# The biharmonic Steklov boundary value problem: positivity preserving and eigenvalues

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This minicourse is motivated on one hand from equations in continuum mechanics, biophysics or differential geometry and on the other hand from basic questions in the theory of partial differential equations.

Around 1800 the physicist Chladni was touring Europe and showing, among other things, the nodal line patterns of vibrating plates. Jacob Bernoulli II tried to model these vibrations by the fourth order operator $\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}$ but his model was not accepted, since it is not rotationally symmetric and it failed to reproduce the nodal line patterns of Chladni. The first use of $\Delta^2$ for the modeling of an elastic plate is attributed to a correction of J.L. Lagrange of a manuscript by Sophie Germain from 1811.

In order to have physically meaningful and mathematically well-posed problems the plate equation $\Delta^2 u = f$ has to be complemented with a suitable set of boundary data. The most commonly studied boundary value problems for second order elliptic equations are named Dirichlet, Neumann and Robin. These three types appear since they have a physical meaning. For fourth order differential equations, such as the plate equation, the variety of possible boundary conditions is much larger. Although the main purpose of this minicourse are Steklov boundary conditions, we will first shortly address some other boundary conditions that are physically relevant.

Assume that the plate, the vertical projection of which is the planar region $\Omega \subset \mathbb{R}^2$, is free to move horizontally at the boundary. Then a simple model for the elastic energy is

$$J(u) = \int_\Omega \left( \tfrac{1}{2} \left(\Delta u\right)^2 + (1-\sigma)\left(u_{xy}^2 - u_{xx}u_{yy}\right) - f\ u \right) dxdy, \tag{1}$$

where $f$ is the external vertical load, $u$ is the deflection of the plate in vertical direction and first order derivatives are left out which indicates that the plate is free to move horizontally. In (1) $\sigma$ is the Poisson ratio, which is defined by $\sigma = \frac{\lambda}{2(\lambda+\mu)}$ with the so-called Lamé constants $\lambda, \mu$ that depend on the material. For physical reasons it holds that $\mu > 0$ and usually $\lambda \geq 0$ so that $0 \leq \sigma < \frac{1}{2}$. Moreover, it always holds true that $\sigma > -1$ although some exotic materials have a negative Poisson ratio. For $\sigma > -1$, the quadratic part of the functional (1) is positive.

If we assume that the plate is hinged, several integration by parts yield the following Euler-Lagrange equation

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \Delta u - (1-\sigma)\,\kappa \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\kappa$ is the mean curvature of $\partial\Omega$. In this course, we consider the general problem in any space dimension $n \geq 2$

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \Delta u - a \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $a \in C^0(\partial\Omega)$. Our purpose is to study the following problems:

**Positivity preserving.** Determine conditions on $\Omega$ and $a$ such that the a vertical upward force ($f \geq 0$) generates a vertical upward deflection ($u \geq 0$). We determine several necessary and/or sufficient conditions for the validity of this principle and we analyze several limit situations.

**Eigenvalues.** In the case where $f \equiv 0$, study the constant values of $a$ (eigenvalues) for which (2) admits nontrivial solutions. On one hand we wish to study the whole spectrum, on the other hand much attention will be devoted to the least eigenvalue. Its important properties will be emphasized and the problem of its minimization among bounded domains will be addressed.

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