Stochastic and worst-case shape optimization problems

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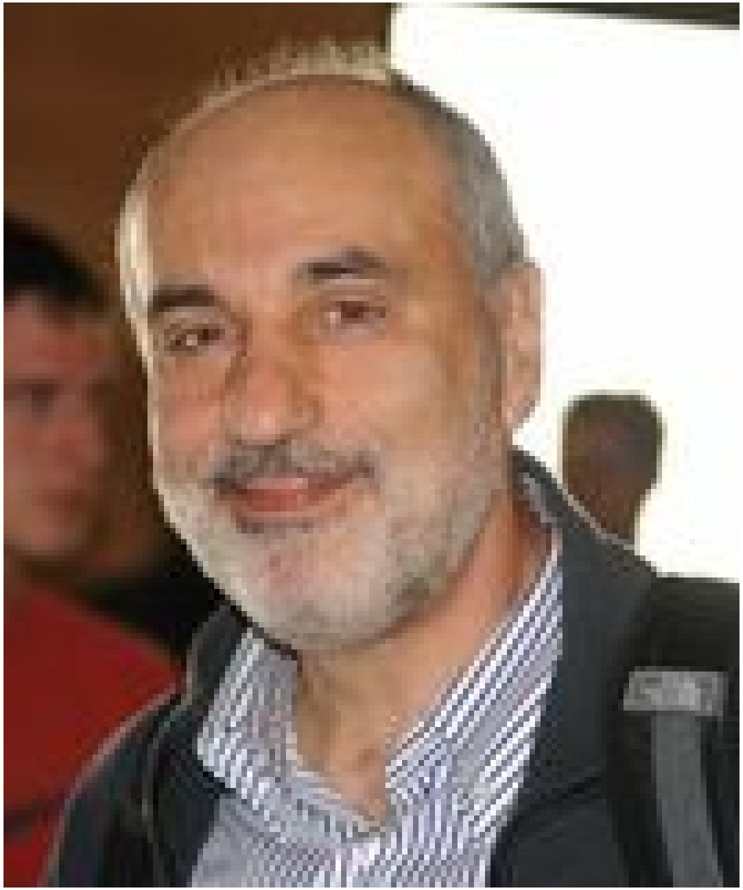
On the occasion of the 60th birthday of Piermarco Cannarsa

Bedlewo 2008 - social dinner

Benasque 2013 - excursion



Benasque 2013 - welcome reception

Joint works

- J.C. Bellido, G. Buttazzo, B. Velichkov: *Worst-case shape optimization for the Dirichlet energy.* Nonlinear Analysis (2017).

- G. Buttazzo, B. Velichkov: *A shape optimal control problem and its probabilistic counterpart.* Paper submitted.

- G. Buttazzo, F. Maestre, B. Velichkov: *Optimal potentials for problems with changing sign data.* Paper in preparation.

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papers available at:

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The problem we deal with is an optimal control problem governed by the state equation

$$-\Delta u = f \text{ in } \Omega, \qquad u \in H_0^1(\Omega);$$

the state variable is $u \in H_0^1(\mathbb{R}^d)$ (extended by zero outside Ω), while the control variable is the domain Ω. The cost function is of the form

$$\int_\Omega j(x,u)\,dx$$

and the class of admissible controls is

$$F(u,\Omega) = \Big\{\Omega \subset D,\ |\Omega| \le m\Big\},$$

where D is a fixed bounded domain of $\mathbb{R}^d$.

Problems of this kind have been studied a lot from the point of view of the existence of an optimal domain; the standard situation is a competition:

homogenization vs shape optimization

In general homogenization wins and no optimal domain exists, since minimizing sequences tend to create fine perforations (Cioranescu-Murat example) and optimal solution exist only in a suitable relaxed sense (capacitary measures introduced by Dal Maso-Mosco 1987). However, in some cases optimal shapes exist.

A first situation in which optimal shapes exist is when geometrical constraints are added to admissible controls, as for instance:

convexity, equi-Lipschitz condition, equi-bounded perimeter, uniform exterior cone condition, uniform capacity condition, uniform Wiener estimates, topological conditions (in dim. 2)...

that rule out the homogenization. In our case we only have the Lebesgue measure constraint $\{|\Omega| \le m\}$ which is not sufficient to provide enough compactness to enforce the existence of an optimal Ω.

Another case in which the existence of an optimal domain occurs is when the cost functional verifies a monotonicity condition.

Theorem [Buttazzo-Dal Maso (ARMA 1993)]
Let $F(\Omega)$ be such that:

- F *is γ-lower semicontinuous;*
- F *is decreasing for set inclusion.*

Then the shape optimization problem

$$\min\Big\{F(\Omega)\ :\ |\Omega|\le m\Big\}$$

admits a solution Ω_{opt}, and $|\Omega_{opt}|=m$.

Let us stress that the monotonicity condition above is rather restrictive and, even if some interesting problems (spectral optimization) verify it, in the linear quadratic case

$$F(u,\Omega) = \int_\Omega |u - u_0|^2 \, dx$$

homogenization wins (i.e. no existence of Ω_{opt}).

We consider the case when the cost integrand j is linear; if R_Ω is the resolvent operator of the Dirichlet Laplacian in Ω, our problem can be rewritten as

$$\min \Big\{ \int_\Omega h(x) R_\Omega(f) \, dx \ : \ |\Omega| \le m \Big\}.$$

We are interested in the case when the datum f is only known only up to some degree of uncertainty; nevertheless, we still want to find an optimal solution in some sense.

A possibility is to assume that f is known with a given probability P on the space of data; we are then in the framework of stochastic optimization and we have to minimize the average cost functional

$$\mathcal{F}_{ave}(\Omega) = \int \left[\int_\Omega h(x) R_\Omega(f)\, dx \right] P(df)$$

in the admissible class $\{|\Omega| \leq m\}$.

Another possibility to handle problems with uncertainty in the data is the so called worst case analysis, in which we optimize the worst possible situation, assuming that the right-hand side f is known up to an error δ. In our case this amounts to minimize the worst case functional

$$\mathcal{F}_{wc}(\Omega) = \sup_{\|g\|_{L^p} \le \delta} \left[\int_\Omega h(x) R_\Omega(f+g)\, dx \right],$$

Roughly speaking we are replacing the P-average by a supremum.

Stochastic shape optimization

We want to determine the shape of a thermic conductor of a given measure m which minimizes the energy, but we only know the heat sources f up to a probability P on $L^2(\mathbb{R}^d)$. We have then the problem

$$\min\left\{\int E(\Omega,f)\,dP(f)\ :\ |\Omega|\le m\right\}$$

where

$$E(\Omega,f)=\min_{u\in H^1_0(\Omega)}\int_\Omega\left(\frac{1}{2}|\nabla u|^2-fu\right)dx.$$

This problems admits an optimal shape.

Since the Sobolev spaces are monotonically included as

$$\Omega_1 \subset \Omega_2 \implies H^1_0(\Omega_1) \subset H^1_0(\Omega_2)$$

the energy $E(\Omega, f)$ is decreasing for the set inclusion, so is also its average

$$\int E(\Omega, f)\, dP(f)$$

and the existence theorem above applies.

Note that in this case, an integration by parts gives the cost functional in the form

$$E(\Omega, f) = -\frac{1}{2}\int_\Omega f R_\Omega(f)\, dx.$$

A more interesting (and realistic) situation is when the cost functional is perfectly determined and the uncertainty occurs only in the PDE. the problem is then

$$\min_{|\Omega|\le m} \int \left[\int_\Omega h(x) u_{\Omega,f} \, dx \right] dP(f)$$

where the function h is perfectly known, while the heat source f is only known up the probability P.

For instance, we want to maximize the average temperature (i.e. $h = 1$) varying the domain Ω, under only a partial information about the heat sources f.

Using the fact that the resolvent operator R_Ω is self-adjoint, we have, denoting by B_P the barycenter of P

$$\int dP(f) \int_\Omega h R_\Omega(f)\, dx = \int dP(f) \int_\Omega R_\Omega(h) f\, dx$$
$$= \int_\Omega R_\Omega(h) \Big(\int f dP(f) \Big) dx = \int_\Omega R_\Omega(h) B_P\, dx.$$

Notice that, by the maximum principle, the monotonicity occurs when

$$B_P \geq 0 \text{ and } h \leq 0 \quad \text{or} \quad B_P \leq 0 \text{ and } h \geq 0.$$

Handling this case is rather more difficult (Buttazzo-Velichkov `arxiv` 2017), when the functions h and B_P may change sign. Nevertheless the existence of an optimal shape still holds, together with some necessary conditions of optimality.

In this case, if the admissible class of domains is $\{|\Omega| \le m\}$, in general we should not expect the optimal domain Ω_{opt} saturates the constraint. In fact, one of the optimality conditions gives

$$\text{either } |\Omega_{opt}| = m \qquad \text{or } \Omega_{opt} \supset \{hB_P \le 0\}.$$

Worst-case shape optimization

We deal here with the case when the control is a domain; other cases of worst case optimization problems can be found in Allaire-Dapogny (M3AS 2014). We want to show the existence of an optimal domain for

$$\min\Big\{\mathcal{F}_{wc}(\Omega)\ :\ \Omega\subset D,\ |\Omega|\le m\Big\}$$

where $\mathcal{F}_{wc}$ is the worst-case functional

$$\mathcal{F}_{wc}(\Omega)=\sup_{\|g\|_{L^p}\le\delta}\Big[\int_\Omega h(x)R_\Omega(f+g)\,dx\Big],$$

Again, in general the cost $\mathcal{F}_{wc}(\Omega)$ is not decreasing with respect to the set inclusion.

We start by considering as F the energy functional

$$E(\Omega, f) = \inf_{u \in H_0^1(\Omega)} \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - fu\right) dx.$$

The worst-case functional $\mathcal{F}_{wc}$ is:

$$\begin{aligned}\mathcal{F}_{wc}(\Omega) &= \sup_{\|g\|_{L^p(D)} \leq \delta} E(\Omega, f+g) \\ &= \inf_{u \in H_0^1(\Omega)} \int_D \left(\frac{1}{2}|\nabla u|^2 - fu\right) dx + \delta \|u\|_{L^{p'}(D)}\end{aligned}$$

and the worst-case shape optimization problem becomes

$$\min \Big\{ \mathcal{F}_{wc}(\Omega) \;:\; \Omega \subset D,\; |\Omega| \leq m \Big\}.$$

The monotonicity assumptions of the existence theorem are verified in the worst-case shape optimization problem, and so we have that for every δ and m there exists an optimal domain $\Omega_{\delta,m}$ solving

$$\min\Big\{\mathcal{F}_\delta(\Omega)\ :\ \Omega\subset D,\ |\Omega|\le m\Big\}$$

where

$$\mathcal{F}_\delta(\Omega)=\inf_{u\in H^1_0(\Omega)}\int_D\left(\frac{1}{2}|\nabla u|^2-fu\right)dx+\delta\|u\|_{L^{p'}(D)}.$$

Radial case

We consider the case of a right-hand side f of radial type; more precisely, we assume $f = f(|x|)$ with $f(r)$ decreasing.

Theorem If D is large enough (to contain a ball of measure m) the optimal domain $\Omega_{\delta,m}$ is a ball of measure m (centered at the origin).

Uncertainty only in the state equation

We consider the case of a shape optimal control problem

$$\max_{|\Omega|\le m} \int_\Omega h(x) u_\Omega \, dx$$

where $h \ge 0$ and u_Ω is the solution of

$$-\Delta u = f \quad \text{in } \Omega, \qquad u \in H_0^1(\Omega).$$

We assume that h is perfectly known, while f is uncertain.

Example: best shape for the average temperature under partially known heat sources.

Again, by the fact that the resolvent operator R_Ω is self-adjoint, we can write the worst case functional as

$$\begin{aligned}\mathcal{F}_\delta(\Omega) &= \sup_{\|g\|_p \le \delta} -\int_D hR(f+g)\,dx \\ &= \sup_{\|g\|_p \le \delta} -\int_D \Big(fR(h) + gR(h)\Big)\,dx \\ &= \int_D -f(x)w_\Omega\,dx + \delta\|w_\Omega\|_{L^{p'}(D)}\end{aligned}$$

where

$$-\Delta w_\Omega = h \quad \text{in } \Omega, \qquad w_\Omega \in H_0^1(\Omega).$$

Notice that $\mathcal{F}_\delta$ is **still** γ-lower semicontinuous but it is **not** monotone decreasing. Then the Buttazzo-Dal Maso theorem for the existence of an optimal shape cannot be used. Nevertheless, the following result holds.

Theorem Assume:

- $h \geq 0$ and $h \in L^d(D)$;
- $f \in L^p(D)$ with $p \geq 2d/(d+2)$;
- $f \geq c > 0$ on D.

Then, there exists $\bar{\delta} > 0$ such that for every $0 < \delta \leq \bar{\delta}$, there exists a solution Ω_δ to the worst-case shape optimal control problem.

A numerical example

$$D = [0,1] \times [0,1], \quad p = 2, \quad \delta = 0.25$$

$$f = \begin{cases} 1 \text{ on } [0,\frac{1}{2}] \times [0,1] \\ 2 \text{ on } [\frac{1}{2},1] \times [0,1] \end{cases}$$

It is numerically convenient to simulate a domain Ω by a potential $V(x)$ taking the value 0 in Ω and $+\infty$ outside. The measure $|\Omega|$ is then simulated through the quantity

$$\int_D e^{-\alpha V(x)}\, dx \quad \text{with } \alpha \text{ small.}$$

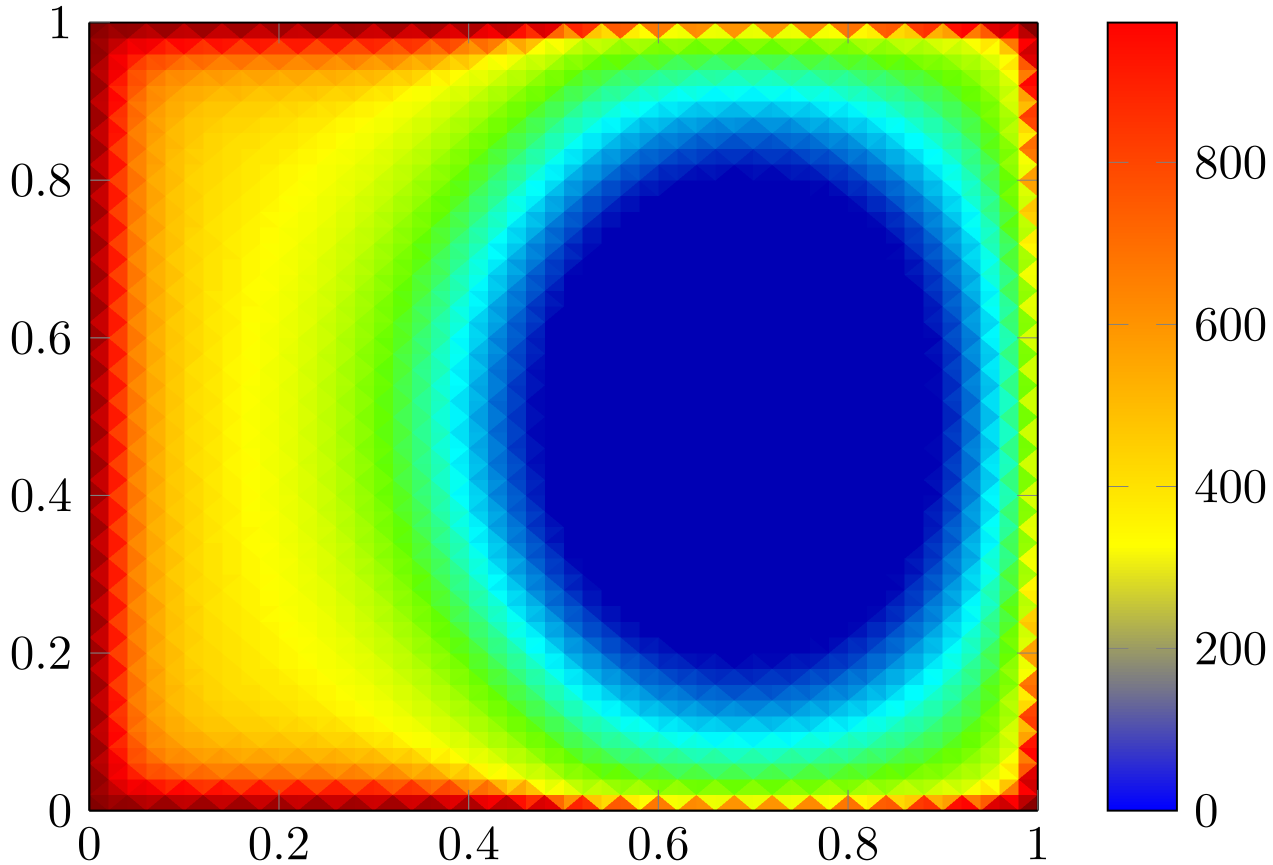
More precisely this approximation has to be stated in terms on Γ-convergence, proved in [BGRV, JEP 2014].

The simulation has been made by J.C. Bellido using:

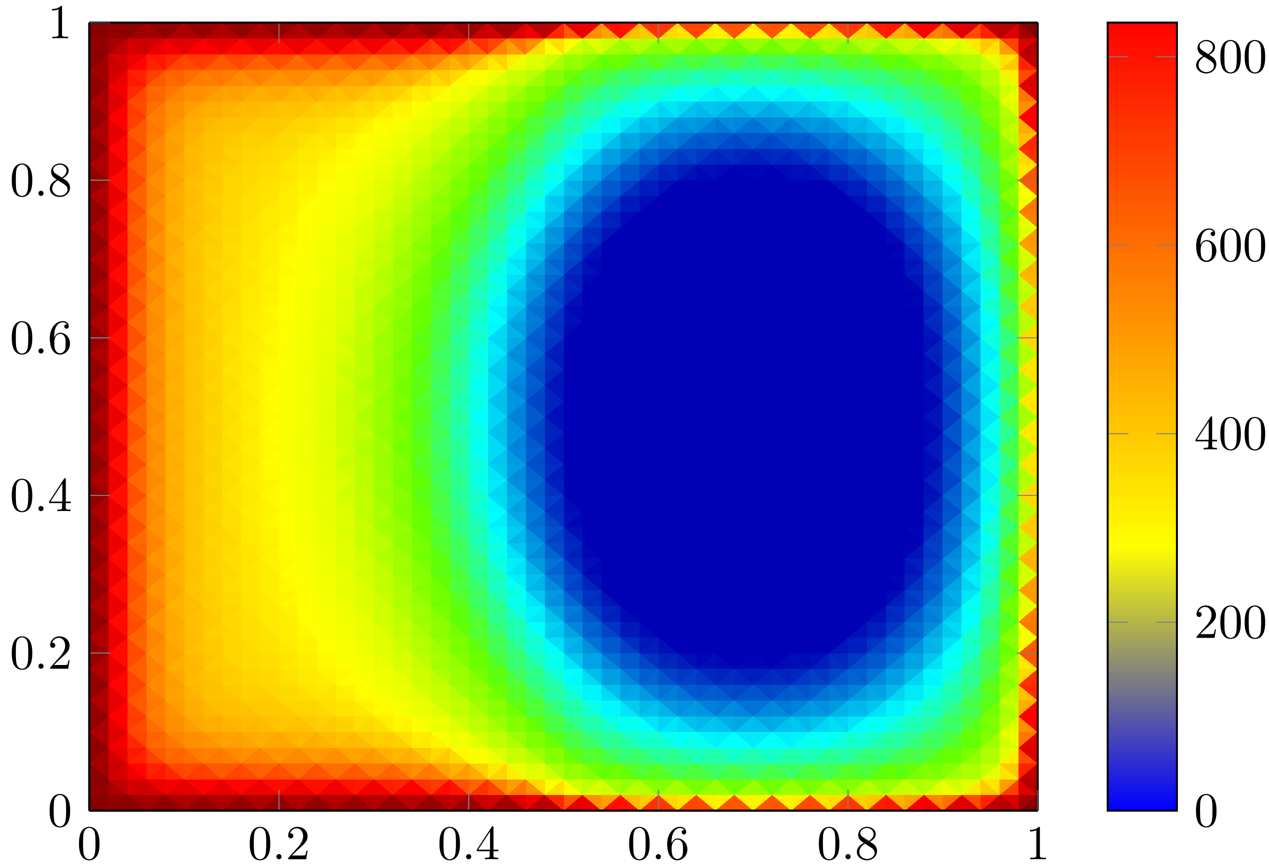
• `FreeFEM++`

• the *Method of Moving Asymptotes* (a kind of gradient method widely used for Topology and Structural Optimization problems)

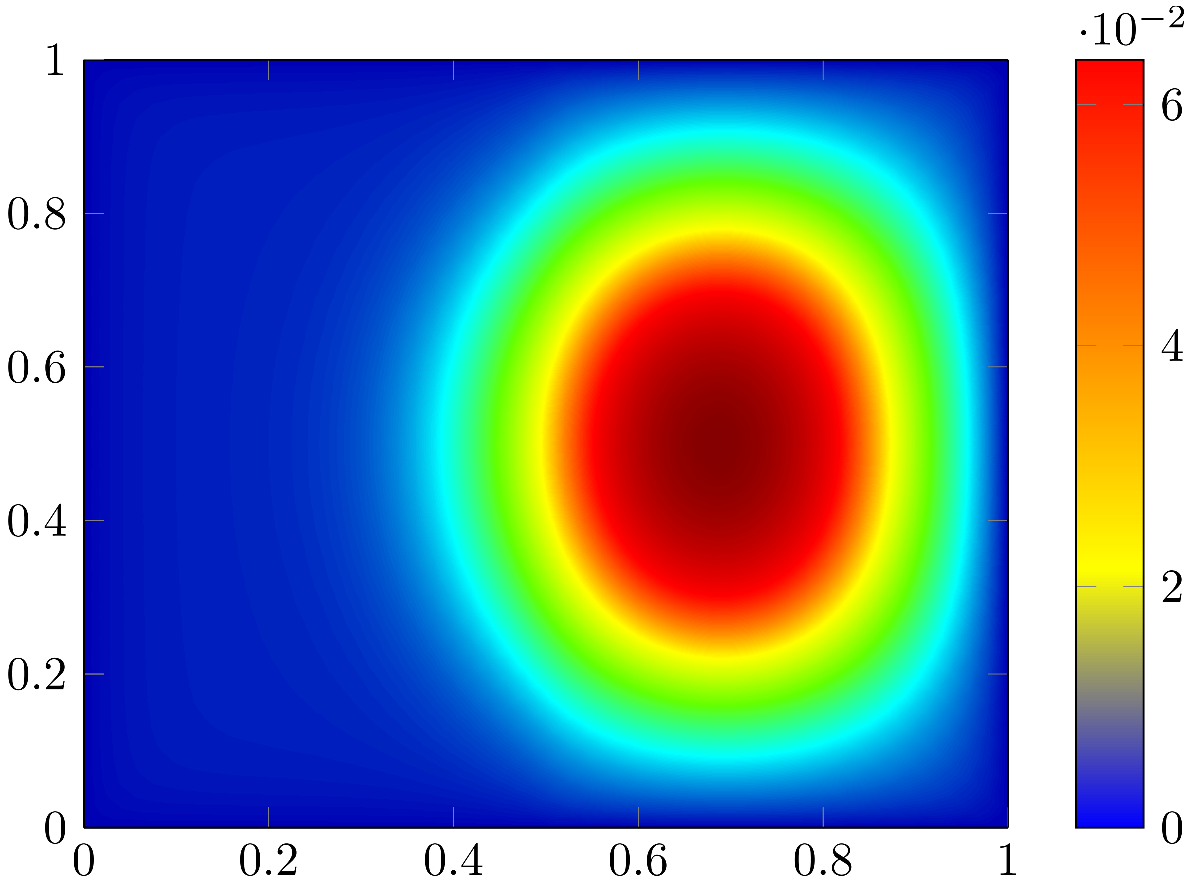
• a mesh of 50×50 elements.



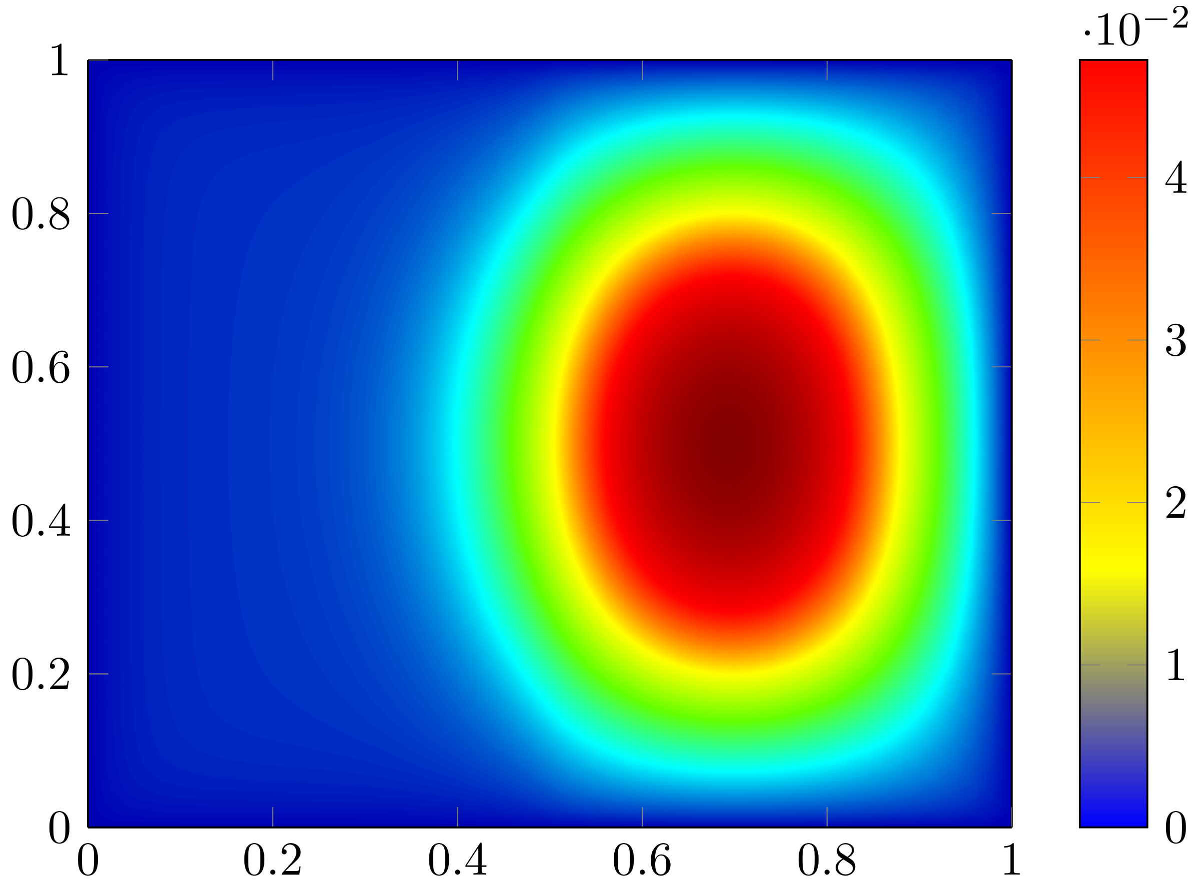
Optimal potential for the unperturbed case



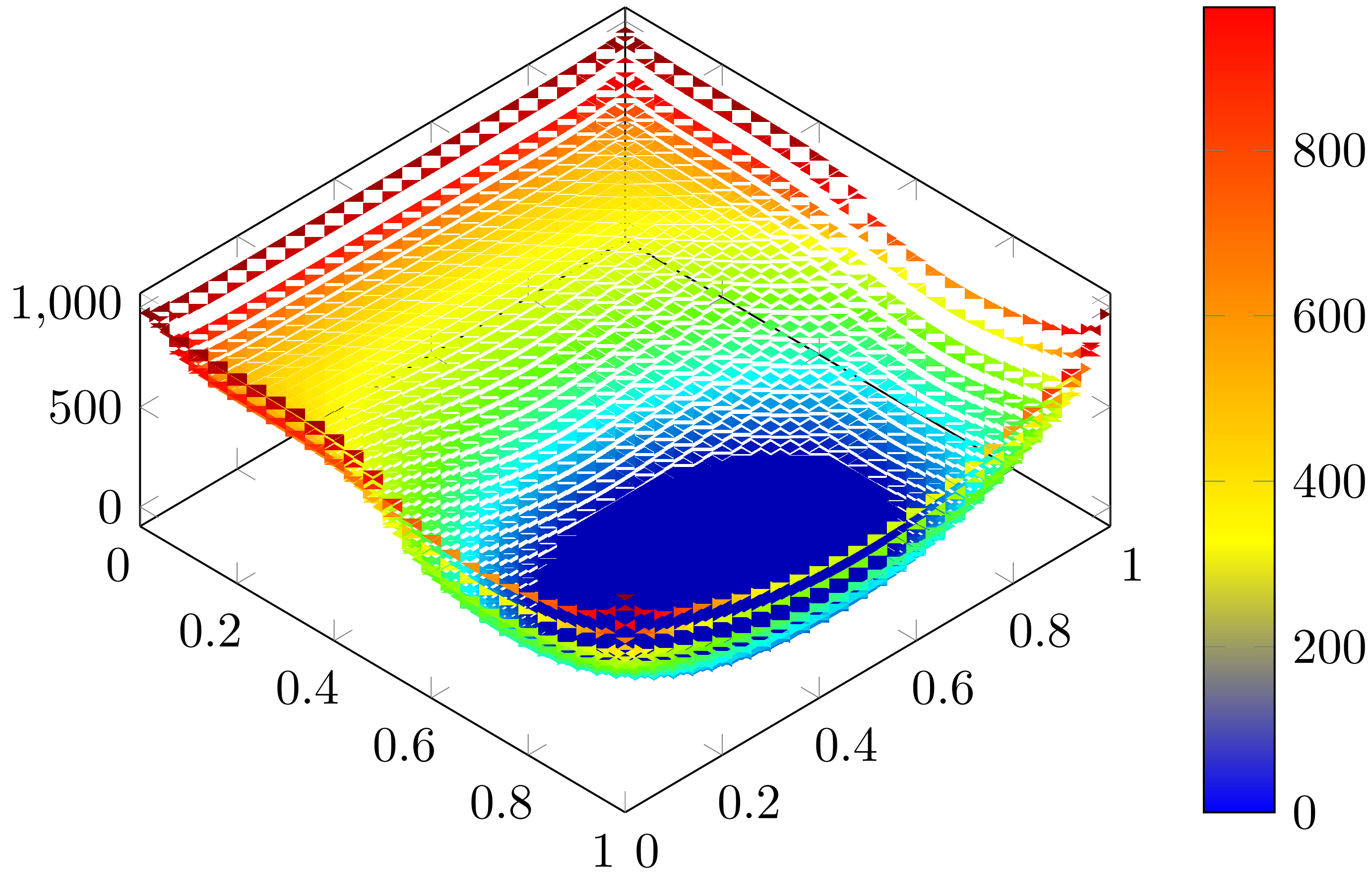
Results for the perturbed case with $\delta = 0.25$



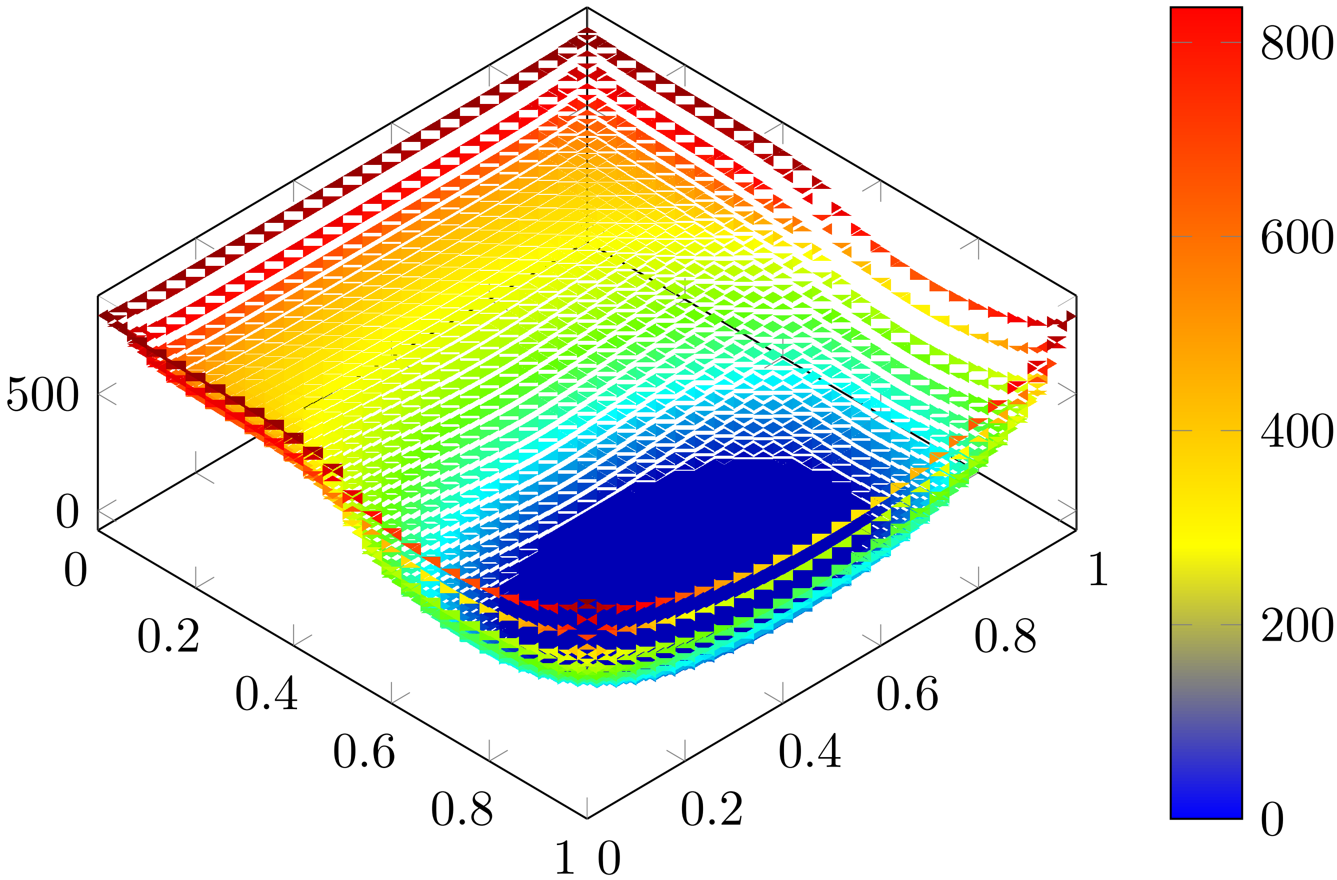
Optimal state for the unperturbed case



Optimal state for the perturbed case with $\delta = 0.25$

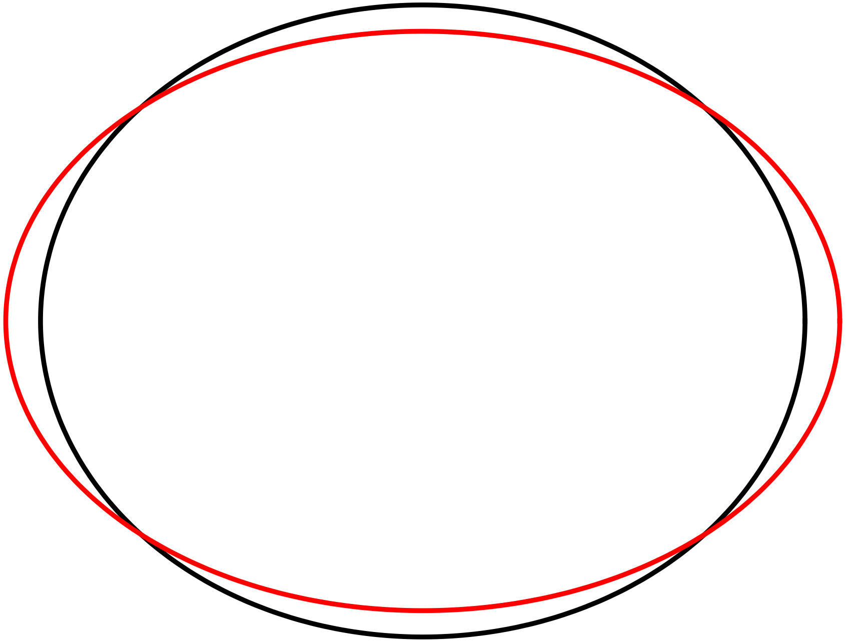


Optimal potential (3D view) for the unperturbed case



Optimal potential (3D view) for the case with $\delta = 0.25$

In progress: It would be very interesting to make an asymptotic analysis (often called Γ development) of the sets Ω_δ for δ small.



the sets Ω (black) and Ω_δ (red)

The expected result is that Ω_δ is (asymptotically) equal to Ω with a boundary layer Σ_δ of local thickness $\delta h(\sigma)$

$$\Sigma_\delta = \left\{ x = t\nu(\sigma),\ \sigma \in \partial\Omega,\ -\delta h^{-}(\sigma) < t < \delta h^{+}(\sigma) \right\}$$

for a suitable function h to be characterized, with $\int_{\partial\Omega} h \, d\sigma = 0$.

Happy birthday Piermarco

and

welcome among seniors...