The biharmonic Steklov boundary value problem: positivity preserving and eigenvalues

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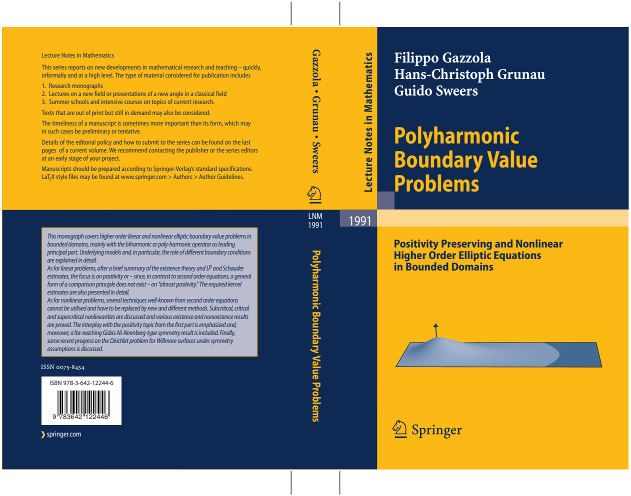
OUTLINE

1. The physical model.
2. Boundary conditions.
3. Positivity preserving.
4. Steklov eigenvalues.
5. Minimisation of the least Steklov eigenvalue.

Except for the very last arguments, all the contents of this minicourse and much more can be found in [GGS]



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THE KIRCHHOFF-LOVE MODEL FOR A THIN PLATE

Consider a plate, the vertical projection of which is the planar region $\Omega \subset \mathbb{R}^2$. A simple model for its **elastic energy** is

$$J(u) = \int_\Omega \left(\frac{1}{2} (\Delta u)^2 + (1-\sigma) \left(u_{xy}^2 - u_{xx} u_{yy} \right) - f \ u \right) dxdy,$$

where f = external vertical load, u = vertical deflection.

σ is the Poisson ratio: $\sigma = \frac{\lambda}{2(\lambda+\mu)}$ with the Lamé constants $\lambda \geq 0$, $\mu > 0$ that depend on the material, hence $0 \leq \sigma < \frac{1}{2}$. Usually $\sigma > -1$ and some exotic materials have a negative Poisson ratio.

For $-1 < \sigma < 1$, the quadratic part of the functional is positive.

BOUNDARY CONDITIONS: ONE DIMENSIONAL BEAM

Figure: The depicted boundary condition for the left endpoint of the beam is **clamped** whereas for the right endpoint it is **hinged**.

Clamped: $u(a) = u'(a) = 0$, also known as homogeneous Dirichlet boundary conditions.

Hinged: $u(b) = u''(b) = 0$, also known as homogeneous Navier boundary conditions. This **is not** the real hinged situation in 2D, due to the boundary curvature!

HINGED BOUNDARY CONDITIONS IN 2D

For hinged boundary conditions the natural setting is the Hilbert space $H^2 \cap H^1_0(\Omega)$. Minimising the energy functional leads to the weak Euler-Lagrange equation

$$\int_\Omega \left(\Delta u \Delta \varphi + (1-\sigma)\left(2u_{xy}\varphi_{xy} - u_{xx}\varphi_{yy} - u_{yy}\varphi_{xx}\right) - f\,\varphi\right)\, dxdy = 0$$

for all $\varphi \in H^2 \cap H^1_0(\Omega)$. Formally, an integration by parts leads to

$$\begin{aligned} 0 \;=\;& \int_\Omega \left(\Delta^2 u - f\right)\;\varphi\; dxdy \;-\int_{\partial\Omega}\left(\frac{\partial}{\partial\nu}\Delta u\right)\;\varphi\; ds \\ &+ (1-\sigma)\int_{\partial\Omega}\left(\left(\nu_1^2-\nu_2^2\right)u_{xy} - \nu_1\nu_2\left(u_{xx}-u_{yy}\right)\right)\frac{\partial}{\partial\tau}\varphi\; ds \\ &+ \int_{\partial\Omega}\left(\Delta u + (1-\sigma)\left(2\nu_1\nu_2 u_{xy} - \nu_2^2 u_{xx} - \nu_1^2 u_{yy}\right)\right)\frac{\partial}{\partial\nu}\varphi\; ds. \end{aligned}$$

One has **$u = 0$ on $\partial\Omega$ and $\Delta u - (1-\sigma)\,\kappa u_\nu = 0$ on $\partial\Omega$.**

Here κ is the mean curvature of the boundary ($\kappa \geq 0$ for convex boundaries).

THE STRONG EULER-LAGRANGE EQUATION

It reads

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \Delta u - (1-\sigma)\,\kappa u_\nu = 0 & \text{on } \partial\Omega. \end{cases}$$

In this situation, with an integration by parts, the elastic energy becomes

$$J(u) = \int_\Omega \left(\tfrac{1}{2}\,(\Delta u)^2 - f\ u \right)\, dx - \frac{1-\sigma}{2} \int_{\partial\Omega} \kappa\, u_\nu^2\, d\omega.$$

This functional has to be minimised over the space $H^2 \cap H^1_0(\Omega)$.

CLAMPED BOUNDARY CONDITIONS IN 2D

For clamped boundary conditions the natural setting is the Hilbert space $H^2_0(\Omega)$. Minimising the energy functional leads to the weak Euler-Lagrange equation

$$\int_\Omega (\Delta u \Delta \varphi + (1-\sigma)\,(2u_{xy}\varphi_{xy} - u_{xx}\varphi_{yy} - u_{yy}\varphi_{xx}) - f\,\varphi)\; dxdy = 0$$

for all $\varphi \in H^2_0(\Omega)$. Formal integration by parts yield

$$\int_\Omega u_{xy}\varphi_{xy}\, dxdy = -\int_\Omega u_x \varphi_{xyy}\, dxdy = \int_\Omega u_{xx}\varphi_{yy}\, dxdy$$

$$\int_\Omega u_{xy}\varphi_{xy}\, dxdy = -\int_\Omega u_y \varphi_{xxy}\, dxdy = \int_\Omega u_{yy}\varphi_{xx}\, dxdy$$

so that the weak Euler-Lagrange equation becomes

$$\int_\Omega \left(\Delta^2 u - f\right)\; \varphi \; dxdy \;=\; 0 \qquad \forall \varphi \in H^2_0(\Omega).$$

THE STRONG EULER-LAGRANGE EQUATION

It reads

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = u_\nu = 0 & \text{on } \partial\Omega. \end{cases}$$

In this situation, the elastic energy becomes

$$J(u) = \int_\Omega \left(\frac{1}{2} (\Delta u)^2 - f \; u \right) dx.$$

This functional has to be minimised over the space $H^2_0(\Omega)$.

BOUNDARY CONDITIONS

Clamped (Dirichlet): $u = u_\nu = 0$ on $\partial\Omega$.
Hinged (Steklov): $u = \Delta u - (1-\sigma)\kappa u_\nu = 0$ on $\partial\Omega$.
We may write the fourth order equation as a second order system:

$$\begin{cases} -\Delta v = f & \text{in } \Omega, \\ v = -(1-\sigma)\kappa u_\nu & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u = v & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Hinged (Navier): $u = \Delta u = 0$ on $\partial\Omega$.
We may write the fourth order equation as a second order system:

$$\begin{cases} -\Delta v = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u = v & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Mixed Navier-Neumann: $u_\nu = \Delta u = 0$ on $\partial\Omega$.
We may write the fourth order equation as a second order system:

$$\begin{cases} -\Delta v = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u = v & \text{in } \Omega, \\ u_\nu = 0 & \text{on } \partial\Omega. \end{cases}$$

No uniqueness, solvability only under the condition $\int_\Omega v = 0$... too complicated!

Neumann-Neumann: $u_\nu = (\Delta u)_\nu = 0$ on $\partial\Omega$.
We may write the fourth order equation as a second order system:

$$\begin{cases} -\Delta v = f & \text{in } \Omega, \\ v_\nu = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u = v & \text{in } \Omega, \\ u_\nu = 0 & \text{on } \partial\Omega. \end{cases}$$

One first needs $\int_\Omega f = 0$; among solutions v one should choose the one satisfying $\int_\Omega v = 0$; then infinitely many solutions u... too complicated!

Even worse: $\Delta u = (\Delta u)_\nu = 0$ on $\partial\Omega$ do not satisfy the complementing condition by Agmon-Douglis-Nirenberg. Well-posedness and elliptic regularity fail!

To see this, consider the problem

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega, \\ \Delta u = (\Delta u)_\nu = 0 & \text{on } \partial\Omega. \end{cases}$$

Any harmonic function is a solution so that the space of solutions **does not** have finite dimension.

If we take any point $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$, the fundamental solution u_0 of $-\Delta$ having pole in x_0 (namely, $u_0(x) = \log|x - x_0|$ if $n = 2$ and $u_0(x) = |x - x_0|^{2-n}$ if $n \geq 3$) is a solution. This shows that it is not possible to obtain uniform a priori bounds in any norm.

Indeed, as x_0 approaches the boundary $\partial\Omega$ it is clear that (for instance!) the H^1-norm of the solution cannot be bounded uniformly in terms of its L^2-norm.

Summarising... we consider the following boundary conditions for a bounded domain $\Omega \subset \mathbb{R}^n$, with $n \geq 2$ and $a \in C^0(\partial\Omega)$:

Steklov: $u = \Delta u - au_\nu = 0$ on $\partial\Omega$.

Dirichlet: $u = u_\nu = 0$ on $\partial\Omega$ (case $a \equiv -\infty$).

Navier: $u = \Delta u = 0$ on $\partial\Omega$ (case $a \equiv 0$).

PPP = POSITIVITY PRESERVING PROPERTY

Consider the boundary value problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ \text{boundary conditions} & \text{on } \partial\Omega. \end{cases}$$

After defining what is meant by **weak solution** $u \in H^2(\Omega)$ ($\cap$...) we address the following

QUESTION: Under which conditions the assumption $f \geq 0$ implies that the solution u exists and is positive?

Does upwards pushing of a plate yield upwards bending?

Remark: Elliptic regularity yields $f \in L^2(\Omega) \Rightarrow u \in H^4(\Omega)$.

Before tackling the PPP...

DEFINITION A bounded domain $\Omega \subset \mathbb{R}^n$ satisfies a outer ball condition if $\forall y \in \partial\Omega$ $\exists$ a ball $B \subset \mathbb{R}^n \setminus \Omega$ s.t. $y \in \partial B$. It satisfies a uniform outer ball condition if the radius B can be taken independently of $y \in \partial\Omega$.

In particular, convex domains or domains with smooth boundary are Lipschitz domains which satisfy a uniform outer ball condition.

THEOREM 1 Assume that $\Omega \subset \mathbb{R}^n$ is a **Lipschitz** bounded domain which satisfies a **uniform outer ball condition**. Then the space $H^2 \cap H^1_0(\Omega)$ is a Hilbert space when endowed with the scalar product

$$(u,v) \mapsto \int_\Omega \Delta u \Delta v \, dx \qquad \text{for all } u,v \in H^2 \cap H^1_0(\Omega).$$

This scalar product induces a norm equivalent to $\| \,.\, \|_{H^2}$.

Proof: Under the above assumptions, Adolfsson (Math. Scand. 1992) proved that $\exists C > 0$ independent of u, such that

$$\|u\|_{H^2} \leq C\|\Delta u\|_{L^2} \qquad \text{for all } u \in H^2 \cap H^1_0(\Omega).$$

For all $u \in H^2 \cap H^1_0(\Omega)$ we also have

$$|D^2u|^2 = \sum_{i,j=1}^{n} (\partial_{ij}u)^2 \geq \sum_{i=1}^{n} (\partial_{ii}u)^2 \geq \frac{1}{n}|\Delta u|^2 \qquad \text{a.e. in } \Omega.$$

This shows that the two norms are equivalent. $\square$

If the domain has a **reentrant corner** then $u \mapsto \|\Delta u\|_2$ is not a norm in $H^2 \cap H^1_0(\Omega)$.

For $\alpha \in \left(\frac{1}{2}\pi, \pi\right)$ fix the domain

$$\Omega_\alpha = \left\{(r\cos\varphi, r\sin\varphi) \in \mathbb{R}^2; 0 < r < 1 \text{ and } |\varphi| < \alpha\right\}.$$

For $\rho = \frac{\pi}{2\alpha} \in \left(\frac{1}{2}, 1\right)$ the function

$$v_\alpha(r,\varphi) = \left(r^{-\rho} - r^\rho\right)\cos(\rho\varphi)$$

satisfies $-\Delta v_\alpha = 0$ in Ω_α, $v_\alpha = 0$ on $\partial\Omega_\alpha \setminus \{0\}$ and $v_\alpha \in L^2(\Omega_\alpha)$. Then $\exists! b_\alpha \in H^1_0(\Omega_\alpha)$ solving

$$-\Delta b_\alpha = v_\alpha \text{ in } \Omega_\alpha, \quad b_\alpha = 0 \text{ on } \partial\Omega_\alpha.$$

One has $\Delta b_\alpha \notin H^1_0(\Omega_\alpha)$ and $b_\alpha \notin H^2(\Omega_\alpha)$.

PPP: Navier boundary conditions: $u = \Delta u = 0$ on $\partial\Omega$.

The problem may be decomposed into a **system**.

$$\begin{cases} -\Delta v = f & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u = v & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Regardless of the regularity of the boundary $\partial\Omega$, we may apply twice the Lax-Milgram Theorem:

$$\forall f \in H^{-1}(\Omega)\ \exists! v \in H^1_0(\Omega) \quad \Longrightarrow \quad \exists! u \in H^1_0(\Omega)\ (\Delta u \in H^1_0(\Omega)).$$

Applying twice the maximum principle for $-\Delta$ we obtain that

$$L^2(\Omega) \ni f \gneqq 0 \quad \Longrightarrow v \gneqq 0 \quad \Longrightarrow u \gneqq 0.$$

The solution u so found is called the **system solution**. Hence, $\exists!$ system solution and PPP holds.

If Ω satisfies a uniform outer ball condition then u has finite energy: $u \in H^2(\Omega)$.

However,... if Ω has a **reentrant corner** (nonsmooth) the problem may also admit **a finite energy** sign changing solution.

For $\alpha \in \left(\frac{1}{2}\pi, \pi\right)$ consider again Ω_α. Let $f \in L^2(\Omega_\alpha)$ and consider the homogeneous Navier problem

$$\Delta^2 u = f \text{ in } \Omega_\alpha, \quad u = 0 \text{ on } \partial\Omega_\alpha, \quad \Delta u = 0 \text{ on } \partial\Omega_\alpha \setminus \{0\}.$$

Let u be the **system solution** obtained by applying twice the Lax-Milgram Theorem and, $\forall c \in \mathbb{R}$ let $u_c = u + cb_\alpha$.

We know that:

- $\forall c \in \mathbb{R}$ we have $u_c \in H^1_0(\Omega_\alpha)$ and $\Delta u_c \in L^2(\Omega_\alpha)$.
- $\forall c \in \mathbb{R}$ u_c is a solution and $u_c \in C^0(\overline{\Omega_\alpha})$, $\Delta u_c \in C^0(\overline{\Omega_\alpha} \setminus \{0\})$.
- $\Delta u_c \in H^1_0(\Omega_\alpha)$ iff $c = 0$.
- $\forall f \in L^2(\Omega_\alpha)$ $\exists! c_\alpha(f) \in \mathbb{R}$ s.t. $u_c \in H^2 \cap H^1_0(\Omega_\alpha) \Leftrightarrow c = c_\alpha(f)$.

Hence, $u \notin H^2(\Omega_\alpha)$ whenever $c_\alpha(f) \neq 0$.

See Kondratiev (Trudy Moskovskogo Matematičeskogo Obščestva, 1967) and Nazarov-Plamenevsky (de Gruyter, 1994).

Now let f be positive. Nazarov-Sweers (JDE, 2007) show that $u_{c_\alpha(f)} \in H^2(\Omega_\alpha)$ but $u_{c_\alpha(f)} \not\geq 0$ when $\alpha > \frac{3}{4}\pi$ and $c_\alpha(f) \neq 0$.
For $\alpha \in (\frac{1}{2}\pi, \frac{3}{4}\pi)$ there is only numerical evidence of sign-changing energy solutions:

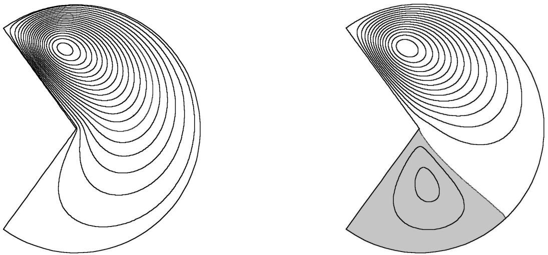


Figure: The level lines of u and $u_{c_\alpha(f)}$ for $f \geq 0$ having a small support near the left top of the domain. Grey region $= \{x : u_{c_\alpha(f)}(x) < 0\}$; here, a different scale is used for the level lines.

PPP: Dirichlet boundary conditions: $u = u_\nu = 0$ on $\partial\Omega$.
Consider the boundary value problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = u_\nu = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, f a datum in a suitable functional space and u denotes the unknown solution.

If Ω is smooth, a unique **Green function** G_Ω exists and

$$u(x) = \int_\Omega G_\Omega(x, y) f(y)\, dy \qquad \forall x \in \Omega.$$

PPP $\iff G_\Omega \geq 0$.

For $x, y \in \overline{B}$ we write

$$[XY] := \sqrt{|x|^2\,|y|^2 - 2x\cdot y + 1} = \left|\, |x|y - \frac{x}{|x|} \right| = \left|\, |y|x - \frac{y}{|y|} \right| .$$

Then

$$G_B(x,y) = k_n\,|x-y|^{4-n} \int_1^{[XY]/|x-y|} (\sigma^2-1)\sigma^{1-n}\,d\sigma > 0\ .$$

- T. Boggio, Rend. Circ. Mat. Palermo, 1905

Hence, the implication $f \gneqq 0 \implies u \gneqq 0$ is true in **balls**.

A more general formula is available for the polyharmonic Dirichlet Green function.

Boggio conjectured that the Green function is always positive (in any domain!).

In 1908, Hadamard already knew that this conjecture fails in annuli with small inner radius. He writes that Boggio had mentioned to him that the conjecture was meant for simply connected domains. In the same publication he writes:

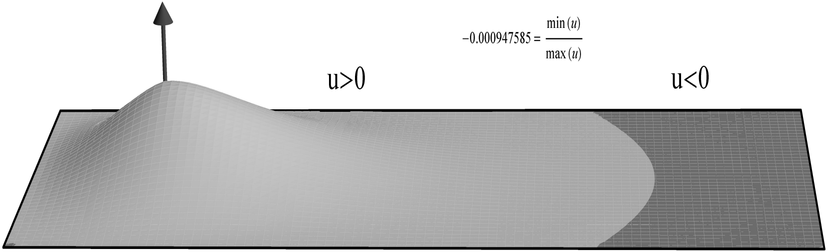
> *Malgré l'absence de démonstration rigoureuse,*
> *l'exactitude de cette proposition*
> *ne paraît pas douteuse pour les aires convexes.*

The Boggio-Hadamard conjecture may be formulated as follows:

> *The Green function G_Ω for the clamped plate boundary value problem on convex domains is positive.*

However, this **conjecture is wrong.**

Duffin (J. Math. Phys. 1949, Bull. AMS 1974) showed that the Green function changes sign **on a long rectangle**.



Garabedian (Pacific J. Math. 1951) showed change of sign of Green's function in **ellipses with ratio of half axes** ≈ 1.6.
Hedenmalm-Jakobsson-Shimorin (J. Reine Angew. Math. 2002) mention that sign change occurs already in **ellipses with ratio of half axes** ≈ 1.2.
Nakai-Sario (J. Reine Angew. Math. 1977) give a construction how to extend Garabedian's example also to **higher dimensions**.

Sign change is also proved by Coffman-Duffin (Adv. Appl. Math. 1980) in any **bounded domain containing a corner**, the angle of which is not too large; in particular, **squares**.

Conclusion: **neither in arbitrarily smooth uniformly convex nor in rather symmetric domains Green's function needs to be positive**.

The Green's function is **positive** in suitable **perturbations of a planar disc** (Grunau-Sweers, Math. Nachr. 1996 & Sassone, Ann. Mat. Pura Appl. 2007).

Using the explicit formula from for the **limaçons de Pascal**, Hadamard also claimed to have proven positivity of the Green function G_Ω when Ω is such a limaçon.
However, Dall'Acqua-Sweers (Ann. Mat. Pura Appl. 2005) showed that this is not the case.

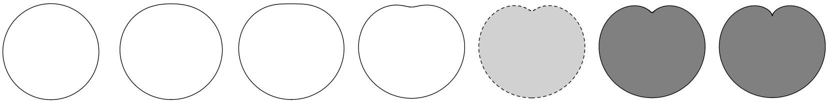


Figure: Limaçons vary from circle to cardioid. The fifth limaçon from the left is critical for a positive Green function.

PPP: Steklov boundary conditions: $u = \Delta u - a u_\nu = 0$ on $\partial\Omega$.

Let Ω be a bounded domain of $\mathbb{R}^n$ ($n \geq 2$) with $\partial\Omega \in C^2$ and consider the boundary value problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \Delta u - a u_\nu = 0 & \text{on } \partial\Omega, \end{cases}$$

where $a \in C^0(\partial\Omega)$, $f \in L^2(\Omega)$.

We say that u is a **weak solution** if $u \in H^2 \cap H^1_0(\Omega)$ and

$$\int_\Omega \Delta u \Delta v \; dx - \int_{\partial\Omega} a \; u_\nu v_\nu \; d\omega = \int_\Omega fv \; dx \qquad \forall v \in H^2 \cap H^1_0(\Omega).$$

SUPERHARMONICITY

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ with $\partial\Omega \in C^2$, $\mathcal{H} := [H^2 \cap H_0^1] \setminus H_0^2(\Omega)$ and

$$d_1(\Omega) := \min_{u \in \mathcal{H}} \frac{\displaystyle\int_\Omega |\Delta u|^2}{\displaystyle\int_{\partial\Omega} u_\nu^2} .$$

The minimum is achieved and $d_1^{-1/2}$ is the norm of the compact linear operator

$$H^2 \cap H_0^1(\Omega) \to L^2(\partial\Omega) \qquad u \mapsto u_\nu|_{\partial\Omega}.$$

THEOREM 2 Let $a \in C^0(\partial\Omega)$, $f \in L^2(\Omega)$, and consider

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \Delta u - a u_\nu = 0 & \text{on } \partial\Omega. \end{cases}$$

If $a < d_1$ it admits a unique solution $u \in H^2 \cap H_0^1(\Omega)$. If also $a \geq 0$ and $f \gneqq 0$, then the solution u is strictly superharmonic in $\overline{\Omega}$.

Sketch of the proof: On the space $H^2 \cap H_0^1(\Omega)$ the functional

$$I(u) := \frac{1}{2}\int_\Omega |\Delta u|^2 - \frac{1}{2}\int_{\partial\Omega} a u_\nu^2 - \int_\Omega fu$$

is strictly convex because $a < d_1$. The solution is the unique minimiser of I.
If $a \geq 0$ and $f \gneqq 0$, then for all $u \in H^2 \cap H_0^1(\Omega) \setminus \{0\}$ the solution $w \in H^2 \cap H_0^1(\Omega)$ to the problem

$$\begin{cases} -\Delta w = |\Delta u| & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

satisfies $w > 0$ in Ω, $w_\nu < 0$ on $\partial\Omega$, and $I(w) \leq I(u)$. $\square$

POSITIVITY

THEOREM 3 Let $a \in C^0(\partial\Omega)$, $f \in L^2(\Omega)$, and consider

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \Delta u - a u_\nu = 0 & \text{on } \partial\Omega. \end{cases}$$

There exists a number $\delta_c := \delta_c(\Omega) \in [-\infty, 0)$ such that:

1. If $a \geq d_1$ and if $0 \lneqq f \in L^2(\Omega)$, then $\nexists$ positive solutions.

2. If $a = d_1$, then $\exists$ a positive eigenfunction $u_1 > 0$ in Ω for $f = 0$. Moreover, u_1 is unique up to multiples.

3. If $a \lneqq d_1$, then $\forall f \in L^2(\Omega)$ $\exists!$ solution u.

4. If $\delta_c \leq a \lneqq d_1$, then $0 \lneqq f \in L^2(\Omega)$ implies $u \gneqq 0$ in Ω.

5. If $\delta_c < a \lneqq d_1$, then $0 \lneqq f \in L^2(\Omega)$ implies $u \geq c_f\, d_{\partial\Omega} > 0$ in Ω for some $c_f > 0$.

6. If $a < \delta_c$, then there are $0 \lneqq f \in L^2(\Omega)$ with $0 \not\leq u$.

Flavour of the proof:

$\Delta^2 u = f$ in $\Omega\,,\quad u = \Delta u - a u_\nu = 0$ on $\partial\Omega$.

EQUIVALENT SYSTEM

$$\begin{cases} -\Delta v = f & \text{in } \Omega, \\ v = -a u_\nu & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u = v & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

OPERATOR FORMULATION

Consider the Green operator $\mathcal{G}$ and the Poisson kernel $\mathcal{K}$, formally:

$$w = \mathcal{G}f + \mathcal{K}g \Longleftrightarrow \begin{cases} -\Delta w = f & \text{in } \Omega, \\ w = g & \text{on } \partial\Omega. \end{cases}$$

Let $(\mathcal{P}w)(x) := -\nu \cdot \nabla w(x) = -w_\nu(x)$ for $x \in \partial\Omega$.

$$u = \mathcal{G}v = \mathcal{G}(\mathcal{G}f + \mathcal{K}a\mathcal{P}u) = \mathcal{G}\mathcal{G}f + \mathcal{G}\mathcal{K}a\mathcal{P}u,$$

$$u = (\mathcal{I} - \mathcal{G}\mathcal{K}a\mathcal{P})^{-1}\,\mathcal{G}\mathcal{G}f. \qquad \square$$

BACK TO THE HINGED PLATE

Positivity preserving property for the hinged plate in planar domains.

Recall the physical bounds for the Poisson ratio: $-1 < \sigma < 1$.

COROLLARY 1 Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with $\partial\Omega \in C^2$ and assume $-1 < \sigma < 1$; $\forall f \in L^2(\Omega)$ $\exists! u \in H^2 \cap H^1_0(\Omega)$ minimiser of the elastic energy functional

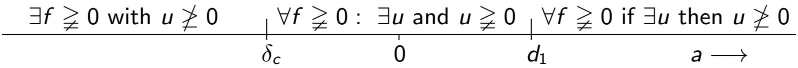
$$J(u) = \int_\Omega \left(\frac{|\Delta u|^2}{2} - f\, u \right) dx - \frac{1-\sigma}{2} \int_{\partial\Omega} \kappa\, u_\nu^2 \, d\omega.$$

The minimiser u is the unique weak solution to

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \Delta u - (1-\sigma)\,\kappa u_\nu = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, $f \gneqq 0$ implies that there exists $c_f > 0$ such that $u \geq c_f\, d_{\partial\Omega}$ and u is superharmonic in Ω.

$\Delta^2 u = f$ in Ω, $\qquad u = \Delta u - a u_\nu = 0$ on $\partial\Omega$.
We saw that $\exists \delta_c \in [-\infty, 0)$ such that:



QUESTIONS:

- What happens if $a - d_1$ changes sign on $\partial\Omega$?
- Are there cases where $\delta_c = -\infty$, $\delta_c > -\infty$?

For the first question: the function

$$a(x) = n + \frac{2x_1}{1 + \frac{2}{n} + \varepsilon + x_1}$$

satisfies $a \in C^0(\partial B)$, $a - n$ changes sign, $f \gneqq 0$ implies $u \gneqq 0$.

THEOREM 4 Let $a \in C^0(\partial\Omega)$, $f \in L^2(\Omega)$, and consider

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \Delta u - a u_\nu = 0 & \text{on } \partial\Omega. \end{cases}$$

If for every $m \in \mathbb{N}$ and $0 \lneqq f \in L^2(\Omega)$ the solution with $a \equiv -m$ is positive, then for every $0 \lneqq f \in L^2(\Omega)$ the solution $u \in H^2_0(\Omega)$ of the Dirichlet problem ($u = u_\nu = 0$ on $\partial\Omega$) satisfies $u \gneqq 0$.

This shows that

If the Dirichlet problem is not positivity preserving then $\delta_c > -\infty$.

The full converse statement is not known: only under additional assumptions on the behaviour of the Green function of the Dirichlet problem.

THEOREM 5 If $\Omega = B$, the unit ball in $\mathbb{R}^n$ $(n \geq 2)$, then $d_1 = n$. Moreover, for all $0 \lneqq f \in L^2(B)$ and all $a \in C^0(\partial B)$ such that $a \lneqq n$, there exists $c > 0$ such that the weak solution u satisfies $u \geq c d_{\partial\Omega}$ in B.

Sketch of the proof: Assume first that $f \in C_c^\infty(B)$ so that $u \in W^{2,p}(B)$ for all $p > 1$. In turn, $u \in C^1(\overline{B})$ and hence $\Delta u = a u_\nu \in C^0(\partial B)$. Therefore, $u \in C^\infty(B) \cap C^2(\overline{B})$.
Consider the auxiliary function $\phi \in C^\infty(B) \cap C^0(\overline{B})$ defined by

$$\phi(x) = (|x|^2 - 1)\Delta u(x) - 4x \cdot \nabla u(x) - 2(n-4)u(x).$$

Then ϕ satisfies the second order Steklov boundary value problem

$$\begin{cases} -\Delta\phi = (1 - |x|^2)f \geq 0 & \text{in } B, \\ \phi_\nu + \frac{1}{2}(n-a)\phi = 0 & \text{on } \partial B. \end{cases}$$

As $a \lneqq n$, by the maximum principle (for this second order problem!) we infer that $\phi > 0$ in $\overline{B}$ and $u_\nu < 0$ on ∂B. Therefore, $-\Delta u > 0$ in B whenever $0 \lneqq f \in C_c^\infty(B)$.
Assume now $0 \lneqq f \in L^2(B)$ and let $u \in H^2 \cap H_0^1(B)$ be the unique weak solution. A density argument shows that $u \gneqq 0$ in B and therefore $\delta_c = -\infty$. Finally, the lower bound $u \geq c d_{\partial\Omega}$ in B follows by comparison with the solutions for smaller a. □

STEKLOV EIGENVALUES

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain with $\partial\Omega \in C^2$, let $d \in \mathbb{R}$ and consider the **boundary eigenvalue problem**

$$\begin{cases} \Delta^2 u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ \Delta u = d u_\nu & \text{on } \partial\Omega\,. \end{cases}$$

EIGENVALUE: A value of d for which the problem admits nontrivial solutions, the corresponding **EIGENFUNCTION** $u \in H^2 \cap H^1_0(\Omega)$ which satisfies

$$\int_\Omega \Delta u \Delta v \, dx = d \int_{\partial\Omega} u_\nu v_\nu \, ds \qquad \text{for all } v \in H^2 \cap H^1_0(\Omega).$$

By taking $v = u$: all the eigenvalues are strictly positive.
The least eigenvalue is the threshold for positivity:

$$d_1 = \min_{u \in \mathcal{H}(\Omega)} \frac{\displaystyle\int_\Omega |\Delta u|^2}{\displaystyle\int_{\partial\Omega} u_\nu^2}\,, \qquad \mathcal{H}(\Omega) := [H^2 \cap H^1_0(\Omega)] \setminus H^2_0(\Omega).$$

THE SPECTRUM OF THE BIHARMONIC STEKLOV PROBLEM

We endow the Hilbert space $H^2 \cap H^1_0(\Omega)$ with the scalar product

$$(u, v) = \int_\Omega \Delta u \Delta v \, dx \ .$$

Consider the space

$$Z = \left\{ v \in C^\infty(\overline{\Omega}) : \Delta^2 u = 0, \ \ u = 0 \text{ on } \partial\Omega \right\}$$

and denote by V the completion of Z with respect to the norm associated to the scalar product $(\cdot, \cdot)$.

THEOREM 6 Assume that $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is an open bounded domain with $\partial\Omega \in C^2$. Then:

- there exist infinitely many (countable) eigenvalues;
- the set of eigenfunctions is a complete orthonormal system in V;
- the only eigenfunction of fixed sign is the one corresponding to the first eigenvalue;
- the space $H^2 \cap H^1_0(\Omega)$ admits the orthogonal decomposition

$$H^2 \cap H^1_0(\Omega) = V \oplus H^2_0(\Omega).$$

- if $v \in H^2 \cap H^1_0(\Omega)$ and if $v = v_1 + v_2$ is the decomposition, then $v_1 \in V$ and $v_2 \in H^2_0(\Omega)$ solve

$$\begin{cases} \Delta^2 v_1 = 0 & \text{in } \Omega \\ v_1 = 0 & \text{on } \partial\Omega \\ (v_1)_\nu = v_\nu & \text{on } \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} \Delta^2 v_2 = \Delta^2 v & \text{in } \Omega \\ v_2 = 0 & \text{on } \partial\Omega \\ (v_2)_\nu = 0 & \text{on } \partial\Omega \, . \end{cases}$$

When $\Omega = B$, much more can be said.

Consider the spaces of harmonic homogeneous polynomials:

$$\mathcal{D}_k := \{P \in C^\infty(\mathbb{R}^n);\ \Delta P = 0 \text{ in } \mathbb{R}^n,\ P \text{ is an homogeneous polynomial of degree } k-1\}.$$

Denote by μ_k the dimension of $\mathcal{D}_k$. In particular, we have

$$\mathcal{D}_1 = \mathrm{span}\{1\}\,, \qquad \mu_1 = 1\,,$$

$$\mathcal{D}_2 = \mathrm{span}\{x_i;\ (i = 1, ..., n)\}\,, \qquad \mu_2 = n\,,$$

$$\mathcal{D}_3 = \mathrm{span}\{x_i x_j;\ x_1^2 - x_h^2;\ (i, j = 1, ..., n,\ i \neq j,\ h = 2, ..., n)\}\,,$$

$$\mu_3 = \frac{n^2 + n - 2}{2}\,.$$

THEOREM 7 If $n \geq 2$ and $\Omega = B$, then for all $k = 1, 2, 3, \ldots$:
(i) the Steklov eigenvalues are $d_k = n + 2(k-1)$;
(ii) the multiplicity of d_k equals μ_k;
(iii) for all $\psi_k \in \mathcal{D}_k$, the function $\phi_k(x) := (1 - |x|^2)\psi_k(x)$ is an eigenfunction corresponding to d_k.

REMARK If $n = 1$, the problem

$$u^{iv} = 0 \quad \text{in } (-1, 1)$$

$$u(\pm 1) = u''(-1) + du'(-1) = u''(1) - du'(1) = 0,$$

admits only **two** eigenvalues, $d_1 = 1$ and $d_2 = 3$, each one of multiplicity 1. The reason of this striking difference is that the "boundary space" has dimension 2, one for each endpoint of the interval $(-1, 1)$. This result is consistent with Theorem 7 since $\mu_1 = \mu_2 = 1$ and $\mu_3 = 0$ whenever $n = 1$.

Proof: An eigenfunction u satisfies $u \in C^\infty(\overline{B})$ and may be written as

$$u(x) = (1 - |x|^2)\phi(x)$$

with $\phi \in C^\infty(\overline{B})$ (this is a nontrivial step!).

Some computations show that the number d is an eigenvalue with corresponding eigenfunction u if and only if ϕ is an eigenfunction of the boundary eigenvalue problem

$$\begin{cases} \Delta\phi = 0 & \text{in } B \\ \phi_\nu = a\phi & \text{on } \partial B, \end{cases}$$

where $a = \frac{d-n}{2}$.

The problem reduces to study the eigenvalues of this second order Steklov problem for which $a \in \mathbb{N}$. $\square$

A PRIORI ESTIMATES FOR HARMONIC FUNCTIONS

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with **smooth boundary**. Let $g \in L^2(\partial\Omega)$ and consider the problem

$$\begin{cases} \Delta v = 0 & \text{in } \Omega \\ v = g & \text{on } \partial\Omega \ . \end{cases}$$

Which is the **optimal constant** $\delta_1(\Omega)$ for the a priori estimate

$$\delta_1(\Omega) \cdot \|v\|^2_{L^2(\Omega)} \leq \|g\|^2_{L^2(\partial\Omega)} \ ?$$

VARIATIONAL CHARACTERISATION OF δ_1

Let $\mathbf{H} :=$ closure of $\{v \in C^2(\overline{\Omega});\ \Delta v = 0 \text{ in } \Omega\}$ with respect to the norm $\|\cdot\|_{\mathbf{H}} := \|\cdot\|_{L^2(\partial\Omega)}$. Let

$$\delta_1 = \delta_1(\Omega) := \min_{h \in \mathbf{H}\setminus\{0\}} \frac{\displaystyle\int_{\partial\Omega} h^2}{\displaystyle\int_{\Omega} h^2} .$$

THEOREM 8 If $\Omega \subset \mathbb{R}^n$ is open bounded with **Lipschitz boundary**, then $\delta_1(\Omega)$ admits a minimiser $h \in \mathbf{H}\setminus\{0\}$.

The **regularity of the boundary** plays a crucial role in the previous statement and in what follows.

Proof: Let $\{h_m\} \subset \mathbf{H} \setminus \{0\}$ be a minimising sequence for $\delta_1(\Omega)$ with $\|h_m\|_{L^2(\partial\Omega)} = 1$. Hence, $\exists h \in \mathbf{H}$ s.t. (up to a subsequence) $h_m \rightharpoonup h$ in $L^2(\partial\Omega)$.

By elliptic estimates (due to Jerison-Kenig), we infer that

$$\exists C > 0 \text{ s.t. } \|w\|_{H^{1/2}(\Omega)} \leq C\|w\|_{L^2(\partial\Omega)} \qquad \forall w \in \mathbf{H}.$$

Hence, $\{h_m\}$ is bounded in $H^{1/2}(\Omega)$, $h_m \rightharpoonup h$ in $H^{1/2}(\Omega)$ up to a subsequence and, by compact embedding, also $h_m \to h$ in $L^2(\Omega)$. Therefore, since $\{h_m\}$ is a minimising sequence, $\|h_m\|_{L^2(\partial\Omega)} = 1$ and $\|h_m\|_{L^2(\Omega)}$ is bounded then $\delta_1(\Omega) > 0$, $h \in \mathbf{H} \setminus \{0\}$ and

$$\|h\|_{L^2(\Omega)}^{-2} = \lim_{m\to\infty} \|h_m\|_{L^2(\Omega)}^{-2} = \delta_1(\Omega).$$

Moreover, by weak lower semicontinuity of $\|\cdot\|_{L^2(\partial\Omega)}$ we also have

$$\|h\|_{L^2(\partial\Omega)}^2 \leq \liminf_{m\to\infty} \|h_m\|_{L^2(\partial\Omega)}^2 = 1$$

and hence $h \in \mathbf{H} \setminus \{0\}$ satisfies $\|h\|_{L^2(\partial\Omega)}^2 \leq \delta_1(\Omega)\|h\|_{L^2(\Omega)}^2$. This proves that h is a minimiser for $\delta_1(\Omega)$. $\square$

THEOREM 9 (FICHERA'S PRINCIPLE OF DUALITY)

Assume that $\partial\Omega$ is Lipschitz and that Ω satisfies a uniform outer ball condition.

Let $\mathbf{H} :=$ closure of $\{v \in C^2(\overline{\Omega});\ \Delta v = 0 \text{ in } \Omega\}$ with respect to the norm $\|\cdot\|_{\mathbf{H}} := \|\cdot\|_{L^2(\partial\Omega)}$.

Let $\mathcal{H} := [H^2 \cap H_0^1(\Omega)] \setminus H_0^2(\Omega)$.

$$\delta_1(\Omega) = \min_{h \in \mathbf{H}\setminus\{0\}} \frac{\displaystyle\int_{\partial\Omega} h^2}{\displaystyle\int_{\Omega} h^2} = \min_{u \in \mathcal{H}} \frac{\displaystyle\int_{\Omega} |\Delta u|^2}{\displaystyle\int_{\partial\Omega} u_\nu^2} = d_1(\Omega).$$

The already mentioned results by Nazarov-Sweers (JDE, 2007) suggest that this result might become **false in domains with a concave corner**.

Proof: (assuming that $\partial\Omega \in C^2$).

We say that δ is an eigenvalue relative to the minimisation problem

$$\min_{h \in \mathbf{H} \setminus \{0\}} \frac{\int_{\partial\Omega} h^2}{\int_{\Omega} h^2}$$

if $\exists g \in \mathbf{H}$ such that

$$\delta \int_{\Omega} gv \, dx = \int_{\partial\Omega} gv \, ds \qquad \forall v \in \mathbf{H}.$$

Note that the Euler equation contains no derivatives !

Clearly, δ_1 is the least eigenvalue.

We show that both $\delta_1 \geq d_1$ and $\delta_1 \leq d_1$.

Let $h \in \mathbf{H} \setminus \{0\}$ be a minimiser for δ_1, then

$$\delta_1 \int_\Omega hv = \int_{\partial\Omega} hv \qquad \forall v \in \mathbf{H} \ . \tag{1}$$

Let $u \in \mathcal{H}$ solve $\Delta u = h$ in Ω, $u = 0$ on $\partial\Omega$. By integrating

$$\int_\Omega hv = \int_\Omega v\Delta u = \int_{\partial\Omega} vu_\nu \quad \forall v \in \mathbf{H} \cap C^2(\overline{\Omega}).$$

By density, this holds $\forall v \in \mathbf{H}$. Inserting into (1) gives

$$\delta_1 \int_{\partial\Omega} vu_\nu = \int_{\partial\Omega} v\Delta u \qquad \text{for all } v \in \mathbf{H} \ .$$

This yields $\Delta u = \delta_1 u_\nu$ on $\partial\Omega$. Therefore,

$$\delta_1 = \frac{\int_{\partial\Omega} h^2}{\int_\Omega h^2} = \frac{\int_{\partial\Omega} |\Delta u|^2}{\int_\Omega |\Delta u|^2} = \delta_1^2 \, \frac{\int_{\partial\Omega} u_\nu^2}{\int_\Omega |\Delta u|^2} \ .$$

In turn, this implies that

$$\delta_1 = \frac{\int_\Omega |\Delta u|^2}{\int_{\partial\Omega} u_\nu^2} \geq \min_{v\in\mathcal{H}} \frac{\int_\Omega |\Delta v|^2}{\int_{\partial\Omega} v_\nu^2} = d_1 \ .$$

Let u be a minimiser for d_1, then $\Delta u = d_1 u_\nu$ on $\partial\Omega$ so that $\Delta u \in H^{1/2}(\partial\Omega) \subset L^2(\partial\Omega)$ and

$$\int_{\partial\Omega} v\Delta u = d_1 \int_{\partial\Omega} v u_\nu \qquad \text{for all } v \in \mathbf{H}\ . \tag{2}$$

Let $h := \Delta u$ so that $h \in L^2(\Omega) \cap L^2(\partial\Omega)$. Moreover, $\Delta h = \Delta^2 u = 0$ (in distributional sense) and hence $h \in \mathbf{H}$. Two integrations by parts (and a density argument) yield

$$\int_\Omega hv = \int_\Omega \Delta u v = \int_{\partial\Omega} v u_\nu \qquad \text{for all } v \in \mathbf{H}\ .$$

Replacing this into (2) gives

$$\int_{\partial\Omega} hv = d_1 \int_\Omega hv \qquad \text{for all } v \in \mathbf{H}\ .$$

This proves that h is an eigenfunction relative to the harmonic problem with corresponding eigenvalue d_1. Since δ_1 is the least eigenvalue, we obtain $d_1 \geq \delta_1$. $\square$

As a byproduct of the proof, we see that the minimisers are related by $h = \Delta\phi$, up to a multiple.

$$\delta_1 = \frac{\int_{\partial\Omega} h^2}{\int_\Omega h^2} = \frac{\int_{\partial\Omega} |\Delta\phi|^2}{\int_\Omega |\Delta\phi|^2} = \frac{d_1^2 \int_{\partial\Omega} \phi_\nu^2}{\int_\Omega |\Delta\phi|^2} = d_1 .$$

MINIMISATION OF THE LEAST EIGENVALUE

For the **second order** Steklov problem

$$\Delta u = 0 \quad \text{in } \Omega \ , \qquad u_\nu = \lambda u \quad \text{on } \partial\Omega \ ,$$

the first (nontrivial) eigenvalue satisfies $\lambda_1(\Omega) \leq \lambda_1(\Omega^*)$.

- F. Brock, ZAMM 2001

But the fourth order Steklov problem appears completely different!

Smith conjectures (and proves!) that for **any** domain Ω, one has $d_1(\Omega) \geq d_1(\Omega^*)$.

- J. Smith, SIAM J. Numer. Anal. 1968

In particular, for planar domains Ω of measure π (as the unit disk), this means that $d_1(\Omega) \geq 2$.

As noticed by Kuttler and Sigillito, this proof contains a gap. In a "Note added in proof" Smith writes:

> *Although the result is probably true, a correct proof has not yet been found.*

- J. Smith, SIAM J. Numer. Anal. 1970

Kuttler also shows that for the square $Q_{\sqrt{\pi}} = (0, \sqrt{\pi})^2$ one has

$$d_1(Q_{\sqrt{\pi}}) < 1.9889 .$$

- J.R. Kuttler, SIAM J. Numer. Anal. 1972

This estimate may be improved to $d_1(Q_{\sqrt{\pi}}) < 1.96256$.

Proof: Let $h(x,y) := x^4 + y^4 - 6x^2y^2 + 2.69$, then $\Delta h = 0$ and

$$\frac{\displaystyle\int_{\partial Q_{\sqrt{\pi}}} h^2}{\displaystyle\int_{Q_{\sqrt{\pi}}} h^2} < 1.96256 . \qquad \square$$

- A. Ferrero, F. Gazzola, T. Weth, Analysis 2005

Kuttler suggests a new and weaker conjecture. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain such that $|\partial\Omega| = |\partial B|$. Then, $n = d_1(B) \leq d_1(\Omega)$.
His numerical results on some rectangles support this conjecture.

THEOREM 10 Let $D_\varepsilon = \{x \in \mathbb{R}^2; \varepsilon < |x| < 1\}$ and let $\Omega_\varepsilon = D_\varepsilon \times (0,1)^{n-2}$. Then $\lim_{\varepsilon \to 0} d_1(\Omega_\varepsilon) = 0$.

This Theorem disproves the conjecture by Kuttler: there is no minimiser to d_1 among all regions having the same perimeter.

Proof: For any $\varepsilon \in (0,1)$ let $w_\varepsilon(x) = \dfrac{1-|x|^2}{4} - \dfrac{1-\varepsilon^2}{4\log\varepsilon}\log|x|$ $\forall x \in D_\varepsilon$. Then $w_\varepsilon \in H^2 \cap H^1_0(D_\varepsilon)$ and

$$\int_{\Omega_\varepsilon} |\Delta w_\varepsilon|^2\, dx = \pi\left(1-\varepsilon^2\right) \ ,$$

$$\int_{\partial\Omega_\varepsilon} (w_\varepsilon)_\nu^2 \ ds = \frac{\pi}{8}\frac{1}{\varepsilon \log^2 \varepsilon} + o\left(\frac{1}{\varepsilon\log^2\varepsilon}\right) \to +\infty \text{ as } \varepsilon \to 0.$$

Hence,

$$\lim_{\varepsilon\to 0} d_1(\Omega_\varepsilon) \le \lim_{\varepsilon\to 0} \frac{\displaystyle\int_{\Omega_\varepsilon} |\Delta w_\varepsilon|^2\, dx}{\displaystyle\int_{\partial\Omega_\varepsilon} (w_\varepsilon)_\nu^2 \ ds} = 0.$$

For $n \ge 3$, let

$$u_\varepsilon(x) = \left(\prod_{i=3}^{n} x_i\,(1-x_i)\right) w_\varepsilon(x_1,x_2) \qquad \forall x \in \Omega_\varepsilon$$

and compute as above. □

BOUNDS IN CONVEX DOMAINS

THEOREM 11 Let $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ be a bounded convex domain with $\partial\Omega \in C^2$.

- For all $x \in \partial\Omega$, let $\kappa(x)$ denote the mean curvature at x and let

$$\underline{K} := \min_{x \in \partial\Omega} \kappa(x) \ ;$$

then $d_1(\Omega) \geq n\underline{K}$ with equality if and only if Ω is a ball.

- The following isoperimetric bound holds

$$d_1(\Omega) \leq \frac{|\partial\Omega|}{|\Omega|}$$

with equality if and only if Ω is a ball.

Proof: For the lower bound, let ϕ be a first Steklov eigenfunction such that $\phi > 0$ in Ω and $\phi_\nu < 0$ on $\partial\Omega$. The boundary condition $\Delta\phi = d_1\phi_\nu$ on $\partial\Omega$ also reads $\phi_{\nu\nu} + (n-1)\,\kappa\phi_\nu = d_1\phi_\nu$ on $\partial\Omega$. Hence,

$$\left(\phi_\nu^2\right)_\nu = 2\phi_{\nu\nu}\phi_\nu = 2\left[d_1 - (n-1)\,\kappa\right]\phi_\nu^2$$

so that if we put $D^2\phi D^2\phi = \sum_{i,j=1}^{n} \left(\partial_{ij}\phi\right)^2$ and integrate by parts

$$2\int_{\partial\Omega}\left[d_1 - (n-1)\kappa\right]\phi_\nu^2\,ds = \int_{\partial\Omega}(\phi_\nu^2)_\nu\,ds = \int_{\partial\Omega}\left(|\nabla\phi|^2\right)_\nu\,ds$$

$$= \int_\Omega \Delta\left(|\nabla\phi|^2\right)\,dx = 2\int_\Omega \nabla\Delta\phi\cdot\nabla\phi\,dx + 2\int_\Omega D^2\phi D^2\phi\,dx$$

$$= -2\int_\Omega \phi\Delta^2\phi\,dx + 2\int_{\partial\Omega}\phi(\Delta\phi)_\nu\,ds + 2\int_\Omega D^2\phi D^2\phi\,dx$$

$$= 2\int_\Omega D^2\phi D^2\phi\,dx \geq \frac{2}{n}\int_\Omega |\Delta\phi|^2\,dx = \frac{2d_1}{n}\int_{\partial\Omega}\phi_\nu^2\,ds$$

the latter since ϕ is the first eigenfunction. Hence, $d_1 \geq n\underline{K}$.

We prove that equality holds if and only if Ω is a ball. If $d_1 = n\underline{K}$, then $d_1 \leq n\kappa(x)$ for $x \in \partial\Omega$ and since $\phi_\nu < 0$ on $\partial\Omega$, from

$$\int_{\partial\Omega} \left(\frac{d_1}{n} - \kappa \right) \phi_\nu^2 \, ds \geq 0,$$

we infer that $\kappa(x) \equiv \frac{d_1}{n}$. Hence, Ω is a ball by Alexandrov's characterisation of spheres (Ann. Mat. Pura Appl. 1962).

The upper bound is obtained by taking $h \equiv 1$ as harmonic test function in Fichera's characterisation:

$$d_1 \, |\Omega| = d_1 \int_\Omega 1^2 \, dx \leq \int_{\partial\Omega} 1^2 \, ds = |\partial\Omega|.$$

If equality holds, then the first eigenfunction ϕ satisfies

$$-\Delta\phi = h = 1 \quad \text{in } \Omega, \qquad \phi = 0, \; \phi_\nu = -d_1^{-1} \quad \text{on } \partial\Omega,$$

and Ω is a ball by a result of J. Serrin (ARMA, 1971). $\square$

EXISTENCE OF OPTIMAL CONVEX SHAPES

THEOREM 12 Among all **convex domains** in $\mathbb{R}^n$ having the **same measure** as the unit ball B, there exists an optimal one, minimising d_1.
Among all **convex domains** in $\mathbb{R}^n$ having the **same perimeter** as the unit ball B, there exists an optimal one, minimising d_1.

These results should be complemented with the description of the optimal convex shapes. This appears quite challenging since, with the measure constraint, **the optimal planar domain is not a disk**.

Sketch of the proof:

STEP 1 **The map $\Omega \mapsto d_1(\Omega)$ is continuous with respect to Hausdorff convergence of convex domains**.

This fact appears nontrivial since there is no monotonicity with respect to inclusions and no obvious extension operator from $H^2 \cap H^1_0(\Omega)$ to $H^2(\mathbb{R}^n)$.

Upper semicontinuity of the map $\Omega \mapsto \delta_1(\Omega)$.

Lower semicontinuity of the map $\Omega \mapsto d_1(\Omega)$.

STEP 2 **A lower bound for d_1 on convex domains**.

By comparison with the solution to the torsion problem, Payne (Indian J. Mech. Math. 1968/69) proved that if ρ_Ω denotes the minimal distance between parallel planes which define a strip containing Ω then $d_1(\Omega) \geq 2\rho_\Omega^{-1}$.

STEP 3 **Conclusion**.

Consider a minimising sequence $\{\Omega_m\} \subset \mathbb{R}^n$ for d_1. By STEP 2 we know that $\exists R > 0$ such that $\Omega_m \subset B_R$ for all m, since otherwise $d_1(\Omega_m) \to +\infty$. This fact, combined with STEP 1 and with Blaschke selection Theorem, shows that the infimum is achieved.

□

NUMERICAL RESEARCH FOR THE OPTIMAL PLANAR SHAPES

We have no theoretical evidence of what the optimal convex shapes could be. Recently, Antunes-Gazzola (ESAIM COCV 2012) performed several numerical experiments.

In the plane, we apply the Method of Fundamental Solutions (MFS). The MFS is a meshfree numerical method for which the approximations are linear combinations of fundamental solution associated to the pde, having pole outside $\overline{\Omega}$.

FIXED AREA

Let Ω_n be a regular n-polygon of measure π having n sides and let D be the unit disk. Then

n	3	4	5	6	7	8	9	10
$d_1(\Omega_n)$	2.02522	1.96179	1.95702	1.96164	1.96733	1.97255	1.97654	1.97974

Remarks.

The equilateral triangle Ω_3 is the maximiser.

We saw before that $d_1(\Omega_4) < 1.96256$ while $d_1(\Omega_4) \approx 1.96179$.

The regular pentagon Ω_5 is the minimiser.

It seems that $n \mapsto d_1(\Omega_n)$ tends monotonically to $2 = d_1(D)$ for $n \geq 5$ and $n \to \infty$.

We tested Reuleaux polygons, irregular polygons up to 8 sides, deformations form regular polygons to the disk, ellipses, stadiums:

Ω_5 remains a good candidate to be the absolute minimiser.

A RELATED PROBLEM? (for a simply supported plate)

$$\Delta^2 u + cu = f \quad \text{in } \Omega, \qquad u = \Delta u = 0 \quad \text{on } \partial\Omega\,,$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain, $c > 0$ is the "stiffness of the resistance to deformation" $f \in L^2(\Omega)$ is the external load, u is the vertical deformation.

PPP: under which conditions on $c > 0$ and Ω the assumption $f \geq 0$ implies that the solution u exists and is positive?

- P.J. McKenna, W. Walter, ARMA 1987
- B. Kawohl, G. Sweers, Indiana UMJ 2002

$\exists$ a maximal interval $c \in (0, c^*(\Omega)]$ where ppp holds. Which is the largest $c^*(\Omega)$ when Ω varies among convex planar domains of given measure? Numerical results show that, among regular polygons, the maximum is attained by the pentagon Ω_5.

- R.F. Bass, J. Horák, P.J. McKenna, Proc. AMS 2004

Since d_1 is the threshold parameter in order to have the ppp for the Steklov problem, are these results somehow connected?

FIXED PERIMETER

Let $\Omega_n^\sharp$ be a regular n-polygon of perimeter 2π having n sides.
Then

n	3	4	5	6	7	8	9	10
$d_1(\Omega_n^\sharp)$	2.60458	2.21364	2.10443	2.05987	2.03791	2.02586	2.01830	2.01336

Remarks.

It seems that now $n \mapsto d_1(\Omega_n^\sharp)$ tends monotonically to $2 = d_1(D)$ for $n \geq 3$ and $n \to \infty$.

The disk D has d_1 smaller than any regular polygon.

We tested Reuleaux polygons, irregular polygons up to 8 sides, deformations form regular polygons to the disk, ellipses, stadiums: **the disk D is a good candidate to be the absolute minimiser.**

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THANK YOU FOR YOUR ATTENTION!