Isoperimetric Inequalities for the Eigenvalues of the Laplacian

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This will be a short course centering on sharp, or *isoperimetric,* inequalities for the eigenvalues of the Laplacian. The basic set-up is to consider the eigenvalues of the Laplacian acting on some bounded domain Ω in Euclidean space $\mathbb{R}^n$ as functionals depending on the domain. Typically we consider Dirichlet, and sometimes Neumann, boundary conditions, and typically our focus is on low eigenvalues, such as the first eigenvalue, $\lambda_1(\Omega)$. If one considers some eigenvalue of the Laplacian, or a combination of eigenvalues (e.g., λ_2/λ_1), a natural question is to ask if, among some class of domains, it has a least, or greatest, value (more precisely a greatest lower bound (*glb,* or *inf*) or a least upper bound (*lub,* or *sup*)). If such a bound exists, one can ask whether it is ever attained, and, if so, for what domains. Natural choices for the class of domains to consider include domains of fixed area (in general, volume), perimeter (surface area), diameter, inradius, or ..., or the class of domains which are convex, simply connected, or satisfy various constraints on the curvature of their boundary. Such problems have come to be called *isoperimetric problems of mathematical physics,* by analogy to the classical isoperimetric inequality, and out of respect for the first book on the subject, by Pólya and Szegő, titled **Isoperimetric Inequalities in Mathematical Physics**.

In particular, in this course we shall focus on what one might think of as the three basic inequalities for low eigenvalues of the Laplacian, (1) the Faber-Krahn inequality, which gives the best lower bound for the first Dirichlet eigenvalue for domains of fixed area (volume), (2) the Segő-Weinberger inequality, which gives the best upper bound to the first nontrivial Neumann eigenvalue for domains of fixed area (volume), and (3) the sharp upper bound to the ratio of the second Dirichlet eigenvalue to the first (which applies to all bounded domains, since this ratio scales with dimension, and hence depends only on the shape of the domain). In all three of these cases, the sharp bound is attained, and the domain which attains it is the disk (or, generally, the ball in n dimensions).

We shall develop the methods that are needed to treat these problems, and go through the proofs in detail. These include the variational characterization of eigenvalues (Rayleigh quotients and the Min-Max Principle), symmetrization/rearrangement techniques (for Steiner and especially Schwarz symmetrization), and sharp center of mass results. In particular, we shall show that the Faber-Krahn inequality follows easily from an understanding of how the Dirichlet norm of a function changes under rearrangement, i.e., the Pólya-Szegő inequality. Additionally, we shall consider Talenti's theorem and how it leads on to Chiti's comparison theorem, which is a key element of the proof of the result for $\lambda_2(\Omega)/\lambda_1(\Omega)$). We will also indicate how the three basic inequalities have precise analogs for the Laplacian acting on domains in the unit sphere, $\mathbb{S}^n$, or in hyperbolic space, $\mathbb{H}^n$ (in the case of the sphere, one typically has to restrict to domains lying in a hemisphere). In making this generalization, we shall see that the "right" analog of the ratio result is that, for fixed λ_1, the disk/ball maximizes λ_2.

Time permitting, we will go on to discuss some further inequalities that build on the three basic inequalities, some other eigenvalue problems for which isoperimetric inequalities are known (such as the Steklov problem and the problem of the buckling or the vibration of a clamped plate), and various related conjectures and open problems. Other possible topics include eigenvalue asymptotics, Pólya's conjectures, and universal inequalities for eigenvalues, but these are most likely to be touched on only in passing, or briefly in the context of open problems.