On the singular dynamics of the viscosity solutions

Joint work with Piermarco Cannarsa & Albert Fathi

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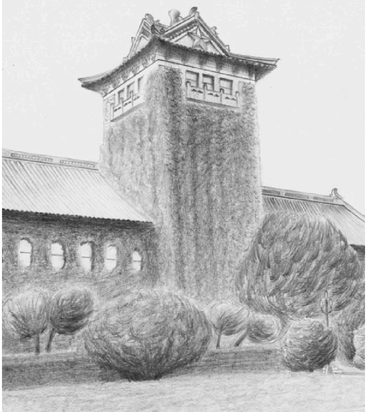
On the occasion of the 60th birthday of Piermarco Cannarsa

July 4, 2017



Introduction

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- if $0 \notin H_p(\mathbf{x}(0), D^+u(\mathbf{x}(0)))$.

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- We developed a intrinsic approach to the propagation of singularities and its connection to generalized characteristics, which leads to a global result.
- This also leads to potential works in a wide range bridging Hamiltonian dynamical systems, PDEs, (sub-)Riemann-Finsler geometry, calculus of variations and optimal controls. We will also interpret some further results we have already obtained recently in this direction.

Literature on propagation of singularities

Reference on the propagation of singularities (possible incomplete)

- Albano-Cannarsa(1999,2000,2002), Albano(2002),
- Bogaevsky(2006),
- Yu(2006,2007),
- Cannarsa-Mazzola−Sinestrari(2015),
- Strömberg(2013), Strömberg-Ahmadzdeh(2014),
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- Cannarsa-Yu(2009),
- Khanin-Sobolevski(2014)

Main reference of this talk:

1. Cannarsa-C-Zhang(2014)
2. Cannarsa-C(2015)
3. Cannarsa-C(preprint 2016)
4. Cannarsa-C-Fathi(2017)
5. Full paper of item 4

A brief review on weak KAM theory

Hamilton-Jacobi equations

Let M be a C^2 smooth closed manifold and $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian. We consider the viscosity solutions of the stationary Hamilton-Jacobi equation

$$H(x, Du(x)) = 0, \quad x \in M, \tag{HJ$_s$}$$

or the evolutionary one

$$D_t u(t,x) + H(x, D_x u(t,x)) = 0, \quad x \in M, t > 0. \tag{HJ$_e$}$$

Value function of Bolza Problem

By dynamical programming principle,

$$u(t,x)=\inf_{y\in M}\{u_0(y)+A_t(y,x)\}$$

is the unique viscosity solution of (HJ_e) with initial data $u(0,x)=u_0$, where

$$A_t(x,y)=\min_{\xi\in\Gamma^t_{x,y}}\int_0^t L(\xi(s),\dot{\xi}(s))ds \qquad (x,y\in M),$$

with

$$\Gamma^t_{x,y}=\{\xi\in W^{1,1}([0,t];M):\xi(0)=x,\xi(t)=y\}$$

Here, $A_t(x,y)$ is called the fundamental solution with respect to (HJ_e), or generating function in the context of symplectic geometry.

Lax-Oleinik semigroups & weak KAM solutions

Let $u_0 \in C(M, \mathbb{R})$, for any $x \in M$, define

$$T_t^+ u_0(x) = \sup_{y \in M} \{u_0(y) - A_t(x, y)\},$$
$$T_t^- u_0(x) = \inf_{y \in M} \{u_0(y) + A_t(y, x)\}.$$

- This is also called Lasry-Lions regularization in PDEs. It is one kind of variational approximation process.

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- This is also called Lasry-Lions regularization in PDEs. It is one kind of variational approximation process.
- A continuous function u is said to be a weak KAM solution of (HJ$_s$) if u is a fixed point of the semigroup $\{T_t^-\}$ for all $t > 0$.

Dominated functions & Calibrated curves

- A function $u : M \to \mathbb{R}$ is said to be dominated by L iff, for each absolutely continuous arc $\gamma : [a, b] \to M$ with $a < b$, one has

$$u(\gamma(b)) - u(\gamma(a)) \leqslant \int_a^b L(\gamma(s), \dot{\gamma}(s))ds.$$

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- An absolutely continuous curve $\gamma : [a, b] \to M$ is said to be u-calibrated if

$$u(\gamma(b)) - u(\gamma(a)) = \int_a^b L(\gamma(s), \dot{\gamma}(s))ds + c[0](b - a).$$

Aubry sets

- The projected Aubry set w.r.t. u is defined by

$$\mathcal{I}(u) = \{x \in M : x = \gamma(0) \text{ for some } u\text{-calibrated curve } \gamma : \mathbb{R} \to M\}.$$

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- The α-limit set of a backward u-calibrated curve is contained in $\mathcal{I}(u)$.

Cut points & Singular points

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- $\Sigma(u) \subset \mathrm{Cut}(u) \subset M \setminus \mathcal{I}(u)$, and $\Sigma(u) \subset \mathrm{Cut}(u) \subset \overline{\Sigma(u)}$.

Regularity properties

Semiconcave functions

- Let $\Omega \subset \mathbb{R}^n$ be a convex open set, a function $u : \Omega \to \mathbb{R}$ is *semiconcave* if there exists a constant $C > 0$ such that

$$\lambda u(x) + (1-\lambda)u(y) - u(\lambda x + (1-\lambda)y) \leqslant \frac{C}{2}\lambda(1-\lambda)|x-y|^2$$

for any $x, y \in \Omega$ and $\lambda \in [0, 1]$.

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  is concave.
- Equivalently, u is semiconcave with constant C if $u = \inf_\alpha u_\alpha$ with each u_α a C^2 functions whose Hessian (in the sense of distribution) is bounded above uniformly by $C\,\mathrm{Id}$.

Superdifferential

Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be a continuous function. We recall that, for any $x \in \Omega$, the closed convex sets

$$D^- u(x) = \left\{ p \in \mathbb{R}^n : \liminf_{y \to x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|y - x|} \geqslant 0 \right\},$$

$$D^+ u(x) = \left\{ p \in \mathbb{R}^n : \limsup_{y \to x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|y - x|} \leqslant 0 \right\}.$$

are called the *subdifferential* and *superdifferential* of u at x, respectively.

Limiting differential

Let $u : \Omega \to \mathbb{R}$ be locally Lipschitz. We recall that a vector $p \in \mathbb{R}^n$ is called a *limiting differential* of u at x if there exists a sequence $\{x_n\} \subset \Omega \setminus \{x\}$ such that u is differentiable at x_k for each $k \in \mathbb{N}$, and

$$\lim_{k\to\infty} x_k = x \quad \text{and} \quad \lim_{k\to\infty} Du(x_k) = p.$$

The set of all limiting differentials of u at x is denoted by $D^*u(x)$.

Properties of superdifferentials

Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be a semiconcave function and let $x \in \Omega$. Then

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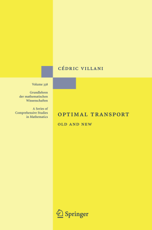
For more information on the semiconcavity, see, e.g.,



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- For each $\lambda > 0$, There exists $t_0 > 0$, the map $y \mapsto A_t(x,y)$ is convex on $B(x,\lambda t)$ with $t \in (0,t_0)$. The constant is C_2/t.

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- If u is a weak KAM solution of (HJ$_s$), by the L-O representation formulae as a marginal function

$$u(x) = \inf_{y \in M} \{u(y) + A_t(y,x)\}, \quad x \in M, t > 0,$$

then u is semiconcave with constant, say C_1.

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- Moreover, for any $t \in (0, t_0]$,

$$
\begin{aligned}
D_y A_t(x, y) =& L_v(\xi(t), \dot{\xi}(t)), \\
D_x A_t(x, y) =& -L_v(\xi(0), \dot{\xi}(0)), \\
D_t A_t(x, y) =& -E_{t,x,y},
\end{aligned}
$$

where $\xi \in \Gamma^t_{x,y}$ is the unique minimizer of $A_t(x, y)$ and $E_{t,x,y}$ is the energy of the Hamiltonian trajectory $(\xi(s), p(s))$ with $p(s) = L_v(\xi(s), \dot{\xi}(s))$.

An illustrative observation

Proposition

Let $x \in M$ *and* $u : M \to \mathbb{R}$ *be a weak KAM solution of the H-J equation* (HJ_s). *Then* $p \in D^*u(x)$ *if and only if there exists a unique* C^2 *backward calibrated curve* $\gamma : (-\infty, 0] \to M$ *with* $\gamma(0) = x$ *and* $p = L_v(x, \dot{\gamma}(0))$.

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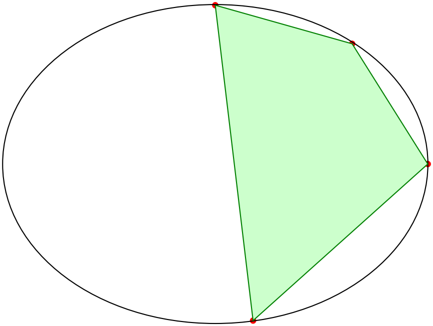

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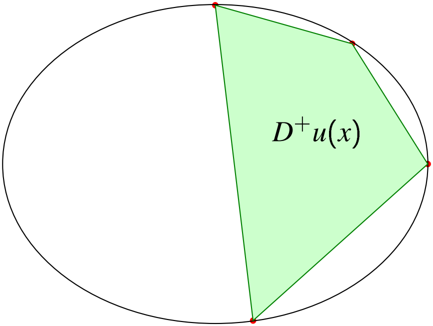

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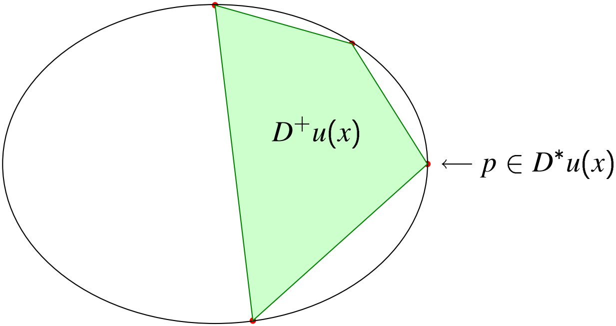

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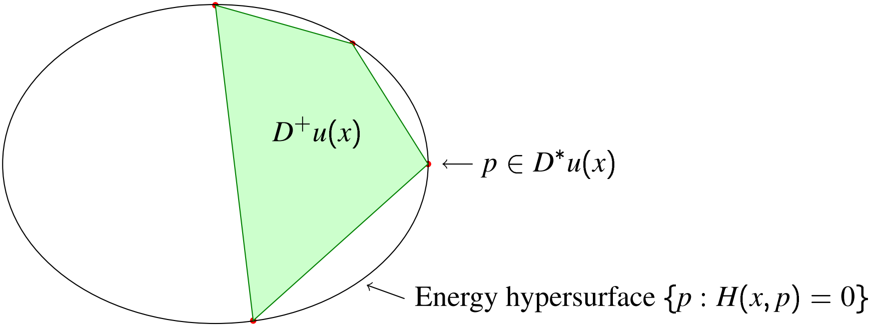

Intrinsic approach of propagation of Singularities

Propagation of Singularities: general case

Singularities for arbitrary $t > 0$ (Cannarsa-C, 2016)

If $x \in \mathrm{Cut}\,(u)$, then any local maximizer of $u(\cdot) - A_t(x, \cdot)$ is contained in $\Sigma\,(u)$ for all $t > 0$. Moreover, There exists $t_0 > 0$ (t_0 is independent of x) such that, if $x \in \mathrm{Cut}\,(u)$, then the function

$$u(\cdot) - A_t(x, \cdot)$$

achieves a unique maximizer $y_{t,x}$ for all $t \in (0, t_0]$. Let the curve is defined by

$$\mathbf{y}(t) := \begin{cases} x & \text{if} \quad t = 0 \\ y_{t,x} & \text{if} \quad t \in (0, t_0], \end{cases} \qquad (\mathrm{GC}_{loc})$$

then $\mathbf{y}(t) \in \Sigma\,(u)$ for all $t \in (0, t_0]$.

Sketch of the proof

Singularities for arbitrary $t > 0$

- For any $t > 0$ and $y_{t,x} \in \operatorname{arg\,max}_{loc}\{u(\cdot) - A_t(x,\cdot)\}$, suppose $y_{t,x}$ is a differentiable point of u. Thus

$$0 \in D^+\{u(\cdot) - A_t(x,\cdot)\}(y_{t,x}) = Du(y_{t,x}) - D^-\{A_t(x,\cdot)\}(y_{t,x}).$$

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- It follows that $A_t(x,\cdot)$ is differentiable at $y_{t,x}$ and

$$p_{t,x} = Du(y_{t,x}) = D_y A_t(x, y_{t,x}).$$

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- There exists two C^2 curves $\xi_{t,x} : [0,t] \to \mathbb{R}^n$ and $\gamma_x : (-\infty, t] \to \mathbb{R}^n$ such that $\xi_{t,x}(0) = x$, $\gamma_x(t) = \xi_{t,x}(t) = y_{t,x}$ and

$$p_{t,x} = L_v(\gamma_x(t), \dot{\gamma}_x(t)) = L_v(\xi_{t,x}(t), \dot{\xi}_{t,x}(t)).$$

  Since $\xi_{t,x}$ and γ_x has the same endpoint condition at t, then they coincide on $[0,t]$. This leads to a contradiction since $x \in \mathrm{Cut}\,(u)$.

Sketch of the proof

Looking for a unique maximizer

- Let $x \in \mathrm{Cut}\,(u)$, denoted by $M_t(x) = \mathtt{arg\,max}_{loc}\{u(\cdot) - A_t(x,\cdot)\}$, the set-valued map $t \mapsto M_t(x) \subset \Sigma\,(u)$ is upper-semicontinuous.

Sketch of the proof

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- Let $x \in \mathrm{Cut}\,(u)$, denoted by $M_t(x) = \mathtt{arg\,max}_{loc}\{u(\cdot) - A_t(x,\cdot)\}$, the set-valued map $t \mapsto M_t(x) \subset \Sigma\,(u)$ is upper-semicontinuous.
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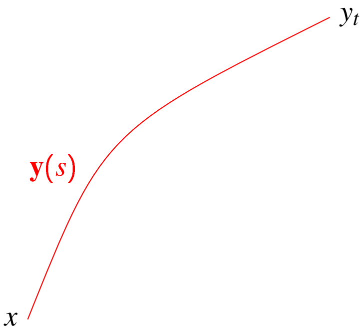
- Whether there exists a continuous selection of the set-valued map $t \mapsto M_t(x)$ is unclear!
- Since $u(\cdot)$ is semiconcave with constant C_1 and $A_t(x,\cdot)$ is convex on $B(x,\lambda t)$, $t \in (0,t_0)$, with constant C_2/t. Therefore $u(\cdot) - A_t(x,\cdot)$ is strictly concave on $B(x,\lambda t)$ if t satisfies

$$C_1 - C_2/t < 0.$$

Then we have a unique maximizer for $t < t_0 = C_2/C_1$.

A picture



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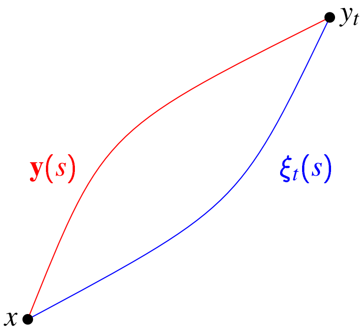

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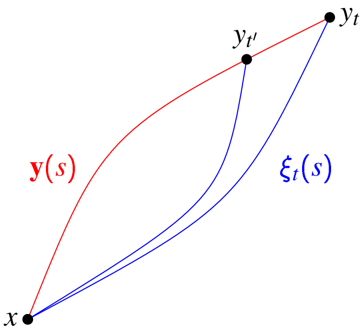

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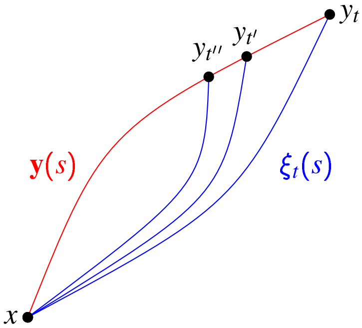

Global singular generalized characteristics

- The arc $\mathbf{y} : [0, t_0] \to M$ is Lipschitz (the constant is independent of x) and it is a generalized characteristic satisfying

$$\dot{\mathbf{y}}(t) \in \operatorname{co} H_p(\mathbf{y}(t), D^+u(\mathbf{y}(t))), \quad \text{a.e. } t \in [0, t_0].$$

Moreover, $\dot{\mathbf{y}}^+(0) = H_p(x, p_0)$, where p_0 is the unique element of minimal energy: $H(x, p) \geqslant H(x, p_0)$ for all $p \in D^+u(x)$.

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- If M is compact, since t_0 is independent of x, then the local defined singular GCs can be extended to a global one.
- It is also true for non-compact M under standard Fathi-Maderna conditions using some local strategy!

Further work

The mechanism in principle

- The basic philosophy of our construction of propagation of singularities is that, under Tonelli-like conditions, if the solution has a representation form of inf-convolution, then the singularities can be interpreted by the associated sup-convolution!

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- It can be applied to Cauchy problem, Dirichlet problem, etc.;
- compact or non-compact manifold;
- Time independent of not;
- Even an implicit representation form recent obtained by Wang-Wang-Yan for the contact type H-J equations $H(x, u(x), Du(x)) = 0$?

The topology of cut locus

A homotopy

One can define a (continuous) homotopy $F : M \times [0, t_0] \to M$,

$$F(x, s) = y_{s,x},$$

which satisfies the following properties:

(a) for all $x \in M$, we have $F(x, 0) = x$;

(b) if $F(x, t) \notin \Sigma(u)$, for some $t > 0$, and $x \in M$, then the curve $s \mapsto F(x, s)$ is u-calibrating on $[0, t]$;

(c) if there exists a u-calibrating curve $\gamma : [0, t] \to M$, with $\gamma(0) = x$, then $s \mapsto F(x, s) = \gamma(s)$, for every $s \in [0, \min(t, t_0)]$.

Homotopy equivalence

Theorem (Cannarsa-C-Fathi, 2017)

The inclusion $\Sigma(u) \subset \mathrm{Cut}\,(u) \subset \overline{\Sigma\,(u)} \cap (M \setminus \mathcal{I}(u)) \subset M \setminus \mathcal{I}(u)$ are all homotopy equivalences.

Corollary

For every connected component C of $M \setminus \mathcal{I}(u)$ the three intersections $\Sigma\,(u) \cap C$, $\mathrm{Cut}\,(u) \cap C$, and $\overline{\Sigma\,(u)} \cap C$ are path-connected.

The key point of the proof is that the cut time function $\tau(x)$ is upper semi-continuous, and the homotopy $G : (M \setminus \mathcal{I}(u)) \times [0,1] \to M \setminus \mathcal{I}(u)$,

$$G(x,s) = F(x, s\alpha(x))$$

is the desired homotopy, where $\alpha : M \setminus \mathcal{I}(u) \to]0,+\infty[$ is a continuous function with $\alpha > \tau$ on $M \setminus \mathcal{I}(u)$.

Local path-conectedness

Theorem (Cannarsa-C-Fathi, 2017)

The spaces $\Sigma(u)$, *and* $\mathrm{Cut}(u)$ *are locally contractible, i.e. for every* $x \in \Sigma(u)$ *(resp.* $x \in \mathrm{Cut}(u)$*) and every neighborhood* V *of* x *in* $\Sigma(u)$ *(resp.* $\mathrm{Cut}(u)$*), we can find a neighborhood* W *of* x *in* $\Sigma(u)$ *(resp.* $\mathrm{Cut}(u)$*), such that* $W \subset V$*, and* W *in null-homotopic in* V.

Therefore $\Sigma(u)$, *and* $\mathrm{Cut}(u)$ *are locally path connected.*

The problem can be attacked by the local homotopy defined above for small t and Hahn-Baire's interpolation theorem (There exists a continuos function α, $u \leqslant \alpha \leqslant v$, with v lower-semicontinuous and u upper semicontinuous).

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The problem can be attacked by the local homotopy defined above for small t and Hahn-Baire's interpolation theorem (There exists a continuos function α, $u \leqslant \alpha \leqslant v$, with v lower-semicontinuous and u upper semicontinuous). Notice that we can choose u to be the cut time function and v to be certain local exit time function.

An example from Mather-Bangert's theory of minimal geodesics

Mather-Bangert theory

generating functions & minimal configurations

The example is based on Bangert's work on minimal geodesics on 2-torus $\mathbb{T}^2$.

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Generating functions & minimal configurations

Let $\mathbb{R}^{\mathbb{Z}}$ be the set of bi-infinite sequences of real numbers with the product topology, and an element in $\mathbb{R}^{\mathbb{Z}}$ will be denoted by $\{x_i\}_{i\in\mathbb{Z}}$. Given a function $h : \mathbb{R}^2 \to \mathbb{R}$, which is called a generating function of a variational principle, a segment $(x_j, \dots, x_k)$, $j < k$, is called minimal if

$$h(x_j, x_{j+1}, \dots, x_k) := \sum_{i=j}^{k-1} h(x_i, x_{i+1}) \leqslant \sum_{i=j}^{k-1} h(y_i, y_{i+1})$$

for all $(y_j, \dots, y_k)$ with $y_j = x_j$ and $y_k = x_k$. A bi-infinite sequence $\{x_i\}$ is said to be minimal if every finite segment of $\{x_i\}$ is minimal. We always call a minimal bi-infinite sequence $\{x_i\} \in \mathbb{R}^{\mathbb{Z}}$ a minimal configuration, and we denote by $\mathcal{M} = \mathcal{M}(h)$ the set of minimal configurations with respect to the generating function h.

Mather-Bangert theory

generating functions & twist maps

Conditions for twist maps

We suppose h satisfies the following conditions:

(h1) $h(x+1, y+1) = h(x, y)$ for all $(x, y) \in \mathbb{R}^2$;

(h2) $\lim_{|y|\to\infty} h(x, x+y) = \infty$ uniformly in x;

(h3) If $x_1 < x_2$ and $y_1 < y_2$, then

$$h(x_1, y_1) + h(x_2, y_2) < h(x_1, y_2) + h(x_2, y_1);$$

(h4) If $(x_{-1}, x_0, x_1) \neq (y_{-1}, y_0, y_1)$ are minimal and $x_0 = y_0$, then $(x_{-1} - y_{-1})(x_1 - y_1) < 0$.

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Area-preserving monotone twist map

When h is smooth, we define $F : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{R}$ by

$$F(x_0, y_0) = (x_1, y_1) \quad \Leftrightarrow \quad y_0 = -D_1 h(x_0, x_1),\ y_1 = D_2 h(x_0, x_1).$$

Mather-Bangert theory

Lifting to $\mathbb{R}^2$

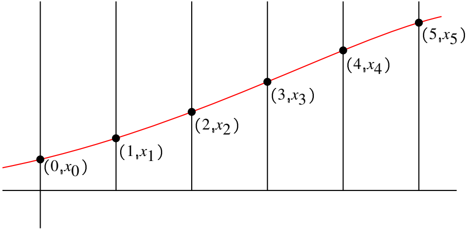

Figure: The minimal geodesics on $\mathbb{T}^2$ and the minimal configurations. The red line is a minimal geodesic in its lift $\mathbb{R}^2$.

Mather-Bangert theory

rotation number

- There exists a continuos map $\tilde{\rho} : \mathcal{M} \to \mathbb{R}$ such that, if $\xi = \{x_i\} \in \mathcal{M}$ then $|x_i - x_0 - i\tilde{\rho}(x)| < 1$ for all $i \in \mathbb{Z}$.

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- Moreover, by the order-preserving properties of the minimal configurations (Lemma of Aubry graphs), we associate a minimal configuration $\xi = \{x_i\}$ with a (order-preserving) circle map $f(x_i) = x_{i+1} \pmod 1$, $i \in \mathbb{Z}$, and the Poincaré's rotation number $\rho(f) = \tilde{\rho}(\xi)$.

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- By well known Denjoy's theorem, when $\rho(f)$ is a irrational number, f is topologically conjugate to a rigid rotation on the circle with the rotation number $\rho(f)$, and the recurrent set of f is the whole circle or a Cantor subset.

Mather-Bangert theory

Denjoy minimal invariant set

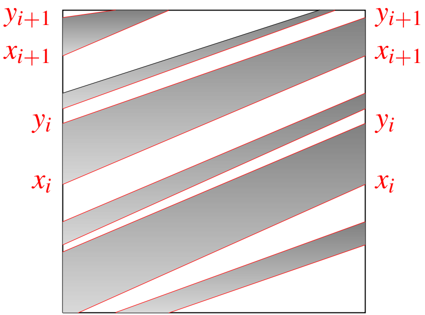

Figure: The minimal geodesics on $\mathbb{T}^2$. The shaded part stands for the complement of the Denjoy set. The red line is a minimal geodesic with rotation number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

Singularities approach Aubry sets

sketch of the proof

- Let $\{(x_i, y_i)\}$ be a sequence of intervals produced by the iteration of any of such an open interval, say (x_0, y_0), then all the iterated open intervals can not intersect each other since order-preserving property of the associated area-preserving monotone twist map.

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- By the periodicity property, we have $\sum_{i\in\mathbb{Z}}(y_i - x_i) \leqslant 1$. Therefore, for any $\varepsilon > 0$, there exists $i_0 \in \mathbb{N}$ such that $\sum_{i>i_0}(y_i - x_i) < \varepsilon$.

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- If $\mathbf{x}$ and $\mathbf{y}$ are the unique generalized characteristics starting from x and y respectively, then there exists $C > 0$ such that

$$|\mathbf{x}(s) - \mathbf{y}(s)| \leqslant C|x - y|, \quad s \in [0, 1).$$

Singularities approach Aubry sets

Formulation of the result

We define a Busemann weak KAM solution

$$u(x) = \inf_{t<0}\{A_t(\gamma(t), x) - A_t(\gamma(t), \gamma(0))\}, \tag{5.1}$$

where $\gamma : (-\infty, 0] \to \mathbb{T}^2$ is semi-static and $A_t(x, y)$ is the minimal action (see, for instance, [Contreras2001]).

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Proposition

Let $\gamma : \mathbb{R} \to \mathbb{T}^2$ be a minimal geodesic with respect to a big bump Riemannian metric and the rotation number of γ is irrational. Suppose u defined in (5.1) is a Busemann weak KAM solution of the associated Hamilton-Jacobi equation. Then, $\mathrm{Sing}\,(u)$ intersects any neighborhood of the projected Aubry set $\Sigma(L)$ nonempty.

semi-dynamics of generalized characteristics

Construction of global GCs

Here we concentrate on the case when M is compact.

- For any $x \in M$, there exists $t_0 > 0$ independent of x such that the curve $\mathbf{y}(\cdot, x) = \mathbf{y} : [0, t_0] \to M$ defined by (GC$_{loc}$) is Lipschitz continuous and $\mathbf{y}(t, x) \in \Sigma(u)$ for all $t \in [0.t_0]$.

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- For each $\tau \in (0, t_0]$, one can define a global GC $\mathbf{x}_\tau : [0, +\infty) \to M$ inductively as

$$\mathbf{x}_\tau(t) = \mathbf{y}(t - i\tau, \mathbf{y}(i\tau, x)), \quad \forall t \in [i\tau, (i+1)\tau],\ i = 1, 2, \ldots. \qquad \text{(GC)}$$

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Our aim is to study the ω-limit set of such GCs in $\mathcal{GC}(x)$.

Critical points

Recall that $x \in M$ is a *critical point* of a viscosity solution u if

$$0 \in \operatorname{co} H_p(x, D^+u(x)),$$

and a *strong critical point* of u if

$$0 \in H_p(x, D^+u(x)).$$

The ω-limit set is an attractor

If $\lim_{t\to\infty} \mathbf{x}_\tau(t)$ exists for any $\mathbf{x}_\tau \in \mathcal{GC}(x)$, then there exists $z \in M$ such that $0 \in H_p(z, D^+u(z))$.

Recurrence property of GCs

We need some quantity to describe the recurrence properties of GCs.

- Let $\mathbf{x}_\tau \in \mathcal{GC}(x)$ such that $\lim_{t\to\infty} \mathbf{x}_\tau(t)$ does not exists and $z_i^\tau = \mathbf{x}_\tau(i\tau)$, $i \in \mathbb{N}$.

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- For any fixed $\tau > 0$, let $\omega = \{i_1, i_2, \dots, i_k, \dots\}$ be any strictly increasing sequence of natural numbers and let $\mathcal{Z}^\tau$ the set of all convergent subsequences of $\{z_i^\tau\}$.

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- For any ω and the associated convergent sequence $z_\omega^\tau = \{z_{i_k}^\tau\}$, we define

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- The quantity $N_\omega(\tau)$ describe the recurrence property of the associated GCs.

Closed GCs

The ω-limit set contains a closed GC

If there exists ω such that $N_\omega(\tau) < \infty$, then there exists a closed generalized characteristic contained in the ω-limit set of $\mathbf{x}_\tau$.

The ω-limit set contains a sequence of closed GCs shrinking to one point

Let $\tau_n \to 0^+$ as $n \to \infty$. If for each $n \in \mathbb{N}$, there exists an ω_n such that $z^{\tau_n}_{\omega_n} \in \mathcal{Z}^{\tau_n}$ with $N_{\omega_n}(\tau_n) < \infty$, and $\lim_{n\to\infty} \tau_n N_{\omega_n}(\tau_n) = 0$, then there exists $z \in M$ such that $0 \in \operatorname{co} H_p(z, D^+u(z))$.

General case

The ω-limit set contains a global defined GC

Fix any $\tau \in (0, t_0]$ and ω. If $N_\omega(\tau) = \infty$, then there exists a global generalized characteristic $\mathbf{x} : (-\infty, +\infty) \to M$ such that $\{\mathbf{x}(t) : t \in \mathbb{R}\}$ is contained in the ω-limit set of $\mathbf{x}_\tau$.

Uniqueness of GCs

- We say a Hamiltonian H *has the uniqueness property* if one has only a unique generalized characteristic from a given initial point.

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- It is still open if there exists an example of Hamiltonian having the uniqueness property besides the Hamiltonians having the p-term with orders less or equal to 2.
- Uniqueness assumption on H implies all the GC can be explained as an arc produced in the procedure of sup-convolution as T_t^+.

Results under uniqueness assumption

When we have uniqueness assumption

Suppose H has the uniqueness property, $x \in \mathrm{Cut}\,(u)$ and $C(x)$ is the component containing x. Let $\mathbf{x} : [0, +\infty) \to M$ be the unique GC staring from x. If there is no critical points of u w.r.t. H in $\overline{C(x)}$, then $\lim_{t\to\infty} \mathbf{x}(t)$ does not exists. In addition, there exists a global GC $\mathbf{y} : \mathbb{R} \to M$ such that $\{\mathbf{y}(t) : t \in \mathbb{R}\}$ is contained in ω-limit set of $\mathbf{x}$. Moreover, we have either

- $\mathbf{y} : \mathbb{R} \to M$ is a global singular generalized characteristic, or
- $\overline{C(x)}$ intersect the Aubry set $\mathcal{I}(u)$ nonempty.

- If ω_x is closed, then all the closed singular GCs are composed of critical points, since the monotonicity property of u along GCs.
- Therefore, $\overline{C(x)}$ must intersect the Aubry set for the unbounded components if there is no critical points in $C(x)$.

More on the critical points

A model problem

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- Suppose $\tilde{\omega}$, the pullback of ω to $\tilde{M}$, is closed, then it is exact by Poincaré's lemma, say $\tilde{\omega} = dS$. Then the associated generalized characteristics has the form (in local chart):

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- x is a (strong) critical point iff $0 \in D^+v(x)$.

More on the critical points

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- If ω is closed and $C \subset \tilde{M}$ is a bounded component of $\overline{\Sigma(u)} = \overline{\mathrm{Sing}(v)}$, then C contains a strong critical point.
- If ω is exact, then each component of $\Sigma(u)$ contains critical points of v.
- It is unclear now if there exists a critical point in the unbounded components of $\Sigma(u)$.

Remarks & possible extension

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- If $\ker\omega$ is an integrable distribution in the sense of Frobenius, the problem is closely connected to the H-J equations with u-term.
- Along this line, what can we talk about the problem in the context of sub-Riemann-Finslerian geometry or non-holonomic mechanics.

Changing cohomology-homology, I

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determines local propagation of singularities in a small time interval $[0, \tau]$.But there is no quantitative estimate of τ on x!

Changing cohomology-homology, II

In [Cannarsa-Yu, 2008], the authors esentially shown

Theorem

If the following condition

$$\begin{aligned} H(x,\cdot) \textit{ takes minimum on } D^+u(x) \textit{ at the unique point } p_0,\\ \textit{where } p_0 \in D^+u(x) \setminus D^*u(x), \end{aligned} \tag{H}$$

is satisfied, then the curve $\mathbf{x}$ *defined by the maximizers is a GC, i.e,*

$$\dot{\mathbf{x}}(s) \in \operatorname{co} H_p(\mathbf{x}(s), D^+u(\mathbf{x}(s))), \quad a.e.\ s \in [0,\tau].$$

with $\dot{\mathbf{x}}(0) = H_p(x, p_0)$.

Changing cohomology-homology, III

- By taking $H = \frac{1}{2}|p - p_0|^2 + \langle q, p - p_0 \rangle$ in the theorem above, we have

$$\dot{\mathbf{x}}(s) \in q - p_0 + D^+u(\mathbf{x}(s))), \quad \text{a.e. } s \in [0, \tau],$$

where $p_0 \in D^+u(x) \setminus D^*u(x)$, and $q \in \mathbb{R}^n$ satisfies

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- Even for mechanical systems, it is not well understood such a changing cohomology-homology process, which occurs when x is a critical point of u definitely.

Changing cohomology-homology, IV

- If H is of class C^3, and H has the form

$$H = H(x, p - p_0 + \bar{p}(x)) + \langle q, p - p_0 \rangle,$$

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- Observe that, in fact, $\bar{p}(x) = L_v(x, 0)$, which is the 1st-order term in the Lagrangian L. Recall the problem we mentioned before!
- Can one gives an intrinsic explanation of such changing Lagrangian method for both regular and singular dynamics, and also the connection to the structure of the flat part of Mather's α-function?

Thanks for your attention!

Tanti Auguri!　　生日快乐!

Piermarco