On the Solution of the Dirichlet Problem for the Subelliptic Eikonal Equation

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What is this talk about

We study the regularity and the structure of the singular support of the viscosity solution of the homogeneous Dirichlet problem for the eikonal equation associated with a system of Hörmander vector fields.

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Assumptions

We consider

- $\Omega \subset \mathbb{R}^n$ an open, bounded set with "smooth" boundary, Γ;
- $\Omega' \subset \mathbb{R}^n$ is a nbd of Ω and let $X_1, \ldots, X_N$ are $N \geq 2$ "smooth" (real) vector fields on Ω' such that the Lie algebra generated by the fields X_j as well as by their commutators of length up to r has dimension n (i.e. we assume the Hörmander condition).

(For example the commutator $[X_1, [X_1, X_2]]$ has length 3.)

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The Subelliptic Eikonal Equation

Let T be the continuous viscosity solution of the Dirichlet problem

$$\begin{cases} \sum_{j=1}^{N} (X_j T)^2(x) = 1, & \text{in} \quad \Omega, \\ T(x) = 0, & \text{on} \quad \Gamma. \end{cases} \tag{1}$$

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The Subelliptic Eikonal Equation (cont'd)

Remarks:

1. we adopt the notion of viscosity solution compatible with the elliptic regularization:

$$-\varepsilon(\partial_{x_1}^2 + \ldots + \partial_{x_n}^2)T(x) + \sum_{j=1}^{N}(X_j T)^2(x) = 1$$

   (i.e. the concavity of the solution is privileged w.r.t. the convexity);
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Degenerate equations

Typical features of "degenerate" Hamiltonians:

1. *the characteristic set*

$$\mathrm{Char}\{X_1, \dots, X_N\} = \{(x,p) \in \Omega' \times (\mathbb{R}^n \setminus \{0\}) \mid X_j(x,p) = 0,\ j = 1, \dots, N\}$$

   is not the empty set. (Here $X_j(x,p)$ is the symbol of the vector field $X_j(x)$.)

2. There can be *characteristic (boundary) points*, i.e. $E := \{x \in \Gamma \mid \mathrm{span}\{X_1(x), \dots, X_N(x)\} \subseteq T\Gamma_x\}$ may be non empty.

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Degenerate equations (cont'd)

Remarks:

1. $\forall x \in \Omega$, span $\{X_1, \ldots, X_N\}(x) = \mathbb{R}^n \implies$ the equation is nondegenerate, i.e. $\mathrm{Char}\{X_1, \ldots, X_N\} = \varnothing$, $T \in Lip_{loc}(\overline{\Omega})$ and $T \in SC_{loc}(\overline{\Omega})$ (observe that $E = \varnothing$).
2. $\mathrm{Char}\{X_1, \ldots, X_N\} = \varnothing \implies E = \varnothing$ but $E = \varnothing \not\Rightarrow \mathrm{Char}\{X_1, \ldots, X_N\} = \varnothing$.

Example: in $\mathbb{R}^2$ consider $X_1 = \partial_{x_1}$, $X_2 = x_1 \partial_{x_2}$ and $\Omega = \{(x_1 - 1)^2 + x_2^2 < 1\}$. Then $E = \varnothing$ and

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The Regularity Problem

Study the regularity of T in Hölder spaces?

Known results

Theorem (Evans-James, 1989)

T is locally Hölder continuous of exponent $\frac{1}{r}$ ($r =$ the length of the Lie bracket needed to generate the Lie algebra).

Is this regularity result the best one can hope for?

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Known results (cont'd)

Let $M > 0$ and let k be a positive integer. Consider the (unbounded) set

$$\Omega = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} \, : \, y > M|x|^{k+1}\}$$

and the eikonal equation

$$\begin{cases} |\nabla_x T(x,y)|^2 + |x|^{2k}(\partial_y T(x,y))^2 = 1 & \text{in } \Omega, \\ T = 0 \qquad \text{on } \partial\Omega. \end{cases}$$

Theorem (A. 2012)

The nonnegative viscosity solution of the Dirichlet problem above is locally Lipschitz continuous in Ω. Furthermore, T is Hölder continuous of the exponent $1/(k+1)$ at $(0,0)$. Finally, $T \in C^\omega(\Omega \setminus \{(0,y) \, : \, y \geq 0\})$.

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How to improve the E.-J. result?

A representation formula

Let us consider the controlled system

$$\begin{cases} y'(t) = \sum_{j=1}^{N} u_j(t) X_j(y(t)), \quad t \geq 0 \\ \\ y(0) = x \in \Omega \end{cases} \tag{2}$$

$u = (u_1, \ldots, u_N) : [0, \infty[\longrightarrow \overline{B}_1^N(0)$ is a measurable function (the "control"). We denote the solution of Equation (2) by $y^{x,u}(\cdot)$.

The arrival time to the target Γ is defined as

$$\tau(x, u) = \inf\{t \geq 0 \,:\, y^{x,u}(t) \in \Gamma\}.$$

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A representation formula (cont'd)

Then

$$T(x) = \inf_u \tau(x, u),$$

in other words T is the minimum time function associated with the system (2) with target Γ.

u is called an *optimal control* if $T(x) = \tau(x, u)$, the corresponding trajectory $y^{x,u}$ is an *optimal trajectory*.

FACT: for every $x \in \Omega$ there exists an optimal control $u(\cdot)$.

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Singular time-optimal trajectories

Let $x \in \Omega$ and let u be an optimal control. We say that $y^{x,u}$ is a singular time-optimal trajectory if there exists

$$p(\cdot) \in \text{ AC } ([0, T(x)]; \mathbb{R}^n \setminus \{0\}) \quad \text{s.t.}$$

1. $\rho(t) := (y^{x,u}(t), p(t)) \in \mathrm{Char}\{X_1, \ldots, X_N\}, \ \forall t \in [0, T(x)]$;
2. $\rho'(t) = \sum_{j=1}^{N} u_j(t) H_{X_j}(\rho(t))$, for a.e. t in $[0, T(x)]$;
3. $p(T(x)) = \lambda \nu(y^{x,u}(T(x)))$, ($\nu =$ unit exterior normal to Γ and $\lambda > 0$).

(Here $H_{X_j} = (\nabla_p X_j(x,p), -\nabla_x X_j(x,p))$.)

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Let $x \in \Omega$ and let $y^{x,u}$ be a time-optimal trajectory. Then $y^{x,u}$ is a singular time-optimal trajectory $\iff$ $y^{x,u}(T(x)) \in E$.

In particular, a singular time-optimal trajectory is tangent to Γ at the terminal point.

Theorem (Derridj, 1972)

$E \subset \Gamma$ is a closed set of measure zero.

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$$\limsup_{\overline{\Omega} \ni x \to x_0} \frac{|f(x) - f(x_0)|}{|x - x_0|} = \infty.$$

Theorem (A., Cannarsa and Scarinci)

Let $x_0 \in \Omega$. Then T is not Lipschitz continuous at x_0 if and only if there exists a singular-time optimal trajectory $y^{x_0,u}$.

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Singular time-optimal trajectories and regularity

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Sussmann, Agrachev, Trélat, Cannarsa and Rifford, ...

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The following assertions are equivalent

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- *T is locally semiconcave in Ω;*
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Is there a minimum time function T which is not better than Hölder continuous somewhere?

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The following assertions are equivalent

- *the minimum time problem admits no singular time-optimal trajectories;*
- T *is locally semiconcave in* Ω*;*
- T *is locally Lipschitz continuous in* Ω*.*

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A model of Liu-Sussmann, 1994

In $\mathbb{R}^3$ consider the vector fields

$$X_1 = \partial_{x_1}, \qquad X_2 = (1 - x_1)\partial_{x_2} + x_1^2 \partial_{x_3}.$$

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There exists an open bounded set with C^∞ boundary s.t. the solution of the equation

$$\begin{cases} (X_1 T)^2 + (X_2 T)^2 = 1 & \text{in } \Omega, \\ T|_\Gamma = 0, \end{cases}$$

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Let Ω be a bounded convex open set with smooth boundary. Then the solution of the equation

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Boundary regularity

Theorem

- *for every $x \in \Gamma \setminus E$, T is smooth on a nbd of x;*
- *for every $x \in E$, T is Hölder continuous of exponent $1/r(x)$.*

$r(x)$ is the length of the commutators of $X_1, \ldots, X_N$ needed to generate $\mathbb{R}^n$.

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Sufficient conditions for the regularity of T

Theorem (A., Cannarsa and Scarinci)

If $E = \varnothing$ or $\mathrm{Char}\{X_1, \ldots, X_N\}$ is a symplectic manifold then T is locally Lipschitz continuous in Ω.

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Sufficient conditions for the regularity of T (cont'd)

In some cases, even if the characteristic set is not a symplectic manifold but it can be splitted into a disjoint union of symplectic submanifolds, our approach can be applied.

Example

Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with smooth boundary and let k be a positive integer. In $\mathbb{R}^3$, consider vector fields

$$X_1 = \partial_{x_1} - x_2^{2k+1}\partial_{x_3} \quad \text{and} \quad X_2 = \partial_{x_2} + x_1^{2k+1}\partial_{x_3}.$$

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Indeed

$$\begin{aligned}&\mathrm{Char}(X_1, X_2)\\ &= \{(x_1, x_2, x_3, x_2^{2k+1}p_3, -x_1^{2k+1}p_3, p_3) \;:\; x_1, x_2, x_3 \in \mathbb{R},\ p_3 \neq 0\}.\end{aligned}$$

$\mathrm{Char}(X_1, X_2)$ can be split into the connected submanifolds

$$\begin{aligned}\Sigma_{1,\pm} = \{(x_1, x_2, x_3, x_2^{2k+1}p_3, -x_1^{2k+1}p_3, p_3) \;:\;& \\ x_1, x_2, x_3 \in \mathbb{R},\ (x_1, x_2) \neq (0,0),\ \pm p_3 > 0\}&\end{aligned}$$

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All these submanifolds are symplectic (the rank of the symplectic form is constant and the symplectic form is nondegenerate on these sets) $\Longrightarrow$ there are no singular time-optimal trajectories.

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The singular support

We say that a point x is not in $\operatorname{sing\,supp} T$ if T is smooth on a nbd of x.

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A partial regularity result

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$\operatorname{sing\,supp} T$ *is a closed set of measure zero.*

In other words, in the complement of a closed set of measure zero T has the same regularity of the data of the Dirichlet problem. No condition is required on the time-optimal trajectories.

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Happy Birthday Piermarco!

Figure: Piermarco liked to "control" not only odes and pdes but...