m-1}(M_0), \quad m \in \mathbb{Z}_+ := \{0, 1, \ldots\} .$$

Applications: polygonal regularity

Let us consider a polygonal domain Ω_∞ with maximum angle α_{MAX} and the **Poisson problem**

$$\Delta u_\infty = f_\infty \text{ in } \Omega_\infty, \quad u_\infty = 0 \text{ on } \partial\Omega_\infty.$$

It is known that u_∞ has **limited regularity,** less than $H^{1+\pi/\alpha_{MAX}}$: Costabel, Dauge, Griesvard, Jerison-Kennig, Kondratiev, Mazya, ...

Theorem (Limited regularity: Kondratiev, ...)

Let $\delta < \pi/\alpha_{MAX}$. Then there exists $C = C_\delta > 0$ such that the solution $u_n \in H_0^1(\Omega_n)$ of the equation $\Delta u_n = f_n$ satisfies

$$\|u_n\|_{H^{1+\delta}(\Omega_n)} \leq C\|f_n\|_{L^2(\Omega_n)}\,.$$

Applications: well-posedness in weighted spaces

The **"limited regularity"** theorem of the previous slide is **not** very satisfying. It is, in any case, a consequence of the following more satisfying result in the Babuška-Kondratiev (weighted Sobolev) spaces

$$\mathcal{K}_a^m(\Omega) := \{\rho^{|\alpha|-a}\partial^\alpha u \in L^2(\Omega),\ |\alpha| \le m\}$$

$\rho =$distance to vertices (Costabel, Dauge, Kondratiev, Mazya, ...)

Theorem (Unlimited weighted regularity: Kondratiev '67, ...)

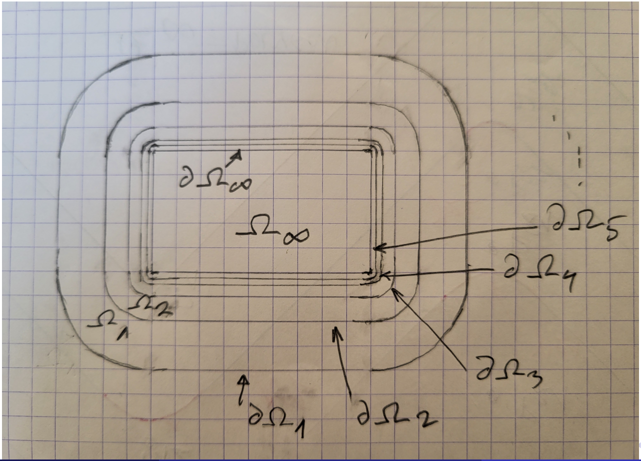
Let $0 \le \delta < \pi/\alpha_{MAX}$ and $m \in \mathbb{Z}_+$. Then there exists $C = C_{m,\delta} > 0$ such that the solution $u_\infty \in H_0^1(\Omega_\infty)$ of the equation $\Delta u_\infty = f_\infty$ satisfies

$$\|u_\infty\|_{\mathcal{K}_{\delta+1}^{m+1}(\Omega_\infty)} \le C\|f_\infty\|_{\mathcal{K}_{\delta-1}^{m-1}(\Omega_\infty)}.$$

This, as well as the next theorem, are consequences of the **well-posedness on manifolds with boundary and relative bounded geometry** (Thm 68 stated earlier).

Applications: "rounding up the corners"

Consider a *sequence of smooth domains* Ω_n, $n \in \mathbb{Z}$, converging to a polygonal domain Ω_∞ (*picture*) and the associated Poisson problems $\Delta u_n = f_n$, $u_n \in H^1_0(\Omega_n)$, for which we want **uniform estimates.**

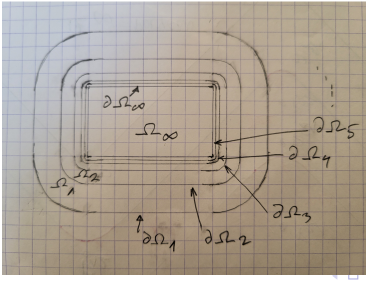


Applications: "rounding up the corners"

Theorem (Daniel-Labrunie-V.N.)

Let $\Omega_n \subset \mathbb{R}^2$ be a sequence of smooth domains "converging" to a polygonal domain Ω_∞, as in the picture (repeated below). Let $\delta < \pi/\alpha_{MAX}$. Then there exists $C = C_\delta > 0$ such that for all $n \in \mathbb{N} \cup \{\infty\}$, the solution $u_n \in H_0^1(\Omega_n)$ of $\Delta u_n = f_n$ satisfies

$$\|u_n\|_{H^{1+\delta}(\Omega_n)} \leq C\|f_n\|_{L^2(\Omega_n)} .$$



The main formula of pseudodifferential theory (repeated)

Recall the **MAIN FORMULA** of **pseudodifferential theory** is:

$$a(x,D)u(x) \ := \ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{\imath \langle x, \xi \rangle} \, a(x,\xi) \hat{u}(\xi) \, d\xi \, ,$$

(**pseudodifferential operator with symbol** $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$.)

Here $\langle x, \xi \rangle := x_1\xi_1 + x_2\xi_2 + \ldots x_n\xi_n$ and

$$\hat{u}(\xi) \ = \ \mathcal{F}u(\xi) \ := \ \int_{\mathbb{R}} e^{-\imath \langle y, \xi \rangle} \, u(y) \, dy \, .$$

is the **Fourier transform** of u.

First example

Let $a : \mathbb{R}^{2n} \to \mathbb{C}$, $a(x,\xi) = a(x)$, (no ξ dependence). Then:

$$\begin{aligned} a(x,D)u(x) \ &:= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{\imath\langle x,\xi\rangle}\, a(x)\hat{u}(\xi)\, d\xi \\ &= a(x)\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{\imath\langle x,\xi\rangle}\, \hat{u}(\xi)\, d\xi \ = \ a(x)u(x)\,, \end{aligned}$$

by the **Fourier inversion formula:**

$$(\mathcal{F}^{-1}w)(x) \ := \ \frac{1}{(2\pi)^n}\int_{\mathbb{R}} e^{i\langle x,\xi\rangle} w(\xi) d\xi\,.$$

Second example

Let $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ be now $a(x,\xi) = a_j(\xi) = \imath\xi_j := \sqrt{-1}\xi_j$, with no x dependence[1] this time. Then:

$$\underline{a_j(D)}u(x) = a(x,D)u(x) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{\imath\langle x,\xi\rangle}\, \imath\xi_j\, \hat{u}(\xi)\, d\xi = \frac{\partial u(x)}{\partial x_j} =: \underline{\partial_j}u(x)\,,$$

because the Fourier transf. interchanges multiplication by $\imath\xi_j$ with ∂_j :

$$\mathcal{F}\,\partial_j = \imath\xi_j\,\mathcal{F} \;\Rightarrow\; \partial_j u = \mathcal{F}^{-1}(\imath\xi_j\,\hat{u}) = \mathcal{F}^{-1}(a_j\mathcal{F}u)\,.$$

[1] $a(D) = \mathcal{F}^{-1}a\mathcal{F} =$convolution operator

Third example

Iterating, we get

$$\partial^\alpha u := \partial_1^{\alpha_1}\partial_2^{\alpha_2}\ldots\partial_n^{\alpha_n} u = \mathcal{F}^{-1}\big((\imath\xi)^\alpha \hat{u}\big).$$

Hence, **for** $a(x,\xi) := \sum_{|\alpha|\le m} a_\alpha(x)\,(\imath\xi)^\alpha$, **polynomial in** ξ:

$$a(x,D)u(x) := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{\imath\langle x,\xi\rangle} \underbrace{\sum_{|\alpha|\le m} a_\alpha(x)\,(\imath\xi)^\alpha}_{a(x,\xi)}\ \hat{u}(\xi)d\xi$$

$$= \sum_{|\alpha|\le m} a_\alpha(x)\,\partial^\alpha u(x)\,,$$

a differential op. (We have used $|\alpha| := \sum \alpha_j$.)

Example $3\frac{1}{2}$

In the particular case $a(\xi) = 1 + |\xi|^2 := 1 + \xi_1^2 + \ldots + \xi_n^2$:

$$a(D)u(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{\imath\langle x,\xi\rangle} (1 + |\xi|^2)\, \hat{u}(\xi)\, d\xi = (1 - \Delta)u(x)\,.$$

It turns out that the inverse of $a(D)$ is of a similar form!

$$(1 - \Delta)^{-1} u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{\imath\langle x,\xi\rangle} \frac{1}{1 + |\xi|^2}\, \hat{u}(\xi)\, d\xi =: b(D)u(x)\,,$$

where $b(x,\xi) = b(\xi) := \frac{1}{1+|\xi|^2}$.[2]

[2] $b(D)$ =convolution with the fundamental solution of $1 - \Delta$.

Symbols and pseudodifferential operators on $\mathbb{R}^n$ (repeated)

Recall the order m, (1,0)-symbols, Hörmander:

$$S^m(\mathbb{R}^{2n}) = S^m_{1,0}(\mathbb{R}^{2n}) := \{a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C} \mid \forall \alpha, \beta, \exists C_{\alpha,\beta} \geq 0 \text{ s.t. } |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|}\}.$$

Also recall:

Theorem

If $a \in S^m(\mathbb{R}^{2n})$, then it defines a continuous map

$$a(x,D) : \mathcal{C}_c^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^n).^{a}$$

[a] (Schwartz' kernel theorem, next!)

Schwartz' kernel theorem

- Let $\langle\ ,\ \rangle : \mathcal{C}_c^\infty(M)' \times \mathcal{C}_c^\infty(M) \to \mathbb{C}$ be the **paring** between distributions and test functions.
- A linear map $T : \mathcal{C}_c^\infty(M) \to \mathcal{C}_c^\infty(M)'$ is **continuous** if $\langle T\phi, \psi\rangle$ is continuous with respect to $\phi \in \mathcal{C}_c^\infty(M)$, for any $\psi \in \mathcal{C}_c^\infty(M)$.

Theorem (Schwartz' kernel theorem)

Let $T : \mathcal{C}_c^\infty(M) \to \mathcal{C}_c^\infty(M)'$ be linear and continuous. Then there exists a unique $k_T \in \mathcal{C}_c^\infty(M \times M)'$ such that

$$\langle T\phi, \psi\rangle = \langle k_T, \psi \boxtimes \phi\rangle ,$$

*where $(\psi \boxtimes \phi)(x, y) = \psi(x)\phi(y)$. ($k_T =$**distribution kernel**.)*

The converse is also true. **Question: Find the kernel $k_{a(x,D)}$.**

Equivalent form using the Fourier transform

If in the **main formula**

$$a(x, D)u(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{\imath\langle x,\xi\rangle}\, a(x,\xi)\hat{u}(\xi)\, d\xi$$

we substitute the Fourier transform

$$\hat{u}(\xi) = \mathcal{F}u(\xi) := \int_{\mathbb{R}} e^{-i\langle y,\xi\rangle}\, u(y)\, dy\,,$$

we obtain the **second main formula**

$$a(x, D)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} e^{\imath\langle x-y,\xi\rangle}\, a(x,\xi)\, u(y)\, dy \right) d\xi\,.$$

The distribution kernel of $a(x, D)$

Theorem

The distribution kernel $k_{a(x,D)}$ of $a(x, D) : \mathcal{C}_c^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^n) \subset \mathcal{C}_c^\infty(\mathbb{R}^n)'$ is:

$$\text{“}k_{a(x,D)}(x, y)\text{”} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{\imath\langle x-y,\xi\rangle} a(x, \xi)\, d\xi = (\mathcal{F}_\xi^{-1} a)(x, x-y)\,.$$

Proof.

The **second main formula** gives (integrating wrt ξ):

$$\langle a(x, D)u, v\rangle = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \underbrace{\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\imath\langle x-y,\xi\rangle} a(x, \xi)\, u(y) dy d\xi}_{a(x,D)u(x)} \, v(x) dx$$

$$= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} k_{a(x,D)}(x, y)\, v(x) u(y)\, dy dx = \langle k_{a(x,D)}, v \boxtimes u\rangle\,,$$

Properties of the kernel $k_{a(x,D)}$

- The Fourier inversion formula allows us to recover $a(x,\xi)$ from
$$k_{a(x,D)}(x,y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{\imath\langle x-y,\xi\rangle} a(x,\xi)\, d\xi\,.$$

Localization: Let $U \subset \mathbb{R}^n$ be open and

$$\Psi_c^m(U) := \{a(x,D) \mid \operatorname{supp} k_{a(x,D)} \Subset U \times U\} \quad \text{and}$$
$$S_c^m(T^*U) := \{a \in S^m(\mathbb{R}^{2n}) \mid \operatorname{supp} a \subset K \times \mathbb{R}^n\,,\ K \Subset U\}\,.$$

The last point shows that

$$a(x,D) \in \Psi_c^m(U) \ \Rightarrow\ a \in S_c^m(T^*U) \subset S^m(\mathbb{R}^{2n}).$$

Diffeomorphism invariance

Recall $\Psi_c^m(U) := \{a(x,D) \mid \text{supp}\, k_{a(x,D)} \text{ compact } \subset U \times U\}$.

Theorem

For $U \subset \mathbb{R}^n$ be open, $\Psi_c^\infty(U) := \cup_{m \in \mathbb{Z}} \Psi_c^m(U)$ is an algebra invariant under diffeomorphisms and under adjoints.*

*More precisely, a diffeomorphism $\phi : U \to W \subset \mathbb{R}^n$ induces bijections:

- $\phi^* : T^*W \to T^*U$, (where $\phi^*(x,\xi) = (\phi^{-1}(x), (d\phi)^T \xi)$);
- $\phi_* : \mathcal{C}_c^\infty(U) \to \mathcal{C}_c^\infty(W)$, (where $\phi_* u := u \circ \phi^{-1}$); and
- $\phi_* : S_c^m(T^*U) \to S_c^m(T^*W)$, (where $\phi_* a := a \circ \phi^*$).

Let $a(x,D) \in \Psi_c^m(U)$, then $\phi_* a := a \circ \phi^* \in S_c^m(T^*W)$ and

$$\phi_* \circ a(x,D) \circ \phi_*^{-1} - (\phi_* a)(x,D) \in \Psi_c^{m-1}(W).$$

Principal symbol

Let $\phi : U \to W$ be a diffeomorphism, as before. Then

$$\begin{array}{ccc} S^m_c(T^*U) \ni a & \xrightarrow{\phi_*} & \phi_*(a) \in S^m_c(T^*W) \\ \downarrow & & \downarrow \\ \Psi^m_c(U) \ni a(x,D) & \longrightarrow & \phi_* a(x,D)\phi_*^{-1} \in \Psi^m_c(W) \end{array}$$

commutes up to **lower order symbols.**

Theorem

If $a(x,D) \in \Psi^m_c(U)$, then the **principal symbol**

$$\sigma_m(a(x,D)) := a + S^{m-1}(T^*U) \in S^m(T^*U)/S^{m-1}(T^*U)$$

is ***well-defined, multiplicative,*** *and* ***diffeomorphism invariant.***

Pseudodifferential operators on manifolds (repeated)

Let M be a **smooth manifold.** Recall the following. If $\phi : U \simeq W \subset \mathbb{R}^n$, $U \subset M$ is open, the **diffeomorphism invariance** allows us to define

$$\Psi_c^m(U) := \phi_*^{-1} \Psi_c^m(W) \phi_* \quad \text{and}$$
$$\sigma_m : \Psi_c^m(U) \to S_c^m / S_c^{m-1}(T^*U).$$

Definition

A linear map $P : \mathcal{C}_c^\infty(M) \to \mathcal{C}^\infty(M)$ is a **pseudodifferential operator** of order $\leq m$ on M if, for any $U \simeq W \subset \mathbb{R}^n$, $U \subset M$, and $\eta \in \mathcal{C}_c^\infty(U)$, we have $\eta P \eta \in \Psi_c^m(U)$.

Let $\Psi^m(M)$ be the set of all **pseudodifferential operators** on M. Then the principal symbol maps extend to **surjective maps**

$$\sigma_m : \Psi^m(M) \to S^m / S^{m-1}(T^*M).$$

Pseudodifferential operators on $\mathbb{R}^n$

For the particular case $M = \mathbb{R}^n$, the space $\Psi^m(\mathbb{R}^n)$ contains:

1. all $a(x, D)$ with $a \in S^m(\mathbb{R}^{2n}) = S^m(T^*\mathbb{R}^n)$.
2. all differential ops. with **smooth** coefficients.
3. all operators with **smooth distribution kernel.**

Not all these operators are of the form $a(x, D)$, with $a \in S^m(\mathbb{R}^{2n})$.

Asymptotic sums

Theorem

Let $a_j \in S^{m_j}(M)$, with m_j decreasing to $-\infty$, $j = 0, 1, 2, \ldots$ Then there exists $a \in S^{m_0}(M)$ such that, for all $N \in \mathbb{N}$, $a - \sum_{j=0}^{N} a_{m_j} \in S^{m_{N+1}}(M)$.

We then write $a \sim \sum_{j=0}^{\infty} a_{m_j}$.

Definition

We say that $a_k : T^*M \to \mathbb{C}$ is **(eventually) homogeneous** of order k if

$$a_k(t\xi) = t^k a_k(\xi) \text{ for } t, |\xi| \geq 1 .$$

An eventually homogeneous function of order m is in $S^m(M)$ if it is smooth enough.

Classical symbols

Definition

A symbol $a \in S^m(\mathbb{R}^n)$ is **classical** if it can be expanded asymptotically $a \sim \sum_{j=0}^{\infty} a_{m-j}$, where $a_k \in S^k(M)$ is homogeneous of order k.

- All results remain true for classical symbols.
- Let $\Psi_{cl}^m(M)$ be the corresponding classical psedodifferential op, then $\sigma_m : \Psi_{cl}^m(M) \to S_{cl}^m / S_{cl}^{m-1}(T^*M)$ has kernel $\Psi_{cl}^{m-1}(M)$.
- If S^*M is the set of vectors of length 1 in T^*M, then

$$S_{cl}^m / S_{cl}^{m-1}(T^*M) \simeq \mathcal{C}^\infty(S^*M)\,,$$

  and we can choose a_m a representative of $\sigma_m(a)$.
- Parametrices of differential operators are classical (index theory).

Vector bundles

We can include **vector bundles:**

- If $E \to M$ is a vector bundle, let $E \subset \mathbb{C}^N$ be a smooth embedding, and let $e \in \mathcal{C}^\infty(M; M_N(\mathbb{C}))$ be the projection $\mathbb{C}^N \to E$.
- $\Psi^m(M; E) := eM_N(\Psi^m(M))e$ acts on $H^s(M; E) \simeq eH^s(M)^N$.
- $\sigma_m : \Psi^m(M; E) \to S^m/S^{m-1}(T^*M; \mathrm{End}(E))$ and the Fredholm theorem remains unchanged.
- Moreover,

$$S^m_{cl}/S^{m-1}_{cl}(T^*M; \mathrm{End}(E)) \simeq \mathcal{C}^\infty(S^*M; \mathrm{End}(E)),$$

where $S^m_{cl} \subset S^m$ denotes the set of **classical symbols,** as before.

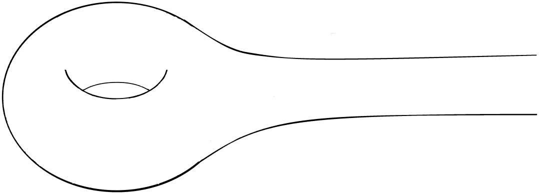
Outline of the rest of this presentation

We use Lie groupoids, Lie algebroids, and C^*-algebras to study operators on Lie manifolds generalization of manifolds with cylindrical ends. These objects are motivated by the case of manifolds with cylindrical ends.

This material is about 2/3 of a sequence of lectures (course) delivered in Bruxelles in 2023. The material on pseudodifferential operators is also from that course.

Motivation: recall manifolds with cylindrical ends

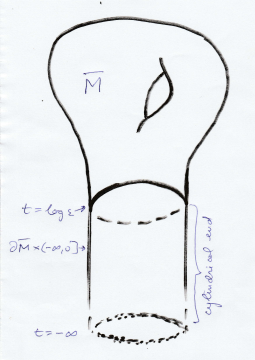
A **model** class of non-compact manifolds M:

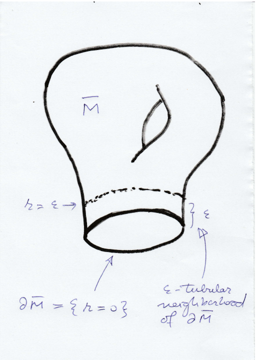


M contains a part (end) that contains a copy of $\partial \overline{M} \times (-\infty, 0]$ with a product metric **isometrically:** $M = \overline{M} \cup (\partial \overline{M} \times (-\infty, 0])$.

We want now to take a quick look at the differential operators that are invariant (in a neighborhood of) infinity.

Going back in the definition of with cylindrical ends





Kondratiev's transform $r := e^t$:

$$(-\infty, \ln \epsilon) \times \partial\overline{M} \ni (t, y) \ \rightarrow \ (r, y) \in (0, \epsilon) \times \partial\overline{M}$$

$$\partial_t \ \rightarrow \ r\partial_r$$

$$a \text{ transl. inv.} \ \rightarrow \ a \text{ smooth on } \overline{M}\,.$$

Differential operators translation invariant operators near infinity

Back to $\overline{M}$, we get differential operators of the form

$$P = \sum_{j,Q} a_Q(e^t, y)(r\partial_r)^j Q(y)$$

- $Q(y)$ differential operator on the boundary $\partial\overline{M} \ni y$.
- $a_Q(e^t, y) = a_Q(y)$ for $r = e^t$ very small.
- $\widetilde{P} = \sum_{j,Q} a_Q(0, y)\partial_t^j Q(y)$ on $\partial\overline{M} \times \mathbb{R}$.

Diff. ops. enerated by $r\partial_r$ and ∂_y, all **tangent to the boundary** $\partial\overline{M}$.

a **Lie algebra.**

Differential operators in general

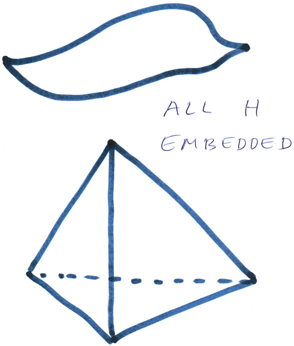
- On a *closed manifold* M, we considered **all differential operators** (all vector fields and all smooth functions).
- In the *non-compact case,* we need to **restrict the choices of vector fields and of smooth functions** in order to obtain meaningful results.
- The choices: **The 'Quantization Program,' vector fields on manifolds with corners** (Cordes, Melrose, Schulze, ...)
- The non-smooth case is treated using the non-compact case, by looking at the set of smooth points M and then choosing a suitable compactification $\overline{M}$ of that set (of smooth points).

Manifolds with corners

A manifold **with corners** $\overline{M}$ is locally of the form $[0,1]^n$.

- A face $H \subset \overline{M}$ of maximal dimension is called a **hyperface.**
- The **boundary** $\partial\overline{M}$ of $\overline{M}$ is the union of all faces of $\overline{M}$ other than its interior.
- A **defining function** of a hyperface H of $\overline{M}$ is a function r such that $H = \{r = 0\}$ and $dr \neq 0$ on H.
- The hyperface $H \subset \overline{M}$ is **embedded** if it has a defining function.

Embedded and non-embedded faces



A word on assumptions and notation

- Our vector bundles $E \to Z$ will always be smooth (and our spaces will be manifolds, possibly with corners).
- $\mathcal{C}^\infty(Z; E)$ denotes the space of **smooth** sections $s : Z \to E$ of E, also $\mathcal{C}^\infty(E)$, if Z is clear from the context.
- $\mathcal{C}_c^\infty(Z; E)$ or $\mathcal{C}_c^\infty(E)$ denotes the space of **compactly supported, smooth** sections of E.
- The Lie bracket will be denoted by $[\, ,\,]$.

The 'Quantization Program'

Main players:

- $\overline{M}$ = compact manifold with **corners.**
- $\mathcal{V}_b(\overline{M}) := \{X \in \mathcal{C}^\infty(\overline{M}; T\overline{M})$ tangent to $\partial\overline{M}\}$.
- We will consider a $\mathcal{C}^\infty(\overline{M})-$ submodule $\mathcal{V} \subset \mathcal{V}_b(\overline{M})$ which is stable for the Lie bracket of vector fields (Lie subalgebra).
- $\mathrm{Diff}(\mathcal{V})$ = differential operators generated by $\mathcal{V}$ and $\mathcal{C}^\infty(\overline{M})$.

The quantization program:

To study the analytic properties of the differential operators in $\mathrm{Diff}(\mathcal{V})$.

Albin, Mazzeo, Melrose, Piazza, Rochon, (the list is v. incomplete ...)

Convenient approach: using **Lie algebroids.**

Lie algebroids (informal definition)

A **Lie algebroid** is a real vector bundle $A \to \overline{M}$ (a manifold with corners, as usual) such that **"$\mathcal{C}^\infty(A)$ is a $\mathcal{C}^\infty(\overline{M})-$Lie algebra."**

More precisely, we are given:

1. A **Lie algebra structure** on the space $\mathcal{C}^\infty(A) = \mathcal{C}^\infty(\overline{M}; A)$ of global (smooth) sections of A.
2. A **Lie algebra action** of $\mathcal{C}^\infty(A)$ on the base ring $\mathcal{C}^\infty(\overline{M})$:
$$\mathcal{C}^\infty(A) \times \mathcal{C}^\infty(\overline{M}) \ni (X, f) \to X(f) \in \mathcal{C}^\infty(\overline{M}) .$$
3. **Compatibility conditions** (next).

Lie algebroids (informal discussion)

Saying that $\mathcal{C}^\infty(A)$ **is a $\mathcal{C}^\infty(\overline{M})$-Lie algebra** implies the followng **compatibility conditions:**

- The action $\mathcal{C}^\infty(A) \times \mathcal{C}^\infty(\overline{M}) \to \mathcal{C}^\infty(\overline{M})$ is a $\mathcal{C}^\infty(\overline{M})-$**module morphism:**
$$(fX)g \;=\; f(Xg)\,.$$
- The bracket $[\,,\,]$ on $\mathcal{C}^\infty(A)$ satisfies the **Leibnitz rule:**
$$[X, fY] = f[X, Y] + (Xf)Y\,.$$
- Saying that action on the base ring is a **Lie algebra action** means that we have a Lie algebra morphism $\mathcal{C}^\infty(A) \to \mathcal{C}^\infty(T\overline{M})$:
$$[X, Y]f = X(Yf) - Y(Xf)\,.$$

Lie algebroids

Formalizing "$\mathcal{C}^\infty(A)$ is a $\mathcal{C}^\infty(\overline{M})$-Lie algebra"

Definition (Pradines 1967, (Book: Mackenzie))

A **Lie algebroid** is a real vector bundle $A \to \overline{M}$ together with

1. a **Lie algebra** structure $[\,,\,]$ on $\mathcal{C}^\infty(A)$ and
2. a **vector bundle map** $\varrho : A \to T\overline{M}$ (anchor, action), such that,
3. $\forall\ X, Y \in \mathcal{C}^\infty(A),\ f \in \mathcal{C}^\infty(\overline{M}),\ \ [X, fY] = f[X, Y] + (\varrho(X)f)Y.$

- The action $(X, f) \to Xf := \varrho(X)f$ is automatically a Lie action.
- $\varrho = \varrho_* : \mathcal{C}^\infty(\overline{M}; A) \to \mathcal{C}^\infty(\overline{M}; T\overline{M})$ is the induced map and we shall usually write $Xf := \varrho(X)f$.

Examples of Lie algebroids

Let us notice first that if $\varrho = 0$, then the Leibnitz rule

$$[X, fY] = f[X, Y] + (Xf)Y = f[X, Y]$$

simply states that **the Lie bracket is bilinear.**

Example 1. A Lie algebra:

- $\overline{M} = pt$ is reduced to a point and $\mathcal{C}^\infty(A) = A$ is a Lie algebra.
- the anchor map $\varrho = 0$.

Conditions are satisfied: (1) the Lie structure is given; (2) The action of $\mathcal{C}^\infty(A)$ on $\mathcal{C}^\infty(\overline{M})$ is trivial (zero) because $\rho = 0$; (3) Leibnitz is satisfied since the action is trivial ($\varrho = 0$).

Examples of Lie algebroids (cont.)

Example 2. A manifold I (as a topological space):

Let $\overline{M}$ = any manifold with corners and A any vector bundle. We let:

1. $[\,,\,] = 0$ on $\mathcal{C}^\infty(A)$.
2. The action of $\mathcal{C}^\infty(A)$ on $\mathcal{C}^\infty(\overline{M})$ is trivial ($\rho = 0$).
3. The Leibnitz rule is satisfied since $\rho = 0$ and the bracket is bilinear (even zero).

Importance: Lie algebroids encompass both Lie algebras and smooth manifolds (if $E = \overline{M} \times \mathbb{C}$), they “interpolate” between these two classes.

Exemples of Lie algebroids (cont.)

Example 3. A manifold II (tangent space):

This time $\overline{M} = M$ closed manifold (smooth, **compact**, no corners or boundary) and $A = TM$.

- The usual Lie structure (bracket) on $\mathcal{C}^\infty(A) = \mathcal{C}^\infty(TM)$.
- $\mathcal{C}^\infty(TM)$ acts on $\mathcal{C}^\infty(M)$ by derivations, that is

$$\varrho = id : T\overline{M} \to T\overline{M}.$$

- The Leibnitz rule is satisfied by the Lie bracket of vector fields.

This example corresponds to **the first example (0) of the last lecture: closed manifolds.** The simplest setting in the quantization program is then:

"To study the differential operators on M, a closed manifold."

The Serre-Swan theorem and our main examples

The next example correspond to the second example discussed in the last lecture: **manifolds with cylindrical ends (1).**

To make the connection(s):

Theorem (Serre–Swan)

Let $\mathcal{E}$ be a projective, finitely generated $\mathcal{C}^\infty(\overline{M})$-module. Then there exists a smooth, finite-dimensional vector bundle $E \to \overline{M}$ such that $\mathcal{E} \simeq \mathcal{C}^\infty(\overline{M}; E)$ naturally as $\mathcal{C}^\infty(\overline{M})$-modules.

The converse is also true (but not used in this talk).

($\overline{M}$ is a manifold with corners, as usual).

The b-calculus and cylindrical ends

Example 4. The b-calculus (or **cylindrical ends**)

- Let $\overline{M}$ = a manifold with boundary $\partial\overline{M}$.
- $\mathcal{V}_b(\overline{M}) := \{ X \in \mathcal{C}^\infty(\overline{M}; T\overline{M}) \mid X|_{\partial\overline{M}}$ tangent to $\partial\overline{M} \}$.
- Near the boundary of $\overline{M}$, we can assume $\overline{M} = [0, \infty) \times \mathbb{R}^{n-1}$. Then $\mathcal{V}_b(\overline{M})$ is even a free module with basis

$$x_1\partial_1,\ \partial_2,\ \partial_3,\ \dots\ ,\partial_n\,.$$

- Serre-Swan theorem gives that $\mathcal{V}_b(\overline{M})$ identifies with the sections of a vector bundle usually denoted ${}^bT\overline{M}$. ($\mathcal{V} = \mathcal{V}_b(\overline{M})$).
- ${}^bT\overline{M}$ is a **Lie algebroid** because $\mathcal{V}_b(\overline{M})$ is a Lie algebra.

The Laplacian in **generalized spherical coordinates** corresponds to the this class of Lie algebroids. (Conical points, APS, ...)

The “edge”-calculus

Example 5. Mazzeo’s “edge”-calculus:

- Let $\overline{M}$ = a manifold with boundary **plus a fibration $\partial\overline{M} \to B^k$**
- $\mathcal{V}_e := \{ X \mid X|_{\partial\overline{M}}$ tangent to **the fibers of** $\partial\overline{M} \to B^k \}$.
- Near the boundary of $\overline{M}$, we can assume $\overline{M} = [0, \infty) \times \mathbb{R}^{n-1}$. Then $\mathcal{V}_e(M)$ is even a free module with basis

$$x_1\partial_1, \ \partial_2, \ \ldots \ , \partial_{n-k} \ , x_1\partial_{n-k+1}, \ldots, x_1\partial_n.$$

- Serre-Swan theorem gives that $\mathcal{V}_e(\overline{M})$ identifies with the sections of a vector bundle ${}^eT\overline{M}$ (Mazzeo $\mathcal{V} = \mathcal{V}_e(\overline{M})$).
- ${}^eT\overline{M}$ is a Lie algebroid because $\mathcal{V}_e$ is a Lie algebra.

- $\partial\overline{M} \to B^k := pt$: b-calculus ($k = 0$).
- $\partial\overline{M} \to B^k := \partial\overline{M}$: **conformally compact manifolds** ($k = n - 1$).
- The Laplacian in **cylindrical coordinates:** $n = 3, k = 1$.

Lie algebroids and Geometry

Many examples of Lie algebroids often arise from **Geometry:**

- **Foliations.**
  - This time we have a subbundle $A = \mathcal{F} \subset TM$ such that $\mathcal{C}^\infty(M; \mathcal{F})$ is stable for the Lie bracket (and hence a Lie algebra).
  - In particular, $\mathcal{C}^\infty(A) = \mathcal{C}^\infty(\mathcal{F})$ consists of vector fields tangent to the leaves of the foliation.
  - Again, $\varrho : \mathcal{F} \to TM$ is the identity (more precisely, the inclusion).

    (Connes. Also Androulidakis, Benameur, Skandalis, ...).
- **Poisson manifolds.** $A = T^*M \to M$.

ATTENTION: the vector associated to some $X \in \mathcal{C}^\infty(A)$ may vanish at some point, without X vanishing at that point. Ex: Poisson not symplectic. ISOTROPY (next).

Isotropy $\mathfrak{g}_x$

Let $\varrho : A \to T\overline{M}$ be a Lie algebroid over $\overline{M}$ and $x \in \overline{M}$. We set:

- $\mathfrak{g}_x := \ker(\varrho_x : A_x \to T_x\overline{M})$; (finite dim.)
- $\mathcal{V} := \mathcal{C}^\infty(A)$, a Lie algebra;
- $\mathcal{V}_x := \{Y \in \mathcal{V} \mid \varrho_x(Y) = 0\} = \{Y \in \mathcal{V} \mid Y(x) \in \mathfrak{g}_x\}$.
- $I_x := \{f \in \mathcal{C}^\infty(\overline{M}) \mid f(x) = 0\}$; and

If $Y \in \mathcal{V}_x$ and $f \in \mathcal{C}^\infty(\overline{M})$, then $(Yf)(x) = 0$, as well.

Let $Y \in \mathcal{V}$, then

$$[X, fY] = X(f)Y + f[X, Y] \in I_x\mathcal{V},$$

and therefore $I_x\mathcal{V}$ is an ideal in $\mathcal{V}_x$.

Consequently, $\mathfrak{g}_x = \mathcal{V}_x/I_x\mathcal{V}$ **is a Lie algebra.** It is the **isotropy** of A at x and will play a **very,very important role** in what follows.

Lie algebroids and isotropy (quick review)

Summary: A **Lie algebroid** (Pradines 1967) is a real vector bundle $A \to \overline{M}$ s. t.

"$\mathcal{C}^\infty(A)$ is a $\mathcal{C}^\infty(\overline{M})$−Lie algebra."

More precisely:

1. $\mathcal{V} := \mathcal{C}^\infty(A)$ is a **Lie algebra.**
2. $\mathcal{V} := \mathcal{C}^\infty(A)$ **acts** on $\mathcal{C}^\infty(\overline{M})$.
3. **Compatibility conditions** (Leibnitz rule, ...).

We usually give the action: $\varrho : A \to T\overline{M}$ (anchor map).

Isotropy $\mathfrak{g}_x$ of $\varrho : A \to T\overline{M}$

Theorem

Let $x \in \overline{M}$, then $\mathfrak{g}_x := \ker(\varrho_x : A_x \to T_x\overline{M})$ is a ***Lie algebra.***

Proof.

$\mathfrak{g}_x = \mathcal{V}_x / I_x\mathcal{V}$, where

- $\mathcal{V}_x := \{Y \in \mathcal{C}^\infty(A) \mid \varrho_x(Y) = 0\}$.
- $I_x := \{f \in \mathcal{C}^\infty(\overline{M}) \mid f(x) = 0\}$.

Leibnitz' rule gives that $I_x\mathcal{V}$ is **an ideal** in $\mathcal{V}_x$.

Consequently, $\mathfrak{g}_x = \mathcal{V}_x / I_x\mathcal{V}$ **is a Lie algebra.** □

$\mathfrak{g}_x$ is the **isotropy** of A at x and will play a **very,very important role.**

Examples of isotropies

Example 1. A Lie algebra:

- $A =$a Lie algebra, $\overline{M} = pt$, $\varrho = 0$.
- The isotropy is $\mathfrak{g}_x = A$, **maximal.**

Example 2. A manifold I (as a topological space):

1. $0 = \varrho : A \to T\overline{M}$ a vector bundle, $[\,,\,] = 0$ on $\mathcal{C}^\infty(A)$.
2. The isotropy is $\mathfrak{g}_x = A_x$, **maximal at each point.**

Example 3. A manifold II (tangent space):

- $A = TM \to TM$ with the Lie bracket and $\varrho = id$.
- The isotropy is $\mathfrak{g}_x = 0$, **minimal at each point.**

The isotropies of the b-calculus

Example 4. ***b*-calculus** (or **cylindrical ends**)

- $\overline{M}$ = a manifold with boundary and $A = {}^bT\overline{M}$, so that $\mathcal{C}^\infty(A) = \mathcal{V}_b(\overline{M}) := \{ X \in \mathcal{C}^\infty(T\overline{M}) \mid X|_{\partial\overline{M}} \text{ tangent to } \partial\overline{M} \}$.
- Near the boundary, $\mathcal{V}_b(\overline{M})$ is a free module with basis

$$x_1\partial_1,\ \partial_2,\ \partial_3,\ \ldots\ ,\partial_n\,.$$

- Let $x \in M := \overline{M} \smallsetminus \partial\overline{M}$, then $\rho_x = id$, so $\mathfrak{g}_x = 0$, **minimal.**
- Let $x \in \partial\overline{M}$, then $\mathfrak{g}_x = \mathbb{R}(x_1\partial_1)$, **NO LONGER minimal.**
- The section $x_1\partial_1$ of $A = {}^bT\overline{M}$ vanishes **nowhere**, but its associated vector field $\varrho(x_1\partial_1) = x_1\partial_1$ vanishes at the boundary. Notice that ∂_1 does **not** define a section of $A = {}^bT\overline{M}$.

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The isotropies of the “edge”-calculus

Example 5. Mazzeo’s “edge”-calculus:

- $\overline{M}$ = a manifold with boundary, $\partial\overline{M} \to B^k$ **smooth fibration.**
- $\mathcal{C}^\infty(A) = \mathcal{V}_e := \{ X \mid X|_{\partial\overline{M}}$ tangent to **the fibers of** $\partial\overline{M} \to B^k \}$.
- Near the boundary, $\mathcal{V}_e(M)$ is a free module with basis

$$x_1\partial_1,\ \partial_2,\ \ldots\ ,\partial_{n-k}\ , x_1\partial_{n-k+1}, \ldots, x_1\partial_n.$$

- In the interior the isotropy vanishes, but if $x \in \partial\overline{M}$, then the isotropy $\mathfrak{g}_x$ is linearly generated by the sections

$$x_1\partial_1\ ,\ x_1\partial_{n-k+1}, \ldots, x_1\partial_n\ .$$

- $\mathfrak{g}_x$ is a **non-commutative** Lie algebra if $k > 0$. (The semi-direct product $\mathbb{R}^k \rtimes \mathbb{R}$.) More difficult!

Outline

We now introduce a generalization of manifolds with cylindrical ends following Ammann-Lauter-V.N.

Definition of ‘Lie manifolds’

Recall $\mathcal{V}_b(\overline{M}) \subset \mathcal{C}^\infty(\overline{M}; T\overline{M})$ = vector fields **tangent to all faces.**

Definition (Ammann-Lauter-V.N.)

A **Lie manifold** is pair $(\overline{M}, \mathcal{V})$ consisting of a compact manifold with corners $\overline{M}$ and a subspace $\mathcal{V} \subset \mathcal{V}_b(\overline{M})$ of vector fields s.t.

1. $\mathcal{V}$ is closed under the Lie bracket $[\ ,\ ]$;
2. $\mathcal{V}$ is a finitely-generated, projective $\mathcal{C}^\infty(\overline{M})$–module;
3. $\mathcal{C}_c^\infty(M; TM) \subset \mathcal{V}$ (recall $M := \overline{M} \smallsetminus \partial\overline{M}$).

(1) and (2) $\Leftrightarrow$ Lie algebroid $A \to \overline{M}$. Addl. prop. (3).

Examples 3–5 (compact, b-calc. and “edge”-calc.).

Compatible Riemannian metric

Let $(\overline{M}, \mathcal{V})$ be a Lie manifold.

- $\mathcal{V} =$projective $\mathcal{C}^\infty(\overline{M})-$module implies that there exists a vector bundle A s.t. $\mathcal{V} \simeq \mathcal{C}^\infty(\overline{M}; A)$. (The Serre-Swan theorem.)
- $A =$Lie algebroid (because $\mathcal{V} =$Lie algebra.) (M, A) also called a Lie manifold.
- The inclusion $\mathcal{C}_c^\infty(M; TM) \subset \mathcal{V}$ is equivalent to $A = TM$ in the interior $M := \overline{M} \smallsetminus \partial\overline{M}$ of $\overline{M}$.
- **Any metric on A will induce a metric on TM** (i.e. on M), called **compatible metric**, and is unique up to Lipschitz equivalence. It is complete and has **positive injectivity radius.**

Properties of compatible metrics

Recall $TM \subset A$, where $M := \overline{M} \smallsetminus \partial \overline{M}$. The Levi-Civita connection

$$\nabla : \mathcal{C}^\infty(TM) \to \mathcal{C}^\infty(T^*M \otimes TM)$$

associated to a compatible metric g on M (comming from a metric on A) **extends to a map** ($A-$connexion, Weinstein and col.)

$$\nabla : \mathcal{C}^\infty(A) \to \mathcal{C}^\infty(A^* \otimes A)\,.$$

(Proof: Koszul's formula ...)

Theorem (Ammann-Lauter-V.N.)

*All **geometric differential operators** associated to a compatible metric g are in $\mathrm{Diff}(A)$.*

For instance, the Laplacian $\Delta_g \in \mathrm{Diff}(A)$.

(Also for the other operators: we need vector bundles.)

Bounded geometry +

- The curvature $\nabla^2 \in \mathcal{C}^\infty(\overline{M}; A^{*\otimes 2} \times \mathrm{End}(A))$ is **bounded.**
- Bounded geometry (also positive injectivity radius, non-trivial).
- If $E, F \to \overline{M}$ are vector bundles, their "right connections" are the A-connections $\nabla : \mathcal{C}^\infty(\overline{M}; A) \to \mathcal{C}^\infty(\overline{M}; A^* \otimes A)$.
- Differential operators $\mathrm{Diff}(A; E, F)$ generated by ∇ and $Hom(E; F)$.
- Clifford bundles $W \to \overline{M}$, with a metric and A-connection, $c : A^* \to \mathrm{End}(W)$, with $c(\xi)c(\eta) + c(\eta)c(\xi) = 2\langle \xi, \eta \rangle I$, then the associated Dirac operator $D\!\!\!/^W \in \mathrm{Diff}(A; W)$ **(Parker's talk).**

Sobolev spaces

- $(M, A) =$ a Lie manifold ($\mathcal{V} = \mathcal{C}^\infty(A)$).
- $g =$ a compatible metric on $M := \overline{M} \smallsetminus \partial\overline{M}$ (from a metric on A).
- $L^p(M)$ is defined wrt the measure induced by g.
- If $s \in \mathbb{R}_+$, we let $H^s(M)$ be the domain of $(1 - \Delta_g)^{s/2}$.
- If $m \in \mathbb{N}$,

$$H^m(M) := \{u \mid X_1 X_2 \ldots X_k u \in L^2(M), k \leq m, X_j \in \mathcal{V}\}$$

Kondratiev 1967, Mazya, Mazzeo, Melrose, Plamenevskij, ...
(Ammann-Lauter-V.N.)

First example (simplest): cylindrical ends

- $\overline{M}$ = a manifold with **smooth boundary** $\partial\overline{M} = \{x = 0\}$.
- $\mathcal{V} = \mathcal{V}_b(\overline{M}) =$ vector fields on $\overline{M}$ **tangent** to $\partial\overline{M}$ (recall projective).
- **No cond. on vector fields in the interior** (all Lie man).
- Choice of **compatible metric** (near the boundary):

$$\frac{(dx_1)^2}{x_1^2} + h(x')\,, \quad h \text{ metric on } \partial\overline{M} \ni x'\,.$$

Kondratiev's transform $x = e^t$ transforms this metric into

$$(dt)^2 + h(x')\,,$$

the Riemannian metric of a **manifold with cylindrical ends**.

Pseudodifferential calculus: b-calculus.
it contains the 'inv'-calculus of the first lecture. (Isotropy $\mathfrak{g}_x \simeq \mathbb{R}$.)

Second example: asymptotically hyperbolic manifolds

- As before, $\overline{M}$ with smooth boundary $\partial\overline{M} = \{x = 0\}$.
- $\mathcal{V} = \mathcal{V}_0 = x\mathcal{C}^\infty(\overline{M}; T\overline{M})$ = the space of vector fields on $\overline{M}$ that **vanish** on the boundary.
- Particular case of the "edge"-calculus ($\partial\overline{M} \to \partial\overline{M}$).
- Local basis: $x\partial_x$, $x\partial_{x_2}$, ... , $x\partial_{x_n}$ (so projective module).
- **Compatible metric:** $g = \frac{h}{x^2}$, where h =metric on $\overline{M}$.

Metric: conformally compact (ex: asymptotically hyperbolic).

Pseudodifferential calculus: Lauter, Mazzeo, Schulze.

(Isotropy $\mathfrak{g}_x \simeq \mathbb{R}^{n-1} \rtimes \mathbb{R}$ at the boundary.)

Third example: asymptotically Euclidean manifolds

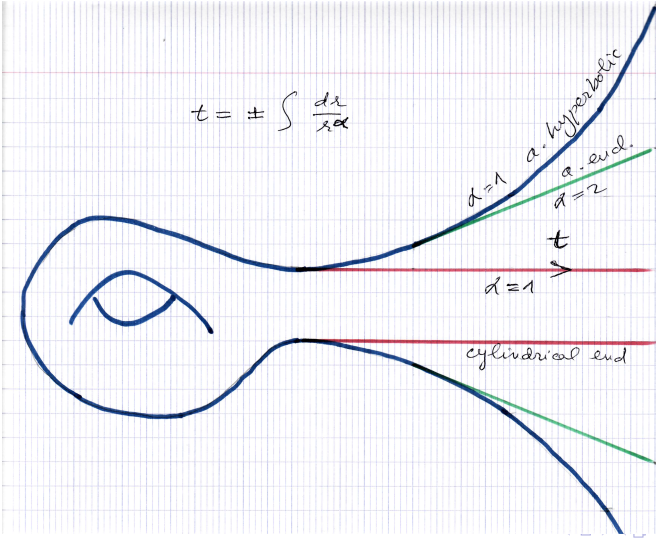
- As before, $\overline{M}$ with smooth boundary $\partial\overline{M} = \{x = 0\}$.
- $\mathcal{V} = x\mathcal{V}_b(\overline{M}) =$ the space of vector fields on $\overline{M}$ that vanish on the boundary $\partial\overline{M}$ and whose **normal covariant derivative to the boundary also vanishes**.
- Local basis $x_1^2\partial_x$, $x\partial_{x_2}$, ... , $x\partial_{x_n}$.
- $\mathfrak{g}_x = \mathbb{R}^n$ at the boundary (trivial inside, all Lie Man.).

The resulting metric and geometry is that of an **asymptotically conical** manifold (includes the **asymptotically Euclidean** case).

If $\overline{M} =$ spherical compactification of $\mathbb{R}^n$, the resulting Riemannian manifold is simply $\mathbb{R}^n$ with the usual metric. ('SG' or sc calculus, Melrose, Parenti, Schrohe, ...) N-body problems (Georgescu, Vasy, Ammann-Mougel-V.N.)

Picture of the examples

$r^{\alpha}\partial_r$:



"No number" example

The metric on M = interior of $\overline{M}$ **does not determine** $\overline{M}$.

Example, $M = \mathbb{R}^n$ with the Euclidean metric:

- $\overline{M}_1 := \overline{\mathbb{R}}^n$, radial compactification, as in the previous example.
- $\overline{M}_2 := \overline{\mathbb{R}}^k \overline{\mathbb{R}}^{n-k}$, product of radial compactifications.
  (Also $\overline{M}_3 := (\overline{\mathbb{R}})^n$, a product of "closed" lines.)
- N-body problems (Kottke, Vasy, Ammann-Mougel-V.N.).

Different differential operators, different analytic properties.

Comparison of the three examples of Lie manifolds

$h =$ (non-singular, true) metric on $\overline{M}$. Examples of compatible metrics:

1. **Cylindrical ends:** $\mathcal{V} = \mathcal{C}^\infty(\overline{M}) r\partial_r + \sum \mathcal{C}^\infty(\overline{M})\partial_y$,

$$g = \frac{(dr)^2}{r^2} + h.$$

2. **Conformally compact:** $\mathcal{V} = \mathcal{C}^\infty(\overline{M}) r\partial_r + \sum \mathcal{C}^\infty(\overline{M}) r\partial_y$,

$$g = \frac{(dr)^2 + h}{r^2} = \frac{h_0}{r^2}.$$

3. **Asymptotically conical:** $\mathcal{V} = \mathcal{C}^\infty(\overline{M}) r^2\partial_r + \sum \mathcal{C}^\infty(\overline{M}) r\partial_y$,

$$g = \frac{(dr)^2}{r^4} + \frac{h}{r^2}$$

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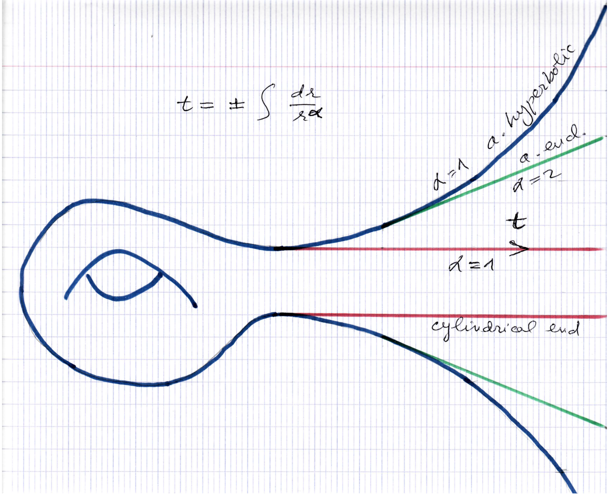
Picture of the examples

$dt := \pm \frac{dr}{r^\alpha}$; a. cyl. & hyp.: $\alpha = 1$; a. euclidean $\alpha = 2$.



Examples: Δ in polar and cylindrical coordinates

1. $r^2\Delta_{\mathbb{R}^2} = (r\partial_r)^2 + \partial_\theta^2$ is the differential operator generated by

$$r\partial_r \quad \text{and} \quad \partial_\theta$$

on $\overline{M} = [0,\infty) \times S^1$. (Totally characteristic operators.)

2. Similarly, $r^{-2}\Delta_{\mathbb{R}^3} = (r\partial_r)^2 + \partial_\theta{}^2 + (r\partial_z)^2$ is the differential operator generated by

$$r\partial_r, \quad \partial_\theta, \quad \text{and} \quad r\partial_z$$

on $\overline{M} := [0,\infty) \times S^1 \times \mathbb{R}$. Edge differential operators.

Limit operators for the *b*-calculus

b-calculus:

($\overline{M}$ = compact manifold with boundary. $\mathcal{V}_b(\overline{M}) := \{X \in \mathcal{C}^\infty(\overline{M}; T\overline{M})$ tangent to $\partial\overline{M}\}$ and $\mathrm{Diff}(\mathcal{V}_b(\overline{M}))$ of differential operators generated by $\mathcal{V}_b(\overline{M})$. $r = x$ distance to the boundary $\partial\overline{M}$.)

Let $x_1 = r, x_2, \ldots, x_n$ be local coordinates near some boundary point. Then, $P \in \mathrm{Diff}(\mathcal{V}_b(\overline{M}))$ if, and only if,

$$Pu(x) = \sum_{|\alpha| \leq m} a_\alpha(x)(x_1\partial_1)^{\alpha_1}\partial_2^{\alpha_2}\ldots\partial_n^{\alpha_n}u(x)\,.$$

(**Totally characteristic operators.** Never elliptic if $\partial\overline{M} \neq \emptyset$.)

Indicial operator $r\partial_r \to \partial_t$ **related to the isotropy!**

Fredholm conditions

- $(\overline{M}, \mathcal{V}) =$ Lie manifold and let $\{Z\}$ be the **orbits** of $\mathcal{V}$ on $\partial\overline{M}$.
- We associate to each orbit Z a simply-connected Lie group G_Z, with $\text{Lie}\, G_Z \simeq \mathfrak{g}_x$, $x \in Z$, and
- associate to each $D \in \text{Diff}(\mathcal{V})$ a differential operator D_Z a G_Z-invariant differential operator on $Z \times G_Z$ **(limit operator).**

Conjecture (Carvalho-V.N.-Qiao, ...)

Let $(\overline{M}, \mathcal{V})$ be a Lie manifold and $D \in \text{Diff}(\mathcal{V})$ have order m.

$D : H^s(M) \to H^{s-m}(M)$ is **Fredholm** $\Leftrightarrow$

D **elliptic** and all $D_Z : H^s(Z \times G_Z) \to H^{s-m}(Z \times G_Z)$ are **invertible** .

Back to the quantization program

Say $D \in \mathrm{Diff}(A)$ is of order $2m$ and we want to prove that

$$D : H^{2m}(M) \to L^2(M)$$

is **invertible** (Hadamard well-posed) or **Fredholm.**

$(1+\Delta_g)^m : H^{2m}(M) \to L^2(M)$ is an isomorphism (the definition).

Our problem is then equivalent to proving that

$$D(1+\Delta_g)^{-m} : L^2(M) \to L^2(M)$$

is invertible (resp. Fredholm).

$D(1+\Delta_g)^{-m}$ is a **pseudodifferential operator** of order zero (no longer differential). **(Technical reasons: norm closure: C^*-algebra; we lose asymptotics, regularity, ... , but OK for Fredholm.)**

Pseudodifferential operators

(Pseudodifferential operators were discussed only briefly.)

- Let $(\overline{M}, A)$ be a Lie manifold, $M := \overline{M} \smallsetminus \partial\overline{M}$, the interior of $\overline{M}$.

The 'Quantization Program'

To construct a **pseudodifferential calculus** $\Psi^m_{pr}(A)$ on M with the usual symbolic properties and which reflects the properties of $\mathrm{Diff}(A)$, the differential operators generated by $\mathcal{V} = \mathcal{C}^\infty(A)$.

Usually many choices! We will use the fact that A extends TM and A^* extends T^*M.

- We fix a compatible metric g on M, as before ($TM \subset A$).
- Let $\exp : TM \to M$ be the exponential map (associated to g), which we know exists since M is complete.

Definition of $\Psi_{pr}^{-\infty}(A)$

Let $0 < r < r_M$, where r_M is the injectivity radius of M, (positive).

Let $\pi : A \to \overline{M}$ be the projection. For $k \in \mathcal{C}^\infty(M^2)$ and $v \in TM$ we set

$$\tilde{k}(v) := k(\pi(v), \exp(v)).$$

Thus $\tilde{k} = k \circ (p, \exp) \in \mathcal{C}^\infty(TM)$.

We let $\Psi_r^{-\infty}(A)$ be the operators with kernels $k : M^2 \to \mathbb{C}$ with support in $\{\mathrm{dist}_g(x, y) < r\}$ such that $\tilde{k}$ **extends to a smooth function on** A.

Definition

Finally, we let $\Psi_{pr}^{-\infty}(A) =$ the **algebra generated by** $\Psi_r^{-\infty}(A)$.

(It consists of properly supported ops; unlike the *b* and edge calculi)

The construction of $\Psi^m_{pr}(A)$

To define the "very small calculus" $\Psi^m_{pr}(A)$, we choose first a "quantization" map $q : S^\infty_{cl}(T^*M) \to \Psi^\infty(M)$ as follows.

Let $\chi : [0, \infty) \to [0, 1]$ be smooth cut-off f. (1 near 0, 0 far from 0) and

$$q : S^m_{cl}(T^*M) \to \Psi^m(M)\,,$$

$$\big[q(a)u\big](x) := (2\pi)^{-n} \int_{T^*_x M} a(\xi) \left(\int_{T_x M} e^{-\imath \langle y, \xi \rangle} \chi(\|y\|) u(\exp(y)) dy \right) d\xi\,.$$

Definition

We then let $\Psi^m_{pr}(A) := q(S^m_{cl}(A^*)) + \Psi^{-\infty}_{pr}(A)\,.$

($S^m_{cl}(E)$ was defined using local trivializations of the v. bundle $E \to \overline{M}$.)

Properties

- $\Psi_{pr}^{m}(A)\Psi_{pr}^{m'}(A) \subset \Psi_{pr}^{m+m'}(A)$.
- $\sigma_m(q(a)) = a + S_{cl}^{m-1}(A^*)$ for all $a \in S_{cl}^{m}(A^*)$; hence, D is a differential operator in $q(S_{cl}^{m}(A))$, iff $D \in \mathrm{Diff}(A)$.
- $D(1+\Delta_g)^{-m} \in \overline{\Psi}^0(A)$ in general, but $D(1+\Delta_g)^{-m} \in \overline{\Psi}^0(A) =$ a **groupoid C^*-algebra** in favorable situations.
- compact operators$= \mathcal{K} \subset \overline{\Psi}^0(A)$ and we obtain a map

$$\overline{\Psi}^0(A)/\mathcal{K} \to \mathcal{B}(L^2(M))/\mathcal{K}\,,$$

which will decide the Fredholm property of our operators. We just need to **decide which operators are invertible in** $\overline{\Psi}^0(A)/\mathcal{K}$.

The role of groupoids (informal)

- Ehresman (Brussels 1959).
- Examples of groupoids $\mathcal{G}$: Melrose's "*b*-**double space**"
- The groupoid $\mathcal{G}$ must satisfy $A(\mathcal{G}) = A$, its Lie algebroid be the one appearing in the quantization program. **(Integration of Lie algebroids!)**
- Depending on what one wants, the groupoid must satisfy some additional conditions ("nice")
- If the groupoid is "nice" ($\mathcal{G}|\partial\overline{M}$ Hausdorff and satisfies the **(ISF)** condition **("Strong Exel property")**) $\Rightarrow$ Fredholm conditions.

Groupoids

Definition

A **groupoid** $\mathcal{G}$ is a small category s.t. all morphisms are invertible.

Notations:

1. $\mathcal{G} \rightrightarrows \overline{M}$ denotes a groupoid with objects (or **units**) M.
2. $d, r : \mathcal{G} \to \overline{M}$ give the **domain** and **range** of a morphism.

Typically, $\overline{M}$ is compact with corners and M is its interior.

A more concrete definition of a groupoid $d, r : \mathcal{G} \rightrightarrows \overline{M}$ is in terms of the **structural morphisms** d, r, μ, u, ι that define it.

Canonical morphisms

The **structural morphisms** d, r, μ, u, ι satisfy:

1. The prod $\mu(g, h) := gh$ is defined if, and only if, $d(g) = r(h)$;
2. $\mu : \mathcal{G}^{(2)} := \{(g, h) | \ d(g) = r(h)\} \to \mathcal{G}$, associative.
3. "Unit map" $u : \overline{M} \to \mathcal{G}$, s.t. $gu(d(g)) = g = u(r(g))g$.
4. $d(u(x)) = r(u(x)) = x$ (so we will identify $x \simeq u(x)$.)

   Above: $\mathcal{G}$ = **category**.Next: "all morphisms are invertible."
5. "Inverse" $\iota : \mathcal{G} \to \mathcal{G}$, s.t.

$$g\iota(g) = r(g) = u(r(g)) \quad \text{and} \quad \iota(g)g = d(g)\,.$$

Outline

We show how to use C^*-algebras, Lie groupoids, their Lie algebroids and associated algebras to characterize Fredholm operators

Fredholm conditions

- $(\overline{M}, \mathcal{V})$ = Lie manifold and $\{Z\}$ = the **orbits** of $\mathcal{V}$ on $\partial\overline{M}$.
- Orbit $Z \to G_Z$, a simply-connected Lie group with $\mathrm{Lie}\, G_Z \simeq \mathfrak{g}_x$,
- $D \in \mathrm{Diff}(\mathcal{V}; E, F) \to$ differential operators $D_Z = \pi_Z(D)$ a G_Z-invariant differential op. on $Z \times G_Z$ **(limit operator).**

Conjecture (Carvalho-V.N.-Qiao, ...)

Let $(\overline{M}, \mathcal{V})$ be a Lie manifold and $D \in \mathrm{Diff}(\mathcal{V}; E, F)$ have order m.

$D : H^s(M; E) \to H^{s-m}(M; F)$ is **Fredholm** $\Leftrightarrow$ D **elliptic** and all $D_Z : H^s(Z \times G_Z; E) \to H^{s-m}(Z \times G_Z, F)$ are **invertible** .

$\sigma_m(D) \in \mathcal{C}^\infty(S^*A; \mathrm{Hom}(E, F))$.

Statement about the $\{\pi_Z\}$, C^*-algebras.

Proving the Fredholm property

$\mathcal{H}$ =Hilbert space, $\mathcal{K}(\mathcal{H})$ =compact operators on $\mathcal{H}$.

- **How to prove that** $T \in \mathcal{B}(\mathcal{H})$ is **Fredholm?**
- **Answer:** Atkinson's theorem (T is invertible modulo $\mathcal{K}(\mathcal{H})$).
- Equivalent formulation, the image of T is invertible in $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$.
- It is convenient to use C^*-algebras.

C^*-algebras

- A **concrete C^*-algebras** is a subalgebra $A = A^* = \overline{A} \subset \mathcal{B}(\mathcal{H})$.
- If $1, a \in A$ and $a^{-1} \in \mathcal{B}(\mathcal{H})$, then $a^{-1} \in A$.
- An **abstract C^*-algebras** is one isometrically isomorphic to a concrete one. (Ex. $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$.)
- How to prove that something is invertible in A?
- **"Invertibility sufficient families of representations"** (Rabinovich-Roch-Silbermann).

Invertibility sufficient families of representations

$A = C^*-$algebra.

- A **representation** of A is a *-morphism $\pi : A \to \mathcal{B}(\mathcal{H}_\pi)$.

Definition

Let $\mathcal{F} := \{\pi\}$ be a family of representations of A. We say that $\mathcal{F}$ is "invertibility sufficient" (ISF) if the following condition is satisfied:

$a \in A$ is invertible in A if, and only if, $\pi(a)$ is invertible in $\mathcal{B}(\mathcal{H}_\pi)$ for all $\pi \in \mathcal{F}$.

- **Example:** $A = \mathcal{C}(X)$, where X is a compact space (model commutative C^*-algebra). We let

$$\mathcal{F} := \{e_x\}, \quad x \in X,$$

$e_x(f) = f(x)$, the **evaluation** at x. Then $\mathcal{F}$ **is (ISF).**

C^*-algebras

In our applications:

- The family $\{\pi\}$ will be the family of **regular representations** of some Lie groupoid $\mathcal{G}|\partial\overline{M}$ (where $\mathcal{G}$ is the "double space").
- $\pi(a)$ will be the "limit operators" of a.
- The algebra $A = C^*(\mathcal{G})/\mathcal{K}$ or $A = \overline{\Psi}^0(\mathcal{G})/\mathcal{K}$.
- We obtain Fredholm conditions when the family of regular representations of $\mathcal{G}|\partial\overline{M}$ is **(ISF)** condition **("Strong Exel property").**

Lie groupoids and convolution

Definition

A **Lie groupoid** $\mathcal{G} \rightrightarrows \overline{M}$ is a groupoid such that $\mathcal{G} =$ manifold (poss. non Hausdorff!), $\overline{M}$ is a Hausdorff manifold (poss. w. corners), all structural maps are smooth, and d is a submersion.

The space $\mathcal{C}_c(\mathcal{G})$ becomes an involutive (pre)Banach algebra:

$$(\varphi_1 * \varphi_2)(g) := \int_{d(h)=d(g)} \varphi_1(gh^{-1})\varphi_2(h)\, \mathrm{d}\lambda_{d(g)}(h)$$

λ_x is a suitable set of measures on $\mathcal{G}_x := d^{-1}(x)$ (**Haar system,** it is the natural volume form for **Lie groupoids:** $\mathcal{G}_x =$ manifold).

(norm $\|f\| := \max_{x\in\overline{M}} \int_{\mathcal{G}_x} |f(h)| d\lambda_x(h)$.)

The Lie algebroid of a Lie groupoid

- Let $\mathcal{G} \rightrightarrows \overline{M}$ be a Lie groupoid with structural map $d, r : \mathcal{G} \rightrightarrows \overline{M}$.
- We consider $\mathcal{V}$ to be the space of d-vertical vector fields (tangent to the fibers of d) that are **right invariant.**
- $\mathcal{V}$ dentifies with the sections of a vector bundle on $\overline{M}$, the **Lie algebroid $A(\mathcal{G})$ of $\mathcal{G}$.**
- The measures on $\mathcal{G}_x := d^{-1}(x)$ comes from a metric on $A(\mathcal{G})$.

To use $\mathcal{G}$, we need $A(\mathcal{G}) = A$, that is, we need to **integrate** the given Lie algebroid (Lie's third theorem). This is the famous construction of the double spaces (Mazzeo, Melrose, ... , the triple space is the space of composable arrows).
There are general theorems, but the resulting groupoid is **useless.**

Groupoid C^*–algebras

Define the **regular representation** π_x, $x \in \overline{M}$,

$$\pi_x : \mathcal{C}_c(\mathcal{G}) \to \mathcal{B}(L^2(\mathcal{G}_x, \lambda_x)), \quad \pi_x(f)g := f * g .$$

Definition (Reduced C^*–algebra $C^*_r(\mathcal{G})$)

$C^*_r(\mathcal{G})$ is the completion of $\mathcal{C}_c(\mathcal{G})$ with respect to

$$\|\varphi\|_r := \sup \|\pi_x(\varphi)\| , \quad x \in \overline{M} .$$

If $\mathcal{G}$ is **metrically amenable** (i.e. $C^*(\mathcal{G}) \simeq C^*_r(\mathcal{G})$), then we have the short exact sequence for all $U = F^c$ open, invariant (Renault)

$$0 \to C^*_r(\mathcal{G}_U) \to C^*_r(\mathcal{G})^+ \to C^*_r(\mathcal{G}_F)^+ \to 0$$

We obtain **Fredholm conditions** when $\mathcal{G}_U := d^{-1}(U) = U \times U$, since $C^*(\mathcal{G}_U) \simeq \mathcal{K}$ (next).

Examples of Lie groupoids I

1. The **pair groupoid** $\mathcal{H} := M \times M$ is the groupoid $\mathcal{H}$ having *exactly* one arrow between any two units. Hence

$$d(x,y) = y,\ r(x,y) = x,\ \text{ and } \ (x,y)(y,z) = (x,z).$$

$A(M \times M) = TM.$

The *Haar measures* λ_x on $\mathcal{G}_x = M \times \{x\}$, $x \in M$, satisfy $\lambda_x = \lambda_y = \lambda$

$$\phi_1 * \phi_2(x,z) \ := \ \int_M \phi_1(x,y)\phi_2(y,z)d\lambda(y)\ ,$$

and hence

$$C^*(\mathcal{H}) \ \simeq \ C_r^*(\mathcal{H}) \simeq \mathcal{K}(L^2(M,\lambda))\,.$$

Finally, π_x is the natural representation on $L^2(M;\lambda)$ (indep. of x).

Other examples of Lie groupoids II

3. A locally compact group: G with its Haar measure, $M = \{e\}$.

   $C^*(G) \cong C_r^*(G) \Leftrightarrow G$ is amenable (this is the case when we have edge calculi or desingularization in general).

4. A **space:** $\mathcal{G} = \overline{M}$ (only units), $C^*(\mathcal{G}) = C_r^*(\mathcal{G}) = \mathcal{C}_0(\overline{M})$.

5. The **product** of the two: $\mathcal{G} = \overline{M} \times G \rightrightarrows \overline{M}$, $d = r$.

6. A **bundle of Lie groups** $\mathcal{G} \to \overline{M}$, in which case $d = r$.

The last example is the main building block for the groupoids in applications.

All these examples satisfy (ISF) if, and only if, the groups involved (if any) are **amenable.**

Other examples of Lie groupoids III

7. The **cross-product groupoid**: obtained from the action of a locally compact group G on manifold with corners $\overline{M}$.

   Then $\mathcal{G} = G \times \overline{M} \rightrightarrows \overline{M}$ with

$$d(g,x) = x\,,\ \ r(g,x) = gx\,,\ \ (h,gx)(g,x) = (hg,x)\,,$$

   and $C^*(\mathcal{G}) = \mathcal{C}(\overline{M}) \rtimes G$ and $C^*_r(\mathcal{G}) = \mathcal{C}_0(\overline{M}) \rtimes_r G$.

(cross-prod: Exel, Georgescu-Iftimovici, Karlovich, Mantoiu, Mougel, Rabinovich, Roch, Silbermann, ...).

Used for the "scattering calculus" and for the N-body problem with $G = \mathbb{R}^n$ and $\overline{M}$ a suitable blow-up of $\overline{\mathbb{R}}^n$, (Georgescu, Vasy, Ammann-Mougel-V.N.)

Other examples of groupoids IV

8. The **"*b*-groupoid"** $\mathcal{G}_b(\overline{M})$ for $\overline{M}$ with smooth boundary $\partial\overline{M}$:

$$\mathcal{G}_b(M) := (M \times M) \sqcup (\partial\overline{M} \times \partial\overline{M}) \times \mathbb{R}\,,$$

$M := \overline{M} \smallsetminus \partial\overline{M}$ is the *interior* of $\overline{M}$.

$\mathcal{G}_b(\overline{M})$ is obtained from the 'double-space' construction by **removing outside faces**

Also Grieser, Lesch, Monthubert, Schrohe, Schulze, ...

$M \subset \overline{M}$ and $\partial\overline{M} \subset \overline{M}$ are invariant subsets of $\overline{M}$. The restriction of $\mathcal{G}_b(M)$ to M is $M \times M$ and to $\partial\overline{M}$ is $\partial\overline{M} \times \partial\overline{M} \times \mathbb{R}$. Each satisfies (ISF), so $\mathcal{G}_b(M)$ satisfies (ISF), next.

The (ISF) property

Assume $\mathcal{G}$ to be Hausdorff (non trivial!).

If $U \subset \overline{M}$ is an invariant open subset and if $\mathcal{G}|_U$ and $\mathcal{G}|_U$ satisfy (ISF,) then $\mathcal{G}$ satisfies (ISF), a consequence of the exact sequence

$$0 \to C_r^*(\mathcal{G}_U) \to C_r^*(\mathcal{G})^+ \to C_r^*(\mathcal{G}_F)^+ \to 0\,.$$

We can "slice and dice" our groupoid (which we could not have done with our operator!)

We prove the Fredholm Conjecture in stages (for sufficiently nice groupoids).

We can "glue" groupoids that satisfy the conjecture (Rémy Côme).

If a groupoid $\mathcal{G}$ satisfies the conjecture, then the groupoid associated to a desingularization also satisfies the conjecture ("blow-up invariant").

The desingularization groupoid I

• Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and $L \subset M$ a **"nice" submanif**, meaning that it has a tubular nbhd in an *algebroid* sense.

• Let $\pi : SNL \to L$ be the **unit sphere bundle** in the normal bundle $\pi : NL \to L$ of L in M ("pies" everywhwere!).

• Let $[M : L] =$ usual **blow-up** obtained by replacing L with SNL.

• On L we consider the bundle of Lie groups $\mathcal{L} := TL \rtimes \mathbb{R}_+^* \xrightarrow{\pi} L$ and define the **fibered pull-back groupoid**

$$\mathcal{H} := SNL \times_L SNL \times_L \mathcal{L} := \{ (x, y, v) \mid \pi(x) = \pi(y) = \pi(v) \}$$

The **desingularization** is then

$$[[\mathcal{G} : L]] := \mathcal{H} \sqcup \mathcal{G}_{M \smallsetminus L}^{M \smallsetminus L} \rightrightarrows [M : L]$$

($\mathcal{G}_A^B = d^{-1}(A) \cap r^{-1}(B)$). Uses res. of Debord-Skandalis.

The desingularization groupoid II

The **desingularization**

$$[[\mathcal{G} : L]] := \mathcal{H} \sqcup \mathcal{G}_{M\smallsetminus L}^{M\smallsetminus L} \rightrightarrows [M : L]$$

was constructed to satisfy the **usual condition ($r_L =$ dist to L):**

$$\mathcal{C}^\infty(A([[\mathcal{G} : L]])) = r_L\mathcal{C}^\infty([M : L])\mathcal{C}^\infty(A((\mathcal{G}))\,.$$

If M has **no** boundary and $\mathcal{G} = M \times M$, then any smooth $L \subset M$ is nice ($A(\mathcal{G})$-tame) and the desingularization $[[\mathcal{G} : L]]$ defines the "edge" calculus for the boundary fibration $SNL := \partial[M : L] \to L$ (Grušin, Schulze, Mazzeo, Krainer).

If, furthermore, L is a point, we obtain the *b*-groupoid for $[M : L]$.

Conclusion

- The Fredholm conjecture may be true for all Lie manifolds. Hard (close to conjectures in C^*-algebras: Effros-Hahn, ...)
- In practice, however: the hardest part is to integrate the Lie algebroid A defining the Lie manifold $\mathcal{C}^\infty(A) = \mathcal{V}$, the vector fields. You want a **Hausdorff** groupoid as "double space."
- There exist general results (Crainic-Fernandez, Debord, V.N.,) but they do not give (in general) Hausdorff groupoids.
- If one is lucky, $\mathcal{G} = \cup_Z Z \times Z \times G_Z \cup M \times M$, (use [V.N.] ≈ 2000).
- Check that on has a stratification of the units $\overline{M} \supset \mathcal{O}_k$, with the orbits Z **closed** in $\mathcal{O}_{k+1} \smallsetminus \mathcal{O}_k$ and **same stabilizer.** (Ex. Compact group actions: Albin-Melrose.) True in all examples that I know.

Thank you for your attention!

See the last slide for some references.

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References for the second part

This list is very incomplete, but may help you get started.

[1] *Pseudodifferential operators on differential groupoids* (with Weinstein and Xu)
https://arxiv.org/pdf/funct-an/9702004.pdf

[2] *Groupoids and the integration of Lie algebroids*
https://arxiv.org/pdf/math/0004084.pdf

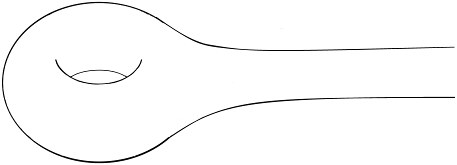
[3] *Fredholm conditions on non-compact manifolds: theory and examples* (with Carvalho and Qiao)
https://arxiv.org/pdf/1703.07953.pdf

[4] *Gluing groupoids: Fredholm conditions and layer potentials* (Carvalho, Côme, and and Qiao)
https://arxiv.org/pdf/1811.07699.pdf

REMOVED STUFF 1

“Pictorial” definition of cylindrical ends (repeated)

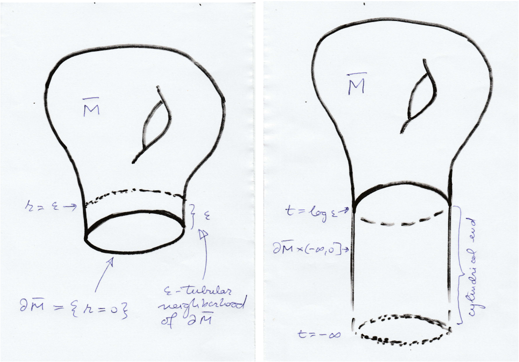
A **model** class of non-compact manifolds: the **manifolds with (straight) cylindrical ends** (picture):



As for manifolds with bounded geometry, the Sobolev spaces on manifolds with cylindrical ends can be defined using **partitions of unity.**

The **formal definition** is on the next slides.

Formal definition of *M* with cylindrical ends



- $\overline{M}$ = a smooth, compact Riemannian manifold, $\partial\overline{M} \neq \emptyset$.
- To $\overline{M}$ we attach the cyl. end $\partial\overline{M} \times (-\infty, 0]$ along $\partial\overline{M} \equiv \partial\overline{M} \times \{0\}$:

$$M := \overline{M} \cup \partial\overline{M} \times (-\infty, 0]$$

Translation invariant operators near infinity

Recall the **partial translations** (isometries) $\Phi_s(x,t) := (x, t-s)$, $s \geq 0$ on the **half-infinite cylindrical end**

$$\Phi_s : \partial\overline{M} \times (-\infty, 0] \to \partial\overline{M} \times (-\infty, -s] \,.$$

Definition

$P : C_c^\infty(M) \to C_c^\infty(M)$ is called **translation invariant at infinity** if

- its distribution kernel is supported in a neighborhood of the diagonal $\{(x,y) \in M \times M : \mathrm{dist}(x,y) < \varepsilon\}$ for some $\varepsilon > 0$, and
- there is $R > 0$ s.t. if $\mathrm{supp}(f) \subset \partial\overline{M} \times (-\infty, -R)$ and $s > 0$,
$$P\Phi_s(f) = \Phi_s P(f) \,.$$

The differential operators that are translation invariant at infinity have **coefficients** in $W^{\infty,\infty}$, and hence, they are bounded on all Sobolev spaces. The same is true of the pseudodifferential operators.

Differential operators

Let $E, F \to M$ be vector bundles, with E endowed with a connection.

Definition

Let $a^{[k]}$ be measurable section of $\mathrm{Hom}(T^{*\otimes k} \otimes E; F)$ and $\nabla^0 = id$. A **∇–differential operator** is a map of the form

$$P = a \cdot \nabla^{tot} := \sum_{j=0}^{\mu} a^{[j]} \nabla^j : \mathcal{C}^\infty(M; E) \to \mathcal{C}^\infty(M; F).$$

$\mathrm{ord}(P)$ is the least μ for which such a writing exists (the *order* of P).

Suitable extensions by continuity of P will also be called ∇–differential operators and will be denoted by the same letter.

Locally, there is no difference between the ∇–differential operators and the usual differential operators.

Mapping properties of differential operators

Let $E, F \to M$ be vector bundles with metrics and metric-compatible with connections.

Theorem

Let $\ell \in \mathbb{Z}_+$ and $a^{[k]} \in W^{\ell,\infty}(M; \mathrm{Hom}(T^{*\otimes k} \otimes E; F))$ and $\nabla^0 = id$. Then the ∇–differential operator

$$P = a \cdot \nabla^{tot} := \sum_{j=0}^{\mu} a^{[j]} \nabla^j : W^{s+\mu,p}(M; E) \to W^{s,p}(M; F).$$

for $0 \leq s \leq \ell$. **(Operator with coefficients in $W^{\ell,\infty}$.)**

Better results (including $|s| \leq \ell$) for **manifolds with bounded geometry,** but first the simpler particular case of manifolds with *cylindrical ends.*

Curvature

It turns out that the **anti-symmetric** part of ∇^2 (i.e. the composite map)

$$\nabla^2 : \mathcal{C}^\infty(E) \to \mathcal{C}^\infty(T^{*\otimes 2}M \otimes E) \to \mathcal{C}^\infty(\Lambda^2 T^*M \otimes E)$$

is actually **linear** (**curvature**, see also next).

Curvature

The **curvature**

$$R \in \Omega^2(M; \mathrm{End}(TM)) = \mathcal{C}^\infty(\Lambda^2 T^*M \otimes \mathrm{End}(TM))$$

of (E, ∇) **(or of** (M, g) **if** $E = TM$**)** is such that

$$\langle \nabla^2 u, X \wedge Y \rangle = R(X, Y)u\,.$$

Explicitly, $\nabla^2 u(X, Y) = \nabla_X \nabla_Y u - \nabla_{\nabla_X Y} u$, and hence the **curvature** R of (E, ∇) is given by

$$R(X, Y)u = \nabla_X \nabla_Y u - \nabla_Y \nabla_X u - \nabla_{[X,Y]} u\,.$$

Definition

We say that M has **totally bounded curvature** if its curvatures $R^M := (\nabla^M)^2$ and all its covariant derivatives $(\nabla^M)^k R^M$ are bounded.

Injectivity radius and geodesics

A C^1-curve $\gamma : (a, b) \to M$ is a **geodesic** if $\nabla^M_{\gamma'(t)}\gamma'(t) = 0$. It is locally distance minimizing and uniquely determined by any $\gamma'(t_0),\ t_0 \in I$. If $M = \mathbb{R}^d$ with the usual metric, then a geodesic is just a straight line.

Let

$$\exp^M(v) := \gamma_v(1),$$

where γ_v is the unique geodesic with $\gamma'(0) = v$, the **(geodesic) exponential map.**

Let $B_r^X(x) := \{y \in X \mid d(x, y) < r\}$, and

$$r_{\mathrm{inj}}^M(p) := \sup\{r \mid \exp^M : B_r^{T_pM}(0) \to B_r^M(p) \text{ is a diffeomorphism}\} \quad \text{and}$$

$$r_{\mathrm{inj}}(M) := \inf_{p\in M} r_{\mathrm{inj}}^M(p).$$

$r_{\mathrm{inj}}(M) =$the **injectivity radius** of the Riemannian manifold (M, g).

Manifolds with bounded geometry

Recall the curvature $R^M(X, Y)u = \nabla_X^M \nabla_Y^M u - \nabla_Y^M \nabla_X^M u - \nabla_{[X,Y]}^M u$ and that M has **totally bounded curvature** if its curvatures $R^M := (\nabla^M)^2$ and all its covariant derivatives $(\nabla^M)^k R^M$ are bounded.

The following concept is crucial.

Definition

A smooth Riemannian manifold (M, g) is said to have **bounded geometry** if M has totally bounded curvature and $rinj(M) > 0$.

Then $H^{-s}(M) := H^{-s}(M)^*$. (**Negative order spaces.**)

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Examples

The following are manifolds with bounded geometry:

(i) A closed manifold (i.e. a smooth, compact manifold without boundary).

(ii) $M = \mathbb{R}^d$ with the standard (Euclidean) metric $dx^2 := (dx_1)^2 + (dx_2)^2 + \ldots + (dx_d)^2$.

(iii) $D \times M$, where M is a manifold with bounded geometry and D is discrete set.

(iv) $M_1 \times M_2$, where M_1 and M_2 have bounded geometry.

(v) However, if M is a manifold with bounded geometry and $U \subset M$ is an open subset, then U has totally bounded curvature, but may not have positive injectivity radius (so no bounded geometry).

Boundary and bounded geometry

Let now M_0 be a Riemannian manifold *with boundary,* then $r_{\rm inj}(M_0) = 0,$ so a manifold with non-empty boundary will never have bounded geometry in the sense of the above definition. The way around this conundrum was found by Schick (2001), who has defined the concept of "manifold with boundary and bounded geometry," (we shall call these manifolds "manifolds with boundary and *relative* bounded geometry," to avoid confusions). We recall the equivalent definition of manifolds with boundary and relative bounded geometry in (Ammann-Grosse-V.N.)

The main point of that definition is to assume that the boundary ∂M_0 of M_0 is a suitable submanifold of a (boundaryless) manifold M with bounded geometry.

Second fundamental form

Let hence M be a (boundaryless) manifold with bounded geometry and let us consider a hypersurface $H \subset M$, that is, a submanifold H of M of codimension $\dim(M) - \dim(H) = 1$. We assume that H carries a globally defined unit normal vector field ν. We let

$$\exp^{\perp}(x, t) := \exp_x^M(t\nu_x)$$

be the exponential in the direction of the chosen unit normal vector. We shall need the **second fundamental form** II^H of H in M, which, we recall, is defined by

$$\mathrm{II}^H(X, Y)\nu := \nabla_X^M Y - \nabla_X^H Y,$$

where ∇^Z is the Levi-Civita connection of Z. Equivalently, since $g(\nu, \nabla_X^H Y) = 0$, we have $\mathrm{II}^H(X, Y) := g(\nu, \nabla_X^M Y)$.

Bounded geometry hypersurface

Definition

Let (M, g) be a Riemannian manifold of bounded geometry and $H \subset M$ be a hypersurface with unit normal vector field ν on H. We say that H is a **bounded geometry hypersurface in M** if:

(i) H is a closed subset of M;

(ii) all covariant derivatives $(\nabla^H)^k \mathrm{II}^H$, $k \geq 0$, are bounded;

(iii) $\exp^\perp : H \times (-\delta, \delta) \to M$ is a diffeomorphism onto its image for some $\delta > 0$.

Then H has bounded geometry.

If $H \subset M$ is a *compact* hypersurface, then H is a bounded geometry hypersurface in M.

Manifolds with boundary and relative bounded geometry

We are ready now to recall the definition of a central concept in analysis on manifolds.

Definition

We shall say that M_0 is a **manifold with boundary and relative bounded geometry** if M_0 is isometrically contained in a (boundaryless) Riemannian manifold M with bounded geometry such that ∂M_0 is a bounded geometry hypersurface in M.

Note that we use the term *"manifold with boundary and relative bounded geometry,"* which we think is more precise than the term *"manifold with boundary and bounded geometry"* used before.

Trace and 'negative' spaces

Let ∂_ν be the normal derivative at the boundary and let

$$H_0^k(M_0) := \cap_{j=0}^{k-1} \ker(\mathrm{res} \circ \partial_\nu^j)$$

denote the joint kernel of the restrictions maps $\mathrm{res} \circ \partial_\nu^j$, $0 \le j \le k-1$.

Theorem (Trace theorem: Grosse-Schneider, Triebel)

Let M_0 be a manifold with boundary and relative bounded geometry. Then, for every $s > 1/2$, the restriction $res\colon \mathcal{C}_c^\infty(M_0) \to \mathcal{C}_c^\infty(\partial M_0)$ extends by continuity to a surjective map

$$\mathrm{res}\colon H^s(M_0) \to H^{s-\frac{1}{2}}(\partial M_0).$$

Moreover, $\mathcal{C}_c^\infty(M_0 \smallsetminus \partial M_0)$ is dense in $H_0^k(M_0)$ and $H^{-k}(M_0)$ identifies with $H_0^k(M_0)^*$, $k \in \mathbb{N}$.

Finite width

Definition

Let (M_0, g) be a Riemannian manifold with boundary ∂M_0. We say that M_0 has *finite width* if:

(i) (M_0, g) is a manifold with boundary and relative bounded geometry and

(ii) The function $M_0 \ni x \to \mathrm{dist}_{M_0}(x, \partial M_0)$ is bounded on M_0.

The last condition is equivalent to

$$\text{“}\exists\, R > 0 \text{ such that } M_0 \subset \{x \in M \mid \exists y \in \partial M_0,\ \mathrm{dist}_{M_0}(x, y) < R\}.\text{”}$$

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Poincaré inequality

Theorem (Ammann-Grosse-V.N., Sakurai)

Let (M_0, g) be a Riemannian manifold with finite width. Then there exists $0 < C_{M_0} < \infty$ such that, for all $f \in \mathcal{C}_c^\infty(M_0)$ (thus $f = 0$ on the boundary of M_0),

$$\|f\|_{L^2(M_0)} \leq C_{M_0} \|df\|_{L^2(M_0)} .$$

A regularity argument then yields the following result.

Theorem (Ammann-Grosse-V.N.)

Let M_0 be a smooth Riemannian manifold with smooth boundary ∂M_0 and finite width. Then Δ induces isomorphisms

$$\Delta_D = \Delta \colon H^{m+1}(M_0) \cap H_0^1(M_0) \to H^{m-1}(M_0) , \quad m \in \mathbb{Z}_+ := \{0, 1, \ldots\} .$$

Applications: polygonal regularity

Let us consider a polygonal domain Ω_∞ with maximum angle α_{MAX} and the **Poisson problem**

$$\Delta u_\infty = f_\infty \text{ in } \Omega_\infty, \quad u_\infty = 0 \text{ on } \partial\Omega_\infty.$$

It is known that u_∞ has **limited regularity,** less than $H^{1+\pi/\alpha_{MAX}}$: Costabel, Dauge, Griesvard, Kondratiev, Mazya, ...

Theorem (Limited regularity: Kondratiev, ...)

Let $\delta < \pi/\alpha_{MAX}$. Then there exists $C = C_\delta > 0$ such that the solution $u_n \in H^1_0(\Omega_n)$ of the equation $\Delta u_n = f_n$ satisfies

$$\|u_n\|_{H^{1+\delta}(\Omega_n)} \leq C\|f_n\|_{L^2(\Omega_n)} .$$

Applications: well-posedness in weighted spaces

The **"limited regularity"** theorem of the previous slide is **not** very satisfying. It is, in any case, a consequence of the following more satisfying result in the Babuška-Kondratiev (weighted Sobolev) spaces

$$\mathcal{K}_a^m(\Omega) := \{\rho^{|\alpha|-a}\partial^\alpha u \in L^2(\Omega),\ |\alpha| \le m\}$$

$\rho =$distance to vertices (Costabel, Dauge, Kondratiev, Mazya, ...)

Theorem (Unlimited weighted regularity: Kondratiev '67, ...)

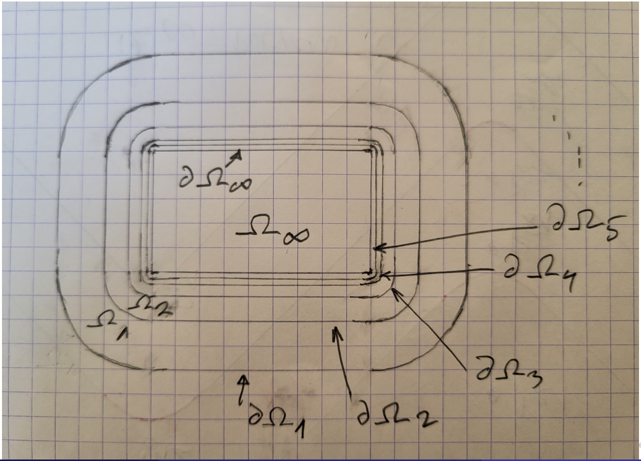
Let $0 \le \delta < \pi/\alpha_{MAX}$ and $m \in \mathbb{Z}_+$. Then there exists $C = C_{m,\delta} > 0$ such that the solution $u_\infty \in H_0^1(\Omega_\infty)$ of the equation $\Delta u_\infty = f_\infty$ satisfies

$$\|u_\infty\|_{\mathcal{K}_{\delta+1}^{m+1}(\Omega_\infty)} \le C\|f_\infty\|_{\mathcal{K}_{\delta-1}^{m-1}(\Omega_\infty)}.$$

This, as well as the next theorem, are consequences of the **well-posedness on manifolds with boundary and relative bounded geometry** (Thm 68 stated earlier).

Applications: "rounding up the corners"

Consider a *sequence of smooth domains* Ω_n, $n \in \mathbb{Z}$, converging to a polygonal domain Ω_∞ (*picture*) and the associated Poisson problems $\Delta u_n = f_n$, $u_n \in H_0^1(\Omega_n)$, for which we want **uniform estimates.**

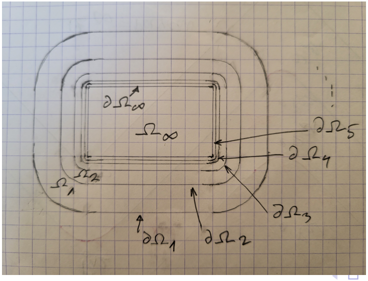


Applications: "rounding up the corners"

Theorem (Daniel-Labrunie-V.N.)

Let $\Omega_n \subset \mathbb{R}^2$ be a sequence of smooth domains "converging" to a polygonal domain Ω_∞, as in the picture (repeated below). Let $\delta < \pi/\alpha_{MAX}$. Then there exists $C = C_\delta > 0$ such that for all $n \in \mathbb{N} \cup \{\infty\}$, the solution $u_n \in H^1_0(\Omega_n)$ of $\Delta u_n = f_n$ satisfies

$$\|u_n\|_{H^{1+\delta}(\Omega_n)} \leq C \|f_n\|_{L^2(\Omega_n)} .$$



REMOVED STUFF 2

Ellipticity

M is still a closed manifold (smooth, compact, without boundary).

Theorem

Let $a \in S^m(T^*M)$. The following are equivalent

1. *There exist $b \in S^{-m}(T^*M)$ such that $ab - 1 \in S^{-1}(T^*M)$*
2. *There are $C, R > 0$ such that $|a(\xi)| \geq C|\xi|^m$ for $\xi \in T^*M$, $|\xi| \geq R$.*

Then a is called **elliptic** *and this property is a property of its class in $S^m(T^*M)/S^{m-1}(T^*M)$.*

Definition

$P \in \Psi^m(M)$ is **elliptic** if $\sigma_m(P) \in S^m/S^{m-1}(T^*M)$ is elliptic.

We can easily **include vector bundles.**

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Fredholm and Atkinson

Recall the following that if X and Y are Banach spaces, an operator $T \in \mathcal{B}(X; Y)$ is **Fredholm** if

$$\dim \ker(P)\,,\ \dim(Y/PX) < \infty.$$

(It implies $PX =$ closed.)

Theorem (Atkinson's theorem)

*Let X and Y be Banach spaces; $T \in \mathcal{B}(X; Y)$ is **Fredholm** if, and only if, there exists $Q \in \mathcal{B}(Y; X)$ such that both $TQ - 1_Y$ and $QT - 1_X$ are compact.*

Fredholm property

Theorem

Assume M is a closed manifold and $P \in \Psi^m(M; E)$ is ***elliptic.*** *Then, for any $s \in \mathbb{R}$,*

$$P : H^s(M; E) \to H^{s-m}(M; E) \quad \textit{is } \textbf{Fredholm.}$$

Proof.

• P is elliptic means that there is $b \in S^{-m}(T^*M)$ such that $\sigma_m(P)b - 1 \in S^{-1}(T^*M)$.

• The surjectivity of σ_{-m} and its multiplicativity give that there exists $Q \in \Psi^{-m}(M)$ (**parametrix**) such that $PQ - 1, QP - 1 \in \Psi^{-1}(M)$.

• Consequently, $PQ - 1$ and $QP - 1$ are compact operators. Atkinson's theorem then gives that P is Fredholm. □

The converse is true (Fredholm for one s implies elliptic).

Extensions of the Fredholmness results

- **A quite satisfying result!** However, much is hidden. The pseudodifferential operators that we defined are not enough, they contain too few regularizing operators!
- We need to enlarge the initial, small calculus (open pr. in gen).
- **Open problem:** find the "right class of regularizing operators."
- **Second part of the lecture notes:** general methods to extend some of the results in this talk (Pseudodifferential operators, Fredholm property, ...) to other classes of manifolds.