Stable solutions in potential Mean Field Game systems

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Mean Field Games (MFG) are Nash equilibria in

- nonatomic games = infinitely many agents having individually a negligible influence on the global system (as in Schmeidler (1973), or Mas-Colell (1983, 1984))

- in a optimal control framework = each agent acts on his state which evolves in continuous time and has a payoff depending on the other's position
(stochastic optimal control)

Pioneering works :

- Models invented by Lasry-Lions (2006)
and Caines-Huang-Malhamé (2006)
- Similar models in the economic literature : heterogeneous agent models
(Aiyagari ('94), Bewley ('86), Krusell-Smith ('98),...)

The typical MFG system

Given a finite horizon $T > 0$, we consider the MFG system

$$(MFG) \qquad \begin{cases} -\partial_t u - \Delta u + H(x, Du) = f(x, m(t)) & \text{in } (0,T) \times \mathbb{T}^d \\ \partial_t m - \Delta m - \mathrm{div}(m D_p H(x, Du)) = 0 & \text{in } (0,T) \times \mathbb{T}^d \\ u(T,x) = g(x, m(T)), \ \ m(0,\cdot) = m_0 & \text{in } \mathbb{T}^d \end{cases}$$

where

- $u = u(t,x)$ is the value function of a typical small player,
- $m = m(t,\cdot)$ is the density of the population at time t.
- $H = H(x,p) : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is a smooth, unif. convex in p, Hamiltonian,
- $f, g : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ are “smooth" coupling functions,
  ($\mathcal{P}(\mathbb{T}^d) =$ the set of Borel probability measures on $\mathbb{T}^d$)
- $m_0 \in \mathcal{P}(\mathbb{T}^d)$ is the initial distribution of the players.

Basic results of the MFG system

For the **MFG equilibrium system :**

$$(MFG)\quad \left\{\begin{array}{lll} (i) & -\partial_t u - \Delta u + H(x, Du) = f(x, m(t)) & \text{in } (0,T)\times\mathbb{T}^d \\ (ii) & \partial_t m - \Delta m - \mathrm{div}(m D_p H(x, Du)) = 0 & \text{in } (0,T)\times\mathbb{T}^d \\ (iii) & m(0,\cdot) = m_0,\ u(T,x) = g(x, m(T)) & \text{in } \mathbb{T}^d \end{array}\right.$$

- **Existence of solutions :** holds under general conditions (Lasry-Lions)
- **Uniqueness** cannot be expected in general,
  - but holds under a monotonicity conditions on f and g (Lasry-Lions) :

$$\int_{\mathbb{T}^d}(f(x,m)-f(x,m'))d(m-m')\geq 0,\ \int_{\mathbb{T}^d}(g(x,m)-g(x,m'))d(m-m')\geq 0.$$

- **Link with differential games** with finitely many players.

  — from the MFG system to the $N-$player differential games
  Many contributions (Huang-Caines-Malahmé, Carmona-Delarue, ...)

  — from Nash equilibria of $N-$player differential games to the MFG system.
  - LQ differential games (Bardi, Bardi-Priuli)
  - Open loop NE (Fischer, Lacker),
  - Closed loop NE (C.-Delarue-Lasry-Lions).

Standing assumptions

- H is $\mathcal{C}^2$ on $\mathbb{T}^d \times \mathbb{R}^d$ with

$$\frac{1}{C} I_d \leq D^2_{pp} H(x,p) \leq C I_d \qquad \forall (x,p) \in \mathbb{T}^d \times \mathbb{R}^d \ .$$

  and

$$\langle D_x H(x,p), p \rangle \geq -C \left(|p|^2 + 1 \right) .$$

- The maps f and g are Lipschitz continuous and regularizing :

  The map $m \to f(\cdot, m)$ is Lipschitz continuous from $\mathcal{P}(\mathbb{T}^d)$ to $\mathcal{C}^1(\mathbb{T}^d)$,
  while the map $m \to g(\cdot, m)$ is Lipschitz continuous from $\mathcal{P}(\mathbb{T}^d)$ to $\mathcal{C}^3(\mathbb{T}^d)$.

- The initial measure m_0 has a smooth density.

Outline

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Potential mean field games

Definition

We say that the MFG system is potential if there exists $F, G : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ of class C^1 such that

$$\frac{\delta F}{\delta m}(m,x) = f(x,m), \qquad \frac{\delta G}{\delta m}(m,x) = g(x,m).$$

Directional derivative : a map $U : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ is C^1 if there exists a continuous map $\dfrac{\delta U}{\delta m} : \mathcal{P}(\mathbb{T}^d) \times \mathbb{T}^d \to \mathbb{R}$ such that, for any $m, m' \in \mathcal{P}(\mathbb{T}^d)$,

$$U(m') - U(m) = \int_0^1 \int_{\mathbb{T}^d} \frac{\delta U}{\delta m}((1-s)m + sm', y)d(m' - m)(y)ds.$$

As $\frac{\delta U}{\delta m}$ is defined up to an additive constant, we adopt the normalization convention

$$\int_{\mathbb{T}^d} \frac{\delta U}{\delta m}(m,y)dm(y) = 0.$$

Potential associated with the MFG system : Let

$$\Phi(m,w) = \int_0^T \int_{\mathbb{T}^d} m(t,x) L\left(x, \frac{w(t,x)}{m(t,x)}\right) dxdt + \int_0^T F(m(t))dt + G(m(T)),$$

where (m, w) solves

$$\partial_t m - \Delta m + \mathrm{div}(w) = 0 \text{ in } (0,T) \times \mathbb{T}^d \qquad m(0) = m_0$$

and L is the "convex conjugate" of H : $L(x,q) = \sup_{p \in \mathbb{R}^d} -\langle p, q\rangle - H(x,p)$.

Proposition

- Φ has a (at least) one minimum.
- If (m, w) is a minimum of Φ, then there exists u such that (u, m) is a solution of the MFG system, with $w = -mD_pH(x, Du)$.
- If f, g are monotone, then Φ is strictly convex and the minimum is unique.

Outline

The Fictitious Play

It runs as follows :

- The agents share the same initial guess $(\overline{m}^0(t))_{t\in[0,T]}$ on the evolution of the population density.

If the game has been played n times, then :

- At the beginning of stage $n+1$, the players have observed the same past and share the same guess $(\overline{m}^n(t))_{t\in[0,T]}$ on the evolving density of the population.
- They compute their corresponding optimal control with value function u^{n+1} accordingly.
- When all players actually implement their optimal strategy, the population density evolves in time and the players observe the resulting evolution $(m^{n+1}(t))_{t\in[0,T]}$.
- At the end of stage $n+1$ the players update their guess according to the rule (the same for all the players), which consists in computing the *average* of their observation up to time $n+1$.

This yields to a sequence $(u^n, m^n, \overline{m}^n)$ satisfying :

$$\begin{cases} -\partial_t u^{n+1} - \Delta u^{n+1} + H(x, Du^{n+1}(t,x)) = f(x, \overline{m}^n(t)), \\ \partial_t m^{n+1} - \Delta m^{n+1} - \mathrm{div}(m^{n+1} D_p H(x, Du^{n+1})) = 0, \\ m^{n+1}(0) = m_0, \ u^{n+1}(x,T) = g(x, \overline{m}^n(T)) \end{cases}$$

where

$$\overline{m}^n = \frac{1}{n}\sum_{k=1}^{n} m^k.$$

and the smooth initial guess $(m^0(t))_{t\in[0,T]}$ is given.

Main result

Theorem [C.-Hadikhanloo (2017)]

Under the previous assumptions :

- The family $\{(u^n, m^n)\}_{n\in\mathbb{N}}$ is uniformly continuous and any cluster point is a solution to the MFG system.
- In particular, if f and g are monotone, then the whole sequence $\{(u^n, m^n)\}_{n\in\mathbb{N}}$ converges to the unique solution of the MFG system.

Remarks :

- The drawback of the result is that players have to know the mechanisms f and g.
- The structure of proof is inspired by ideas of Monderer-Shapley '96.

Ideas of proof

- **Potential associated with the MFG system :** Recall that

$$\Phi(m,w)=\int_0^T\int_{\mathbb{T}^d} m(t,x)L(x,w(t,x)/m(t,x))dxdt+\int_0^T F(m(t))dt+G(m(T)),$$

where (m,w) solves $\partial_t m-\Delta m+\mathrm{div}(w)=0$ in $(0,T)\times\mathbb{T}^d \qquad m(0)=m_0$.

- **Monotonicity of Φ :** Setting $w^n := -m^n D_pH(x,Du^n), \qquad \overline{w}^n=\frac{1}{n}\sum_{k=1}^n w^k$, then

$$\Phi(\overline{m}^{n+1},\overline{w}^{n+1})-\Phi(\overline{m}^n,\overline{w}^n)\le -\frac{1}{C}\frac{a_n}{n}+\frac{C}{n^2},$$

where $a_n=\int_0^T\int_{\mathbb{T}^d}\overline{m}^{n+1}\left|\overline{w}^{n+1}/\overline{m}^{n+1}-w^{n+1}/m^{n+1}\right|^2$.

- This implies that $\sum_n a_n/n<+\infty$ and, as $|a_{n+1}-a_n|\to 0$, that $a_n\to 0$.

- Thus $\|\overline{w}^{n+1}/\overline{m}^{n+1}-w^{n+1}/m^{n+1}\|\to 0$, which yields $\|(\overline{u}^n,\overline{m}^n)-(u^n,m^n)\|\to 0$.

$\square$

Other results

- The fictitious play works also for **first order Mean Field Games** :

$$(MFG) \quad \left\{ \begin{array}{lll} (i) & -\partial_t u + H(x, Du) = f(x, m(t)) & \text{in } [0,T] \times \mathbb{T}^d \\ (ii) & \partial_t m - \mathrm{div}(m D_p H(x, Du)) = 0 & \text{in } [0,T] \times \mathbb{T}^d \\ (iii) & m(0,\cdot) = m_0, \; u(T,x) = g(x, m(T)) & \text{in } \mathbb{R}^d \end{array} \right.$$

- Because the lack of regularity of the solutions, statements and proofs are completely different.
- The same procedure can be adapted to N players : any cluster point is then an approximate MFG equilibrium when N is large.
- Convergence in the monotone, non potential case (Hadikhanloo).

Outline

Aim :

- Understand the multiplicity of solutions
- Show that “most" solutions are isolated
- Prove the full convergence of the Fictitious Play in this setting.

Additional assumptions : $f, g : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ are C^1 with respect to m.

$\longrightarrow$ **Strongly inspired** by similar results in classical optimal control or in optimal control of PDEs. (Cannarsa-Sinestrari (2004) or Cannarsa and Tessitore (1994))

Stable solutions of the MFG system

Definition

Let (u, m) be a solution of the MFG system starting from the initial configuration $m(t_0) = m_0$. We say that (u, m) is stable if, for any $t_1 \in [t_0, T)$, the unique classical solution to the linearized system

$$(LS)_{|_{[t_1,T]}} \quad \begin{cases} -\partial_t v - \Delta v + D_p H(x, Du) \cdot Dv = \dfrac{\delta f}{\delta m}(x, m)(\mu) \text{ in } \mathbb{T}^d \times (t_1, T), \\ \partial_t \mu - \Delta \mu - \mathrm{div}(\mu D_p H(x, Du)) - \mathrm{div}(m D^2_{pp} H(x, Du) Dv) = 0 \text{ in } \mathbb{T}^d \times (t_1, T) \\ \mu(x, t_1) = 0, \;\; v(x, T) = \dfrac{\delta g}{\delta m}(x, m(T))(\mu(T)) \text{ in } \mathbb{T}^d. \end{cases}$$

is given by $(v, \mu) \equiv (0, 0)$.

Stable solutions and uniqueness

Stable solutions are isolated :

Proposition (Briani-C., preprint)

Let (u, m) be a stable solution of the MFG system starting from (t_0, m_0).

Then, for any $\eta > 0$, there is a neighborhood U of m_0 in $\mathcal{P}(\mathbb{T}^d)$ such that : for any $m_1 \in U$, there is a unique solution (u', m') of the MFG system starting from (t_0, m_1) which is close to (u, m) :

$$\|(u, m) - (u', m')\|_{C^{1,0} \times C^0} \leq \eta.$$

Proof : compactness argument.

Existence of stable solutions

Recall that

$$\Phi(m,w) = \int_0^T \int_{\mathbb{T}^d} m(t,x) L(x, w(t,x)/m(t,x)) dxdt + \int_0^T F(m(t))dt + G(m(T)),$$

where (m,w) solves $\partial_t m - \Delta m + \mathrm{div}(w) = 0$ in $(0,T) \times \mathbb{T}^d \qquad m(0) = m_0$.

Theorem (Briani-C., preprint)

Let (u,m) corresponding to a minimizer of the energy Φ. Then for any $t_1 \in (0,T)$, the pair $(u,m)_{|_{[t_1,T]}}$ is stable.

In particular, the set of initial positions (t_0, m_0) from which a stable solution starts is dense.

Sketch of proof.

- For $t \in [t_0, T]$, let $\mathcal{J}(t_0, m, u; \mu, z)$ the second order derivative of the potential Φ in the direction (μ, z) :

$$\begin{aligned}
&\mathcal{J}(t, m, u; \mu, z) \\
&\quad := \int_t^T \int_{\mathbb{T}^d} m^{-1} D^2_{qq} L\left(x, \frac{w}{m}\right)(z + \mu D_p H(x, Du)) \cdot (z + \mu D_p H(x, Du)) \\
&\quad + \int_t^T \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{\delta f}{\delta m}(x, m(t), y)\mu(x,t)\mu(y,t) + \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{\delta g}{\delta m}(x, m(T), y)\mu(x,T)\mu(y,T).
\end{aligned}$$

- By minimality of (u, m), $\mathcal{J}(t, m, u; \mu, z) \geq 0$ for all $t \in [t_0, T]$ and all (μ, z) such that $\partial_t \mu - \Delta \mu + \mathrm{div}(z) = 0$.

- **Lemma.** (v, μ) solves $(LS)_{|_{[t,T]}}$ IFF $\mathcal{J}(t, u, m; \mu, z) = 0$ where $z = -\mu D_p H(x, Du) - m D^2_{pp} H(x, Du) Dv$.

- Let $t_1 \in (t_0, T)$ and (v, μ) solution to $(LS)_{|_{[t_1, T]}}$. To show : $(v, \mu) = 0$.

- Set $z := -\mu D_p H(x, Du) - m D^2_{pp} H(x, Du) Dv$. Then $\mathcal{J}(t_1, m, u; \mu, z) = 0$.

  Extend (μ, z) to $[t_0, T]$ by setting $(\mu, z) = 0$ on $[t_0, t_1]$. Then $\mathcal{J}(t_0, m, u; \mu, z) = 0$.

- Thus (μ, z) solves $(LS)_{|_{[t_0, T]}}$ and so (μ, z) is smooth on $[t_0, T]$: $z(t_1) = 0$ and $Dv(t_1) = 0$.

- So (v, μ) solves $(LS)_{|_{[t_1, T]}}$ with $(Dv(t_1), \mu(t_1)) = 0$.

- A unique continuation argument* à la Lions-Malgrange implies that $(Dv, \mu) = 0$.

(* inspired by Cannarsa and Tessitore (1994))

□

Application to the Fictitious Play

Assume that

- (u, m) is a stable MFG equilibrium in a potential MFG,
- that (u^n, m^n) is the sequence given by the Fictitious Play,
- that the initial guess m^0 is “sufficiently close" to m.

Theorem (C.-Briani, preprint)

Under the above condition, the full sequence (u^n, m^n) converges to (u, m).

Conclusion

So far, we have explained that the fictitious play mechanism works for a potential Mean field games and discussed the stability of solutions.

Open problems

- Convergence rate.
- More general models of learning.
- Learning in Mean field games with common noise (not potential).